Ergod. Th. & Dynam. Sys. (2018), **38**, 961–981
doi:10.1017/etds.2016.55

Hölder continuity of Oseledets splittings for semi-invertible operator cocycles

DAVOR DRAGIČEVIĆ and GARY FROYLAND

School of Mathematics and Statistics, University of New South Wales, Sydney NSW 2052, Australia
(e-mail: d.dragicevic@unsw.edu.au, g.froyland@unsw.edu.au)

(*Received* 18 *February* 2016 *and accepted in revised form* 17 *May* 2016)

Abstract. For Hölder continuous cocycles over an invertible, Lipschitz base, we establish the Hölder continuity of Oseledets subspaces on compact sets of arbitrarily large measure. This extends a result of Araújo *et al* [On Hölder-continuity of Oseledets subspaces *J. Lond. Math. Soc.* **93** (2016) 194–218] by considering possibly non-invertible cocycles, which, in addition, may take values in the space of compact operators on a Hilbert space. As a by-product of our work, we also show that a non-invertible cocycle with non-vanishing Lyapunov exponents exhibits non-uniformly hyperbolic behaviour (in the sense of Pesin) on a set of full measure.

1. *Introduction*

The celebrated Oseledets multiplicative ergodic theorem (MET) [**20**] plays a fundamental role in the modern theory of dynamical systems. At an abstract level, the MET generalizes the notion of eigenvalues and eigenvectors for a single matrix $A \in \mathbb{R}^{d\times d}$ to concatenations of matrices $A(f^{n-1}x)\cdots A(f(x))A(x)$, where $A : X \to \mathbb{R}^{d\times d}$ is an invertible matrix-valued function on a probability space $(X, \mathcal{B}, \mu)$ and $f : X \circlearrowleft$. Under some technical assumptions, the MET guarantees the existence of a finite set of numbers (called Lyapunov exponents) and subspaces of $\mathbb{R}^d$ (called Oseledets subspaces) which either form a decomposition or a filtration of $\mathbb{R}^d$ (depending on whether f is invertible or not) such that Lyapunov exponents describe the asymptotic growth of vectors that belong to Oseledets subspaces under the action of A.

Arguably, the most important applications of this result are in the area of smooth dynamics. For example, the proof of MET initiated the study of non-uniformly hyperbolic dynamical systems: that is, systems with non-zero Lyapunov exponents with respect to some smooth invariant probability measure. Since the landmark works of Pesin in the 1970s, the theory of non-uniform hyperbolicity emerged as an independent, rich and active discipline lying at the heart of dynamical systems theory. Among the most important consequences of non-uniform hyperbolicity is the existence of stable invariant manifolds and their absolute continuity property (see [**21**]). The theory also describes the ergodic properties of a dynamical system with a finite invariant measure that is absolutely continuous with respect to the volume and it expresses the Kolmogorov–Sinai entropy in terms of the Lyapunov exponents by Pesin's entropy formula (see [**21**]). Furthermore, combining the non-uniform hyperbolicity with the non-trivial recurrence guaranteed by the existence of a finite invariant measure, the work of Katok [**17**] revealed a rich and complicated orbit structure, including an exponential growth rate for the number of periodic points measured in terms of the topological entropy and the approximation of the entropy of an invariant measure by uniformly hyperbolic horseshoes. More recently, Barreira, Pesin and Schmeling [**6**] discovered a striking relationship between this theory and a dimension theory of dynamical systems by resolving the long standing Eckmann–Ruelle conjecture. We refer to [**5**] for further references and a detailed exposition of this theory.

Oseledets' MET has not only been re-proved in many different ways, but it has also been generalized several times, including to compact operators on Hilbert spaces by Ruelle [**22**], to compact operators on Banach spaces with some continuity conditions on the base f and the dependence of the operators on $x \in X$ by Mañé [**19**], to quasi-compact operators on possibly non-separable Banach spaces with continuity conditions by Thieullen [**23**] and to quasi-compact operators on separable Banach spaces with weaker continuity conditions by Lian and Lu [**18**]. Prior to the publication of [**12**], all previous work considered the MET in one (or both) of two flavours: either there is no invertibility assumption on the base and the linear actions and one obtains the existence of an equivariant flag or filtration, or there is an invertibility assumption on *both* the base *and* the linear actions and one obtains the much stronger outcome of existence of an equivariant splitting.

Froyland *et al* [**12**] extended the classical Oseledets multiplicative ergodic theorem by proving that if the base is invertible, a unique Oseledets splitting exists even when the matrices are not necessarily invertible.

