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DESCENDINGLY INCOMPLETE ULTRAFILTERS AND THE CARDINALITY OF ULTRAPOWERS

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Let D be an ultrafilter on I, and let κ be a cardinal. D is said to be *κ-descendingly incomplete* (κ-d.i.) if there exists a chain $X_\alpha : \alpha < \kappa$ of elements of D such that $\alpha < \beta \rightarrow X_\alpha \subseteq X_\beta$ and $X_\alpha = \phi$. Such a chain will be called a *κ-chain* for D. The notion of κ-descending incompleteness is due to Chang [**3**].

In this paper we explore the relationship between the cardinality of the ultrapower κ^I/D and the existence of certain chains on D. Since we deal so much with questions of size, we do not ordinarily make a notational distinction between a set and its cardinality. Where such a distinction is useful, the cardinality of a set A will be denoted by $|A|$.

The cardinal κ has a natural well-ordering which we denote by $<$. In the usual way, $<$ induces an order on κ^I/D, which we also denote by $<$. There is a natural (order-preserving) embedding of κ into κ^I/D. We make the usual identification and assume that $\kappa \subseteq \kappa^I/D$.

The following result is already implicit in Chang [**3**].

LEMMA 1. *κ is bounded above in κ^I/D with respect to $<$ if and only if D is κ-d.i.*

Proof. Suppose that κ is bounded in κ^I/D, and let $f/D \in \kappa^I/D$ be an upper bound for κ. For any $\alpha < \kappa$, let $X_\alpha = \{i \mid f(i) > \alpha\}$. It is clear that $\{X_\alpha : \alpha < \kappa\}$ is a κ-chain for D.

Conversely, let $\{X_\alpha : \alpha < \kappa\}$ be a κ-chain for D. Define $f : I \rightarrow \kappa$ by $f(i) = \alpha$ if and only if $i \in X_\alpha - X_{\alpha+1}$. Then f/D is an upper bound for κ in κ^I/D.

For ultrafilters D which are not κ-d.i., we obtain a representation for κ^I/D in terms of ultrapowers of smaller objects.

LEMMA 2. *If D is not κ-d.i., then*

$$|\kappa^I/D| = \sup_{\alpha<\kappa} |\alpha|^I/D.$$

Proof. For any $\alpha < \kappa$, let $C_\alpha = \{f/D \mid f/D < \alpha\}$. By Lemma 1, κ is confinal in κ^I/D with respect to $<$, and so we have the representation $\kappa^I/D = \bigcup_{\alpha<\kappa} C_\alpha$. But from the definition of C_α, $|C_\alpha| = |\alpha|^I/D$.

It is well-known that if D is regular and κ-d.i., then $\kappa^I/D > \kappa$. This is essentially a restatement of the fact that if $|I| = \kappa$ and D is uniform, then $\kappa^I/D > \kappa$. The main result of this paper is a partial converse of this theorem.

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If 2^κ is the nth successor of κ for some integer n, a converse essentially as strong as can be expected will follow.

Let $f : I \to A$. Put $i \backsim i'$ if $f(i) = f(i')$. The relation $\backsim$ partitions I. This partition will be called the partition induced by f. If Π is any partition of I, define $g : I \to \Pi$ by $g(i) = C$, where $C \in \Pi$ is the cell to which i belongs. Let D be an ultrafilter on I. We can now define an ultrafilter E on Π by putting $X \in E$ if and only if $g^{-1}(X) \in D$. E will be called the image of D on Π. In this situation, there is for any A a natural embedding of A^Π/E in A^I/D. For to any $f/E \in A^\Pi/E$ there corresponds an object $\hat{f}/D \in A^I/D$, where $\hat{f}(i) = f(C)$ for any cell C and any $i \in C$. A^Π/E will be treated as a subset of A^I/D.

For any ultrafilter D, let th(D) (the thickness of D) be the smallest cardinal μ such that there exists $X \in D$ with $|X| = \mu$. The following small observation will be needed in the proof of the main result:

LEMMA 3. *Let $\Pi_\alpha : \alpha < \rho$ be a sequence of partitions of I such that if $\alpha < \beta$, Π_β is a proper refinement of Π_α. Then for any ordinal η, $|\Pi_\eta| \geqq |\eta|$.*

Proof. We define a sequence C_α of subsets of I such that for any α, C_α meets a cell of Π_α in at most one point. Let $C_0 = \phi$. For any α, $C_{\alpha+1} = C_\alpha \cup \{p\}$ where p is in a cell of $\Pi_{\alpha+1}$ to which no element of C_α belongs. This is possible since $\Pi_{\alpha+1}$ is a proper refinement of Π_α. For α a limit ordinal,

$$C_\alpha = \bigcup_{\beta<\alpha} C_\beta.$$

Then clearly $|\Pi_\eta| \geqq |C_\eta| = |\eta|$.