THEOREM 1. [**12**] *Let $f : X \circlearrowleft$ preserve an ergodic Borel probability measure μ and assume that $A : X \to \mathbb{R}^{d\times d}$ satisfies*

$$\int_X \log^+ \|A(x)\| \, d\mu(x) < +\infty. \tag{1}$$

Then there exist numbers

$$-\infty \le \lambda_1 < \lambda_2 < \cdots < \lambda_k \tag{2}$$

and, for μ-almost every $x \in M$, a measurable decomposition

$$\mathbb{R}^d = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_k(x) \tag{3}$$

such that

$$A(x)E_i(x) \subset E_i(f(x)) \quad (\textit{with equality if } \lambda_i > -\infty) \tag{4}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log\|A(f^{n-1}x)\cdots A(fx)A(x)v\| = \lambda_i \quad \textit{for } v \in E_i(x)\setminus\{0\},\ i \in \{1, \ldots, k\}. \tag{5}$$

Semi-invertible versions of the Oseledets theorem for quasi-compact operator cocycles were developed in [**13**] and [**15**], generalizing the results of [**23**] and [**18**], respectively. The numbers in (2) are called *Lyapunov exponents* and we will refer to the splitting in (3) as the *Oseledets splitting*. Furthermore, we will say that $E_i(x)$ is the Oseledets subspace that corresponds to a Lyapunov exponent λ_i.

It is well known that, in general, in all the above-mentioned generalizations of MET, the Oseledets subspaces depend only measurably on base points. However, it was recently proved by Araújo *et al* [**2**] that, under the assumptions that $f : X \circlearrowleft$ is a Lipschitz map and that $A : X \to GL(d, \mathbb{R})$ is Hölder continuous, one is able to establish Hölder continuity of the Oseledets subspaces on compact sets of arbitrarily large measure. The arguments in [**2**] build on a previous work of Brin [**9**] who proved (in a particular case of derivative cocycles) that, for Anosov systems, the stable and unstable distributions depend Hölder continuously everywhere and that the same happens for non-uniformly hyperbolic systems, but on a compact set of arbitrarily large measure.

The main objective of this paper is to extend the results from [**2**] by considering possibly non-invertible cocycles as well as compact operator cocycles with values in the space of all bounded linear operators acting on some Hilbert space. In order to describe our main result in a finite-dimensional case, assume that $A : X \to \mathbb{R}^{d\times d}$ is a Hölder continuous cocycle over an invertible Lipschitz transformation $f : X \circlearrowleft$ satisfying (1). We prove that the Oseledets subspaces in Theorem 1 are Hölder continuous on compact sets of arbitrarily large measure. We emphasize that the lack of the invertibility causes substantial complications and that, consequently, crucial parts of our argument differ from the approach developed in [**2**]. In addition, this new setting requires new proofs of versions of some well-known facts from Pesin theory. For example, Theorem 2 establishes upper and lower bounds for the growth of the cocycles when restricted to the subbundles $E(x)$ and $F(x)$ given by

$$E(x) = E_1(x) \oplus \cdots \oplus E_i(x) \quad \text{and} \quad F(x) = E_{i+1} \oplus \cdots \oplus E_k(x),$$

as well as a lower bound on the angle between $E(x)$ and $F(x)$. This result plays an important role in our arguments but is also of independent interest since it, in particular, implies that if all Lyapunov exponents are non-zero, then the cocycle exhibits a non-uniformly hyperbolic behaviour on a set of full measure. To the best of our knowledge, this result had not yet been established before for semi-invertible cocycles.

We emphasize that semi-invertible cocycles arise in two very important situations from the point of view of applications. Firstly, the study of Markov chains in a random environment (MCRNs). Markov chains form the basis of mathematical models for a huge variety of physical, chemical, and biological phenomena, including problems in statistical mechanics, (bio)chemical engineering, epidemic modelling, complex networks and genetics. More typically than not, the underlying transition probabilities in the Markov chain model evolve over time according to some external random or time-dependent environment. Instead of having a single invariant probability measure for a stationary Markov chain, Markov chains in random environments possess a family of (random) invariant measures (see, e.g., [**10**]), which depend on the environment. In the language of Oseledets' MET, X is the environment, $f : X \circlearrowleft$ describes the evolution of the random

https://doi.org/10.1017/etds.2016.55 Published online by Cambridge University Press

environment and $A : X \to \mathbb{R}^{d\times d}$ is a stochastic matrix-valued function. The family of random invariant measures are the top Oseledets spaces, corresponding to the leading Lyapunov exponent $\lambda_k = 0$. The stability of these random invariant measures has been explored in [**11**], where it is shown that, under mild assumptions on perturbations to f or A, the random invariant measure is continuous in probability with respect to the environment. Theorem 5 in the present work will show that if f is Lipschitz and $x \mapsto A(x)$ is Hölder continuous, then the random invariant measure depends Hölder continuously on the environment configuration $x \in X$ on compact sets of arbitrarily large measure. These assumptions on f and A are very reasonable for mathematical models of real-world processes and our result provides the assurance that, on the vast bulk of the environment space, the time-asymptotic distribution of trajectories of the MCRN varies continuously with the environment.

A second application, which was the motivation for [**12**], concerns a program to understand time-dependent dynamical systems through transfer operator cocycles. One begins with a function $x \mapsto T_x$, where each $T_x : M \circlearrowleft$ is a nonlinear map on a smooth Riemannian manifold M. A map cocycle $T_{f^{n-1}x} \circ \cdots \circ T_{fx} \circ T_x$ represents the time-dependent evolution of a nonlinear dynamical system. For example, let M be a three-dimensional manifold representing the ocean, let X be the internal configuration of the ocean (e.g. the distribution of pressure gradients), let f describe how the internal configuration changes over one day and let T_x describe the motion of water particles over one day given that the current configuration is x. Associated with each T_x is a linear operator (the transfer operator; see, e.g., [**4**] for definitions) $\mathcal{L}_x : \mathcal{B} \circlearrowleft$, which acts on a suitable Banach space $\mathcal{B}$. Continuing with our ocean example, if $g(z) : M \to \mathbb{R}$ describes the distribution of some inert, neutrally buoyant chemical in the ocean at 'time' $x \in X$, then $(\mathcal{L}_x g)(z)$ is the distribution of the chemical one day later. That is, the transfer operators $\{\mathcal{L}_x\}_{x\in X}$ transform densities in $\mathcal{B}$ to densities in $\mathcal{B}$ just as the maps $\{T_x\}_{x\in X}$ transform points in M to points in M.

In many areas of nonlinear dynamics, including fluid dynamics and models of geophysical flow such as the ocean and atmosphere, one is interested in structures that decay to equilibrium very slowly: so-called *Lagrangian coherent structures* or *coherent sets*. In fluid dynamics, these represent parts of the fluid that are slow to mix with the rest of the fluid; in the ocean and atmosphere, these structures have physical manifestations as gyres and eddies, and vortices, respectively. It turns out that the second largest Lyapunov exponent (the first non-trivial exponent after $\lambda_k = 0$) describes the time-asymptotic decay rate of the family of most slowly decaying signed distributions $\{g_x(z)\}_{x\in X}$. Furthermore, and crucially for applications, these signed distributions are given by the corresponding second Oseledets spaces (see [**12**, **14**] for details). In numerical experiments, the transfer operators $\mathcal{L}_x$ are represented as large stochastic matrices on computers and the Oseledets spaces are similarly discretized. Theorem 5 in the present paper states that if f is Lipschitz and the linear actions are Hölder continuous, then the corresponding Oseledets spaces, which describe the coherent structures, are Hölder continuous functions on subsets of the base space X of arbitrarily large measure. This establishes the important fact that, in applications, dramatic changes in coherent structures as a function of the driving configuration are extremely rare.

2. *Semi-invertible cocycles and non-uniform hyperbolicity*

In order to make our arguments more transparent and easier to follow, our presentation is for finite-dimensional cocycles. In the final section, we highlight the changes necessary to deal with the infinite-dimensional setting. A measurable map $\mathcal{A} : X \times \mathbb{N}_0 \to \mathbb{R}^{d\times d}$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, is said to be a *cocycle* over f if:

(1) $\mathcal{A}(x, 0) = \mathrm{Id}$ for every $x \in X$; and

(2) $\mathcal{A}(x, n+m) = \mathcal{A}(f^n(x), m)\mathcal{A}(x, n)$ for every $x \in X$ and $n, m \geq 0$.

A map $A : X \to \mathbb{R}^{d\times d}$, defined by $A(x) = \mathcal{A}(x, 1)$, $x \in X$, is called a *generator* of a cocycle $\mathcal{A}$. For an f-invariant set $\Lambda \subset X$, a family of subspaces $E(x) \subset \mathbb{R}^d$, $x \in \Lambda$ is called $\mathcal{A}$-*invariant* if $A(x)E(x) \subset E(fx)$ for each $x \in \Lambda$.

We will now establish several auxiliary results related to Theorem 1 that will be used throughout the paper. We start with the following lemma.

LEMMA 1. *Assume that Λ is an f-invariant set and let $E(x) \subset \mathbb{R}^d$ and $F(x) \subset \mathbb{R}^d$, $x \in \Lambda$ be $\mathcal{A}$-invariant families of subspaces with the property that there exist $\lambda_1 < \lambda_2$, $\varepsilon > 0$ and measurable functions $C, \tilde{C} : \Lambda \to (0, \infty)$ such that:*

(1)
$$\lambda_1 + 3\varepsilon \leq \lambda_2 - 2\varepsilon; \tag{6}$$

(2) $E(x) \cap F(x) = \{0\}$ *for $x \in \Lambda$;*

(3) *for $x \in \Lambda$, $v \in E(x) \oplus F(x)$ and $n \geq 0$,*
$$\|\mathcal{A}(x, n)v\| \leq \tilde{C}(x)e^{(\lambda_2+\varepsilon)n}\|v\|; \tag{7}$$

(4) *for $x \in \Lambda$, $v \in F(x)$ and $n \geq 0$,*
$$\|\mathcal{A}(x, n)v\| \geq \frac{1}{C(x)}e^{(\lambda_2-\varepsilon)n}\|v\|; \tag{8}$$

(5) *for $x \in \Lambda$, $v \in E(x)$ and $n \geq 0$,*
$$\|\mathcal{A}(x, n)v\| \leq C(x)e^{(\lambda_1+\varepsilon)n}\|v\|; \textit{ and} \tag{9}$$

(6) *for $x \in \Lambda$ and $m \in \mathbb{Z}$,*
$$\tilde{C}(f^m(x)) \leq \tilde{C}(x)e^{\varepsilon|m|} \quad \textit{and} \quad C(f^m(x)) \leq C(x)e^{\varepsilon|m|}. \tag{10}$$

Then, there exists a measurable function $K : \Lambda \to (0, \infty)$ satisfying
$$K(f^m(x)) \leq K(x)e^{5\varepsilon|m|} \quad \textit{for } x \in \Lambda \textit{ and } m \in \mathbb{Z} \tag{11}$$

and such that
$$\|v_1\| \leq K(x)\|v_1 + v_2\| \quad \textit{and} \quad \|v_2\| \leq K(x)\|v_1 + v_2\| \tag{12}$$

for $v_1 \in E(x)$ and $v_2 \in F(x)$.