We have now:

THEOREM 1. *Let $\kappa^I/D > \kappa$. Let λ be the smallest cardinal such that $\kappa^\lambda > \kappa$. Then D is μ-d.i. for some μ with $\lambda \leqq \mu \leqq \max(\kappa, 2^\lambda)$.*

Proof. If we can show that the ultrafilter D has an image E such that th$(E) = \mu$, then D is μ-d.i. For any non-principal ultrafilter E is th(E)-d.i., and since E is an image of D, from any μ-chain in E it is easy to construct a μ-chain in D.

Let $f_\alpha/D : \alpha < \kappa^+$ be a sequence of κ^+ distinct elements of κ^I/D. For each α, the partition induced by f_α has cardinality $\leqq\kappa$. Indeed without loss of generality we may assume that for each α the partition induced by f_α has cardinality $<\lambda$. For if the cardinality of the smallest partition induced by a representative of f/D is μ, then D has an image of thickness μ, and hence a μ-chain.

We now define a sequence Π_α of partitions of I. Let Π_0 be the one cell partition. For any α, let $\Pi_{\alpha+1}$ be the common refinement of Π_α and the partition induced by f_β, where f_β/D is the first element of our sequence which does not have a representative constant on the cells of Π_α. If α is a limit ordinal, let Π_α be the common refinement of all the Π_β with $\beta < \alpha$. For some ordinal $\eta \leqq \kappa^+$, every f_β/D has representative constant on the cells of Π_η, and the process of

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choosing the Π_α terminates. For $\alpha \leqq \eta$, let D_α be the image of D on Π_α. It is clear that (under our identification) $f_\alpha/D \in \kappa^{\Pi}\eta/D_\eta$ for all $\alpha < \kappa^+$. If $\eta \leqq \lambda$, we are through. For since $\Pi_{\alpha+1}$ divides any cell of Π_α into $<\lambda$ pieces, $|\Pi_\eta| \leqq 2^\lambda$. But since $\kappa^{\Pi}\eta/D_\eta \geqq \kappa^+$, $\text{th}(D_\eta) \geqq \lambda$, and so

$$\lambda \leqq \text{th}(D_\eta) \leqq |\Pi_\eta| \leqq 2^\lambda.$$

But then D is μ-d.i. for some μ with $\lambda \leqq \mu \leqq 2^\lambda$.

If $\eta \geqq \lambda$, we show that already $\text{th}(D_\lambda) \geqq \lambda$. Since $\text{th}(D_\lambda) \leqq 2^\lambda$, this will complete the argument. Let $\text{th}(D_\lambda) = \rho$, and let $\Pi_\lambda{}^*$ be an element of D_λ of cardinality ρ. For any α we have a natural projection map $\phi_\alpha : \Pi_\lambda \to \Pi_\alpha$. Let $\Pi_\alpha{}^* = \phi_\alpha(\Pi_\lambda{}^*)$. We show that for any $\alpha < \lambda$, $\Pi^*{}_{\alpha+1}$ refines $\Pi_\alpha{}^*$ properly.

For suppose that $\Pi^*{}_{\alpha+1} = \Pi_\alpha{}^*$, and let f_β be any function constant on the cells of $\Pi_{\alpha+1}$. We define a function $g_\beta : I \to \kappa$. Let C be any cell of Π_α, and K be the collection of $i \in C$ which belong to some cell of $\Pi_\lambda{}^*$. Suppose there is some $i_0 \in K$. If $i \in C$, let $g_\beta(i) = f_\beta(i_0)$. If $K = \phi$, let g_β be constant on C.

Now if $i \in K$, since $\Pi^*{}_{\alpha+1} = \Pi_\alpha{}^*$, i and i_0 must belong to the same cell of $\Pi_{\alpha+1}$, and so $g_\beta(i) = f_\beta(i)$. So f_β and g_β agree on an element of D, and hence f_β/D has a respresentative constant on the cells of Π_α, contradicting the choice for $\Pi_{\alpha+1}$. It follows that $\Pi^*{}_{\alpha+1}$ is a proper refinement of $\Pi_\alpha{}^*$.

But now it follows immediately from Lemma 3 that $\text{th}(D_\lambda) = |\Pi_\lambda{}^*| \geqq \lambda$, and so Theorem 1 is proved.

It seems plausible that the upper bound for μ obtained in Theorem 1 can be improved to κ. This would yield a best possible result, since for any μ, if D is a regular ultrafilter on μ, $|\kappa^\mu/D| = \kappa^\mu$. If κ and λ are as in the statement of Theorem 1, and $2^\lambda \leqq \kappa$, then Theorem 1 yields a best possible upper bound directly. Lemma 2 can be used together with Theorem 1 to deal with other rather special cardinals κ, but we have no generally valid argument that will improve our upper estimate in all cases.

If 2^κ is the nth successor of κ for some integer n, then the upper bound for μ can indeed be improved to κ. This is a routine application of the main result of [**6**]. So in particular we have:

Corollary 1 (G.C.H.). *Let κ be regular. If $\kappa^I/D > \kappa$, then D is κ-d.i.*

For κ singular, assuming the G.C.H., it is tempting to believe that if $\kappa^I/D > \kappa$, D is $\text{cf}(\kappa)$-d.i. However, if we assume the existence of measurable cardinals, a counterexample can be exhibited using ideas similar to those of [**1**].