Proof. Let $P(x) : E(x) \oplus F(x) \to E(x)$ and $Q(x) : E(x) \oplus F(x) \to F(x)$ be projections. Set
$$\gamma(x) = \inf\{\|v_1 + v_2\| : v_1 \in E(x),\ v_2 \in F(x),\ \|v_1\| = \|v_2\| = 1\}.$$

https://doi.org/10.1017/etds.2016.55 Published online by Cambridge University Press

For any $v \in E(x) \oplus F(x)$ such that $P(x)v \neq 0$ and $Q(x)v \neq 0$,

$$\begin{aligned}
\gamma(x) &\leq \left\| \frac{P(x)v}{\|P(x)v\|} + \frac{Q(x)v}{\|Q(x)v\|} \right\| \\
&= \frac{1}{\|P(x)v\|} \left\| P(x)v + \frac{\|P(x)v\|}{\|Q(x)v\|} Q(x)v \right\| \\
&= \frac{1}{\|P(x)v\|} \left\| v + \frac{\|P(x)v\| - \|Q(x)v\|}{\|Q(x)v\|} Q(x)v \right\| \\
&\leq \frac{2\|v\|}{\|P(x)v\|}.
\end{aligned}$$

Hence

$$\|P(x)v\| \leq \frac{2}{\gamma(x)} \|v\|.$$

We note that the above inequality is trivially satisfied when $P(x)v = 0$. Finally, if $Q(x)v = 0$, then $P(x)v = v$ and we conclude that

$$\|P(x)\| \leq \max\{1, 2/\gamma(x)\} \quad \text{for } x \in \Lambda. \tag{13}$$

Similarly,

$$\|Q(x)\| \leq \max\{1, 2/\gamma(x)\} \quad \text{for } x \in \Lambda. \tag{14}$$

Now take arbitrary $v_1 \in E(x)$ and $v_2 \in F(x)$ such that $\|v_1\| = \|v_2\| = 1$. By (7)–(9),

$$\begin{aligned}
\|v_1 + v_2\| &\geq \frac{1}{\tilde{C}(x)e^{(\lambda_2+\varepsilon)n}} \|\mathcal{A}(x, n)(v_1 + v_2)\| \\
&\geq \frac{1}{\tilde{C}(x)e^{(\lambda_2+\varepsilon)n}} \left(\frac{1}{C(x)} e^{(\lambda_2-\varepsilon)n} - C(x)e^{(\lambda_1+\varepsilon)n} \right),
\end{aligned} \tag{15}$$

for every $n \geq 0$. Let $n(x)$ be the smallest integer such that

$$\frac{1}{C(x)} e^{(\lambda_2-\varepsilon)n(x)} - C(x)e^{(\lambda_1+\varepsilon)n(x)} \geq \frac{1}{C(x)} e^{(\lambda_2-2\varepsilon)n(x)} \tag{16}$$

or, equivalently,

$$e^{(\lambda_2-\varepsilon)n(x)} - C(x)^2 e^{(\lambda_1+\varepsilon)n(x)} \geq e^{(\lambda_2-2\varepsilon)n(x)}. \tag{17}$$

By (15) and (16),

$$\gamma(x) \geq \frac{1}{\tilde{C}(x)e^{(\lambda_2+\varepsilon)n(x)}} \cdot \frac{1}{C(x)} e^{(\lambda_2-2\varepsilon)n(x)}$$

and thus

$$\frac{2}{\gamma(x)} \leq 2C(x)\tilde{C}(x)e^{3\varepsilon n(x)}. \tag{18}$$

Finally, we claim that $n(f^m(x)) \leq n(x) + |m|$ for each $x \in \Lambda$ and $m \in \mathbb{Z}$. Indeed, using (6) and (10),

$$\begin{aligned}
&e^{(\lambda_2-\varepsilon)(n(x)+|m|)} - C(f^m(x))^2 e^{(\lambda_1+\varepsilon)(n(x)+|m|)} \\
&\quad \geq e^{(\lambda_2-\varepsilon)n(x)} \cdot e^{(\lambda_2-\varepsilon)|m|} - C(x)^2 e^{(\lambda_1+\varepsilon)n(x)} \cdot e^{(\lambda_1+3\varepsilon)|m|} \\
&\quad \geq e^{(\lambda_2-\varepsilon)n(x)} \cdot e^{(\lambda_2-\varepsilon)|m|} - C(x)^2 e^{(\lambda_1+\varepsilon)n(x)} \cdot e^{(\lambda_2-2\varepsilon)|m|} \\
&\quad \geq e^{(\lambda_2-2\varepsilon)n(x)} \cdot e^{(\lambda_2-2\varepsilon)|m|} \quad \text{by (17)} \\
&\quad \geq e^{(\lambda_2-2\varepsilon)(n(x)+|m|)}.
\end{aligned}$$

https://doi.org/10.1017/etds.2016.55 Published online by Cambridge University Press

In order to complete the proof of the lemma, we are going to show that the function $K(x) = \max\{1, 2C(x)\tilde{C}(x)e^{3\varepsilon n(x)}\}$ satisfies (11) and (12). We note that (12) follows directly from (13), (14) and (18). Moreover, using (10),

$$C(f^m(x))\tilde{C}(f^m(x))e^{3\varepsilon n(f^m(x))} \le C(x)\tilde{C}(x)e^{3\varepsilon n(x)} \cdot e^{5\varepsilon|m|},$$

for each $x \in \Lambda$ and $m \in \mathbb{Z}$, which readily implies that (11) holds. □

Suppose that the Lyapunov exponents of the cocycle $\mathcal{A}$ are given by (2). Then, for each $i \in \{1, \ldots, k\}$, we can associate to (3) a new decomposition of $\mathbb{R}^d$ as

$$\mathbb{R}^d = \bigg(\bigoplus_{j \le i} E_j(x)\bigg) \oplus \bigg(\bigoplus_{j > i} E_j(x)\bigg). \tag{19}$$

The following result establishes exponential bounds for $\mathcal{A}$ along the two subspaces forming the decomposition (19) as well as for angles between them. For invertible cocycles, such a result is well known (see [**5**, Theorem 3.3.1], for example). A major difficulty in adapting the arguments in [**5**] is that they rely heavily on the well-known fact that the angles between Oseledets subspaces in the standard (invertible) MET exhibit a subexponential growth along each trajectory. On the other hand, to the best of our knowledge, no such statement was established in relation to the semi-invertible version of MET stated in Theorem 1. This forces us to develop an argument (based on Lemma 1), which is completely different from the one in [**5**], to first establish exponential bounds for $\mathcal{A}$ along the subspaces in (19) and then use this to deduce an appropriate bound for the angle between those subspaces.

THEOREM 2. *Let $\mathcal{A}$ be a cocycle over f satisfying (1) with Lyapunov exponents as in (2) and take $i \in \{1, \ldots, k\}$. Let*

$$E^1(x) = \bigoplus_{j=1}^{i} E_j(x) \quad \text{and} \quad E^2(x) = \bigoplus_{j=i+1}^{k} E_j(x).$$

Then there exists a Borel set $\Lambda \subset X$ such that $\mu(\Lambda) = 1$ and, for each $\varepsilon > 0$, there are measurable functions $C, K : \Lambda \to (0, \infty)$ with the property that, for every $x \in \Lambda$:

(1) *for each $v \in E^1(x)$ and $n \ge 0$,*

$$\|\mathcal{A}(x, n)v\| \le C(x)e^{(\lambda_i+\varepsilon)n}\|v\|, \tag{20}$$

where, if $i = 1$ and $\lambda_1 = -\infty$, λ_1 is replaced by any number that belongs to the interval $(-\infty, \lambda_2)$;

(2) *for each $v \in E^2(x)$ and $n \ge 0$,*

$$\|\mathcal{A}(x, n)v\| \ge \frac{1}{C(x)}e^{(\lambda_{i+1}-\varepsilon)n}\|v\|; \tag{21}$$

(3) *for each $u \in E^1(x)$ and $v \in E^2(x)$,*

$$\|u\| \le K(x)\|u + v\| \quad \text{and} \quad \|v\| \le K(x)\|u + v\|; \text{ and} \tag{22}$$

(4) *for each $n \in \mathbb{Z}$,*

$$C(f^n(x)) \le C(x)e^{\varepsilon|n|} \quad \text{and} \quad K(f^n(x)) \le K(x)e^{\varepsilon|n|}. \tag{23}$$

https://doi.org/10.1017/etds.2016.55 Published online by Cambridge University Press

Proof. Step 1. Upper bound for growth on E^1 and temperedness of the function C.

We begin by establishing property (20). We start with the following lemma.

LEMMA 2.

$$\limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(x,n)|E^1(x)\| \le \lambda_i \quad \text{for } \mu - \text{a.e. } x \in X, \tag{24}$$

where $\mathcal{A}(x,n)|E^1(x)$ denotes the restriction of $\mathcal{A}(x,n)$ onto $E^1(x)$.

Proof of the lemma. Let $\{e_1, \ldots, e_l\}$ be an orthonormal basis for $E^1(x)$. For each $n \in \mathbb{N}$, let $v_n \in E^1(x)$ be such that $\|v_n\| = 1$ and $\|\mathcal{A}(x,n)|E^1(x)\| = \|\mathcal{A}(x,n)v_n\|$. Furthermore, for $n \in \mathbb{N}$, write v_n in the form

$$v_n = \sum_{j=1}^{l} a_{j,n} e_j$$

for $a_{j,n} \in \mathbb{R}$. We note that $|a_{j,n}| = |\langle v_n, e_j\rangle| \le \|v_n\| \cdot \|e_j\| = 1$ and thus

$$\|\mathcal{A}(x,n)|E^1(x)\| \le \sum_{j=1}^{l} |a_{j,n}| \cdot \|\mathcal{A}(x,n)e_j\| \le \sum_{j=1}^{l} \|\mathcal{A}(x,n)e_j\|. \tag{25}$$