Corollary 2 (G.C.H.). *If κ is regular, and $\kappa^I/D > \kappa$, then $|(\kappa^I/D)^\kappa| = |\kappa^I/D|$.*

Proof. Chang [**3**] has shown that if

$$\kappa = \sum_{\delta<\gamma} \kappa^\delta,$$

and D is γ-d.i., then $|(\kappa^I/D)^\gamma| = |\kappa^I/D|$. By Corollary 1, Chang's condition is

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fulfilled with $\gamma = \kappa$. From Keisler's inequality $(\kappa^\lambda)^I/D \leqq (\kappa^I/D)^\lambda$ [4] one can only conclude that $\mathrm{cf}(|\kappa^I/D|) \geqq \kappa$.

COROLLARY 3 (C.G.H.). *Let κ be regular. Then κ is confinal in κ^I/D if and only if $|\kappa^I/D| = \kappa$.*

Proof. The proof follows by Lemma 1.

From the proof of Theorem 4, it is easy to see that (assuming 2^κ is the nth successor of κ for some n) if $|\kappa^I/D| = \kappa$, there is an ultrafilter E on a set J with $|J| < \kappa$ such that $\kappa^I/D \simeq \kappa^J/E$. So if we think of κ as being equipped with its full structure (all relations and functions on κ), κ^I/D is a simple extension of κ [**2**]. It is natural to ask here whether in the proof of this purely algebraic result special assumptions about exponentiation of cardinals can be eliminated. It is also reasonable to expect that if $|\kappa^I/D| \leqq 2^\kappa$, there is an ultrafilter E on a set J with $|J| \leqq \kappa$ such that $\kappa^I/D \simeq \kappa^J/E$. At this moment these questions remain open.

Define a function f from ordinals to cardinals by putting $f(0) = \omega$, $f(\alpha + 1) = |(f(\alpha))^I/D|$, and for limit ordinals β, $f(\beta) = \sup_{\alpha<\beta} f(\alpha)$. The function f reaches a maximum $\mu \leqq |2^I|$.

COROLLARY 4 (G.C.H.). *μ is the smallest cardinal such that D is not μ-d.i. In particular, if $|\omega^I/D| = |\omega^I|$, then D is κ-d.i. for all infinite $\kappa \leqq |I|$.*

In the proof of the next result, we need the fact that if $(2^\kappa)^I/D > 2^\kappa$, then $\kappa^I/D > \kappa$. Without any additional trouble we can prove the slightly stronger

LEMMA 4. $(A^B)^I/D \geqq (A^I/D)^{B^I/D}$.

Proof.† Any second order existential sentence true in a model $\mathscr{C}$ is true in every ultrapower of $\mathscr{C}$. Consider the model $\mathscr{C} = \langle A^B, A, B, R\rangle$ where $R(a, b, f)$ if and only if $f(b) = a$. In the model $\mathscr{C}^I/D$ we have

$$(A^B)^I/D \subseteq (A^I/D)^{B^I/D}$$

with the obvious identification induced by R^I/D.

Lemma 4 quickly yields that if $(2^\kappa)^I/D > 2^\kappa$, then $\kappa^I/D > \kappa$. For let $A = 2$, $B = \kappa$. Then $(2^\kappa)^I/D \leqq 2^{\kappa^I/D}$ and so if $(2^\kappa)^I/D > 2^\kappa$, we must have $\kappa^I/D > \kappa$.

Our final result gives a very weak estimate for the cardinality of κ^I/D when κ is a limit cardinal in terms of cardinalities of ultrapowers of cardinals smaller than κ.

THEOREM 2 (G.C.H.). *Let κ be a limit cardinal. Suppose there is a sequence λ_α of regular cardinals such that $\lambda_\alpha \rightarrow \kappa$ and $\lambda_\alpha{}^I/D > \lambda_\alpha$. Then $\kappa^I/D > \kappa$.*

†We thank the referee for suggesting this simple proof.

Proof. By Corollary 1, D has a λ_α-chain for all α. From a chain $\{X_\beta : \beta < \lambda_\alpha\}$ one obtains a partition Π_α of I whose cells are the sets $X_{\beta+1} - X_\beta$. Since we have a λ_α-chain and λ_α is regular, the image E_α of D on Π_α is uniform. Let Π be the common refinement of the partitions Π_α, and let E be the image of D on Π. $|\Pi| \leqq 2^\kappa = \kappa^+$. Since each E_α is an image of E, and $\text{th}(E_\alpha) = \lambda_\alpha$, $\text{th}(E) \geqq \kappa$. If $\text{th}(E) = \kappa$, E is κ-d.i., so $\kappa^I/D > \kappa$. If $\text{th}(E) = \kappa^+ = 2^\kappa$, $(2^\kappa)^I/D > 2^\kappa$, and so $\kappa^I/D > \kappa$.

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