Since $e_j \in E^1(x)$, it follows, from (5), that

$$\limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(x,n)e_j\| \le \lambda_i \quad \text{for } j \in \{1, \ldots, l\}. \tag{26}$$

Finally, we note that (25) and (26) readily imply (24). □

It follows from (24) that, for $\varepsilon > 0$,

$$D(x) := \sup_{n\ge 0}\{\|\mathcal{A}(x,n)|E^1(x)\| \cdot e^{-(\lambda_i+\varepsilon)n}\} < \infty \tag{27}$$

for μ almost every $x \in X$.

LEMMA 3.

$$\lim_{n\to\pm\infty} \frac{1}{n} \log D(f^n(x)) = 0 \quad \text{for } \mu\text{-almost every } x \in X. \tag{28}$$

Proof of the lemma. For $n \ge 1$,

$$\begin{aligned}\|\mathcal{A}(x,n) \mid E^1(x)\| &\le \|\mathcal{A}(f(x), n-1) \mid E^1(f(x))\| \cdot \|A(x)|E^1(x)\| \\ &\le \|\mathcal{A}(f(x), n-1) \mid E^1(f(x))\| \cdot \|A(x)\|.\end{aligned}$$

By multiplying the above inequality by $e^{-(\lambda_i+\varepsilon)n}$, we obtain

$$e^{-(\lambda_i+\varepsilon)n}\|\mathcal{A}(x,n)|E^1(x)\| \le e^{-(\lambda_i+\varepsilon)(n-1)}\|\mathcal{A}(f(x), n-1)|E^1(f(x))\| \cdot e^{-(\lambda_i+\varepsilon)}\|A(x)\|.$$

Hence

$$D(x) \le D(f(x)) \cdot \max\{e^{-(\lambda_i+\varepsilon)}\|A(x)\|, 1\}.$$

It follows, from (1), that there exists a non-negative and integrable function $\psi : X \to \mathbb{R}$ such that

$$\log D(x) - \log D(f(x)) \le \psi(x). \tag{29}$$

https://doi.org/10.1017/etds.2016.55 Published online by Cambridge University Press

Set

$$\tilde{D}(x) = \log D(x) - \log D(f(x)).$$

We note that

$$\frac{1}{n}\log D(f^n(x)) = \frac{1}{n}\log D(x) - \frac{1}{n}\sum_{j=0}^{n-1}\tilde{D}(f^j(x)) \tag{30}$$

for each $x \in X$ and $n \in \mathbb{N}$. By (29), $\tilde{D}^+$ is integrable. Hence we can apply the Birkhoff ergodic theorem (see [**3**, p. 539]) and conclude that there exists $a \in [-\infty, \infty)$ such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\tilde{D}(f^j(x)) = a \tag{31}$$

for μ-almost every $x \in X$. It follows, from (30) and (31), that

$$\lim_{n\to\infty}\frac{1}{n}\log D(f^n(x)) = -a.$$

On the other hand, since μ is f-invariant, for any $c > 0$,

$$\lim_{n\to\infty}\mu(\{x \in X : \log D(f^n(x))/n \geq c\}) = \lim_{n\to\infty}\mu(\{x \in X : \log D(x) \geq nc\}) = 0,$$

which immediately implies that $a \geq 0$. Thus

$$\lim_{n\to\infty}\frac{1}{n}\log D(f^n(x)) \leq 0.$$

Since $D(x) \geq 1$ for μ almost every $x \in X$, by (27), we conclude that (28) holds when $n \to \infty$.

Now we establish (28) for the case $n \to -\infty$. Set

$$D'(x) = \log D(f^{-1}(x)) - \log D(x).$$

Obviously,

$$\frac{1}{n}\log D(f^{-n}(x)) = \frac{1}{n}\log D(x) + \frac{1}{n}\sum_{j=0}^{n-1}D'(f^{-j}(x)) \tag{32}$$

for each $x \in X$ and $n \in \mathbb{N}$. By (29), D'^+ is integrable. Hence we can apply the Birkhoff ergodic theorem and conclude that there exists $a \in [-\infty, \infty)$ such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}D'(f^{-j}(x)) = a \tag{33}$$

for μ-almost every $x \in X$. We can now proceed as in the previous case and obtain that $a = 0$, which implies (28). □

It follows, from (28) and Proposition 4.3.3(ii) in [**3**], that there exists a non-negative and measurable function C defined on a set of full measure satisfying the first inequality in (23) such that $D(x) \leq C(x)$, which, together with (27), implies that (20) holds.

Step 2. Lower bound for growth on E^2 and temperedness of the function $1/C$.

https://doi.org/10.1017/etds.2016.55 Published online by Cambridge University Press

We now show (21). By Theorem 1, the cocycle $\mathcal{A}$ is invertible along the subbundle E^2. We would like to apply Step 1 to the inverse of the cocycle obtained by restricting $\mathcal{A}$ onto the subbundle E^2 to conclude that (21) holds for some function C satisfying the first inequality in (23) on a set of full measure. In order to do this, we first require the integrability condition of Lemma 4. The arguments in the proof of Lemma 4 are partly inspired by those in the proof of Corollary 3.8 [**8**].

LEMMA 4.
$$\int_X \log^+\|(A(x)|E^2(x))^{-1}\|\,d\mu(x) < \infty.$$

Proof of the lemma. Take an arbitrary $v \in E^2(x)$ such that $\|v\| = 1$ and find an orthonormal basis $\{v_1, \ldots, v_m\}$ of $E^2(x)$ such that $v_1 = v$. Then,

$$|\det(A(x)|E^2(x))| \le \|A(x)v\| \cdot \prod_{i=2}^{m}\|A(x)v_i\| \le \|A(x)v\| \cdot \|A(x)\|^{m-1}.$$

Hence
$$\|A(x)v\| \ge \|A(x)\|^{1-m} \cdot |\det(A(x)|E^2(x))|. \tag{34}$$

Moreover, by (34),

$$\begin{aligned}\|(A(x)|E^2(x))^{-1}\| &= \sup_{w\in E^2(fx),\|w\|=1}\|A(x)^{-1}w\| \\ &= \sup_{v\in E^2(x),\|v\|=1}\frac{1}{\|A(x)v\|} \le \frac{\|A(x)\|^{m-1}}{|\det(A(x)|E^2(x))|},\end{aligned}$$

and therefore

$$\log\|(A(x)|E^2(x))^{-1}\| \le (m-1)\log\|A(x)\| - \log|\det(A(x)|E^2(x))|.$$

In view of (1), setting
$$\psi(x) = \log|\det(A(x)|E^2(x))|, \tag{35}$$

it remains to prove $\psi^- \in L^1(\mu)$. We first note that $\psi(x) \le \log\|A(x)\|^m = m\log\|A(x)\|$, which, together with (1), implies that $\psi^+ \in L^1(\mu)$. It follows, from Birkhoff's ergodic theorem, that there exists $a \in \mathbb{R} \cup \{-\infty\}$ such that

$$a = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\psi(f^i(x)) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\psi(f^{-i}(x))$$

for μ-almost every $x \in X$. We note that

$$\begin{aligned}a = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\psi(f^{-i}(x)) &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\log|\det(A(f^{-i}(x))|E^2(f^{-i}(x)))| \\ &= \lim_{n\to\infty}\frac{1}{n}\log|\det(\mathcal{A}(f^{-n}(x)), n)|E^2(f^{-n}(x)))| \\ &= \lim_{n\to\infty}\frac{1}{n}\log\frac{1}{|\det(\mathcal{A}(f^{-n}(x)), n)|E^2(f^{-n}(x)))^{-1}|}.\end{aligned}$$

https://doi.org/10.1017/etds.2016.55 Published online by Cambridge University Press

Again, let $\{v_1, \ldots, v_m\}$ be an orthonormal basis for $E^2(x)$. Then, by going backwards in (2) (which is possible by [**13**, Lemma 20]), we can find $\lambda \in \mathbb{R}$ such that, for sufficiently large n,

$$|\det(\mathcal{A}(f^{-n}(x), n)|E^2(f^{-n}(x)))^{-1}| \le \prod_{i=1}^{m} \|(\mathcal{A}(f^{-n}(x), n)|E^2(f^{-n}(x)))^{-1} v_i\| \le e^{mn\lambda},$$

which implies that $a \ge -m\lambda$ and thus $a \in \mathbb{R}$. Moreover, by Kingman's subadditive ergodic theorem,

$$a = \inf_{n\in\mathbb{N}} \frac{1}{n} \int_X \log|\det(\mathcal{A}(f^{n-1}(x), n)|E^2(x))|\, d\mu(x) \le \int_X \log|\det(A(x)|E^2(x))|\, d\mu(x),$$

which implies the integrability of ψ^-. □

To finish this step, we now establish the existence of function C satisfying (21) and the first inequality in (23). By Theorem 1, the map $A(x)$ is invertible along the direction $E^2(x)$ and we will denote the inverse of this map by $A^{-1}(x)$. Let $\mathcal{B}$ be a cocycle over f^{-1} defined on a subbundle $E^2(x)$ with generator $A^{-1} \circ f^{-1}$. It follows, from Theorem 1, Lemma 4 and [**13**, Lemma 20], that the Lyapunov exponents of the cocycle $\mathcal{B}$ are given by

$$-\lambda_k < \cdots < -\lambda_{i+1}.$$

Furthermore, $E_j(x)$ is the Oseledets subspace corresponding to $-\lambda_j$ for $i+1 \le j \le k$. We can now apply Step 1 to $\mathcal{B}$ to conclude that there exists a function $C : \Lambda \to (0, \infty)$ such that

$$\|\mathcal{B}(x, n)v\| \le C(x)e^{(-\lambda_{i+1}+\varepsilon/2)n} \quad \text{for } x \in \Lambda,\ n \ge 0 \text{ and } v \in E_{j+1}(x) \oplus \cdots \oplus E_k(x) \tag{36}$$

and

$$C(f^m(x)) \le C(x)e^{(\varepsilon/2)|m|} \quad \text{for } x \in \Lambda \text{ and } m \in \mathbb{Z}. \tag{37}$$

It follows readily from (36) and (37) that (21) holds.

Step 3. Lower bound for K and temperedness of K.

The existence of function K satisfying (22) and (23) follows by applying Lemma 1 successively. For $i = k-1$, it is sufficient to apply Lemma 1 to $E(x) = E_1(x) \oplus \cdots \oplus E_{k-1}(x)$ and $F(x) = E_k(x)$ using the properties (20) (both for $E(x)$ and $E(x) \oplus F(x)$) and (21) from Steps 1 and 2. For $i = k-2$, we again apply Lemma 1 to $E(x) = E_1(x) \oplus \cdots \oplus E_{k-1}(x)$ and $F(x) = E_k(x)$ and obtain a function K_1, as in the statement of Lemma 1. Further, we apply Lemma 1 to $E(x) = E_1(x) \oplus \cdots \oplus E_{k-2}(x)$ and $F(x) = E_{k-1}(x)$ and obtain a function K_2, as in the statement of Lemma 1. Now take an arbitrary $v \in E_1(x) \oplus \cdots \oplus E_{k-2}(x)$ and $w \in E_{k-1}(x) \oplus E_k(x)$, $w = w_1 + w_2$, $w_1 \in E_{k-1}(x)$, $w_2 \in E_k(x)$. By (12),

$$\|v\| \le K_2(x)\|v + w_1\| \le K_1(x)K_2(x)\|v + w\|. \tag{38}$$

Similarly,

$$\|w_2\| \le K_1(x)\|v + w\|$$

and

$$\|w_1\| \le K_2(x)\|v + w_1\| \le K_1(x)K_2(x)\|v + w\|.$$

https://doi.org/10.1017/etds.2016.55 Published online by Cambridge University Press

Hence

$$\|w\| \le 2\max\{K_1(x), K_1(x)K_2(x)\}\|v+w\|. \tag{39}$$

It follows, from (38) and (39), that (22) holds for $K(x) = 2\max\{K_1(x), K_1(x)K_2(x)\}$, which, in view of (11), satisfies the second inequality in (23) with ε replaced by some $a\varepsilon$ for some $a > 0$ (this is possible since $\varepsilon > 0$ can be made arbitrarily small). Proceeding inductively, we can establish the appropriate bounds for the angle in the general case when $1 \le i \le k$. □

As we have already noted in the introduction, Theorem 2 plays a central role in the proof of our main result. However, it is also a result of independent interest. For example, it shows that the non-invertible cocycles with all non-zero Lyapunov exponents are non-uniformly hyperbolic in the sense of Pesin (see [**5**] for details) on a set of full measure. Furthermore, it shows that the notion of a non-uniform exponential dichotomy for not necessarily invertible discrete time dynamics (introduced by Barreira and Valls in [**7**]) is ubiquitous in the context of ergodic theory. In order to formulate an explicit result, we recall that we say that a sequence $(A_n)_{n\in\mathbb{Z}}$ of operators on $\mathbb{R}^d$ admits a *non-uniform exponential dichotomy* if:

(1) there exist projections $P_n : \mathbb{R}^d \to \mathbb{R}^d$ for each $n \in \mathbb{Z}$ satisfying

$$A_n P_n = P_{n+1} A_n$$

for $n \in \mathbb{Z}$, such that each map

$$A_n | \ker P_n : \ker P_n \to \ker P_{n+1}$$

is invertible; and

(2) there exist a constants D, $\lambda > 0$ and $\varepsilon \ge 0$ such that

$$\|\mathcal{A}(m, n)P_n\| \le De^{-\lambda(m-n)+\varepsilon|n|} \quad \text{for } m \ge n$$

and

$$\|\mathcal{A}(m, n)Q_n\| \le De^{-\lambda(n-m)+\varepsilon|n|} \quad \text{for } m \le n,$$

where $Q_n = \mathrm{Id} - P_n$ and

$$\mathcal{A}(m, n) = (\mathcal{A}(n, m)| \ker P_m)^{-1} : \ker P_n \to \ker P_m$$

for $m < n$.

The following result is a direct consequence of Theorem 2.

THEOREM 3. *Let $\mathcal{A}$ be a cocycle satisfying (1) with non-vanishing Lyapunov exponents. Then there exists a Borel set $\Lambda \subset X$ of full μ-measure such that, for each $x \in \Lambda$, the sequence $(A_n)_{n\in\mathbb{Z}}$ defined by $A_n = A(f^n(x))$, $n \in \mathbb{Z}$ admits a non-uniform exponential dichotomy.*

3. *An Oseledets splitting of the adjoint cocycle*

The other crucial ingredient in the proof of Hölder continuity of the Oseledets splitting (Theorem 5) is the use of the adjoint cocycle. Suppose that $\mathcal{A}$ is a cocycle over f whose generator A satisfies (1). Moreover, assume that the Lyapunov exponents of $\mathcal{A}$ and the corresponding Oseledets decomposition are given by (2) and (3). We denote by $\mathcal{A}^*$ the cocycle over f^{-1} with generator $A^* \circ f^{-1}$. The following result identifies Lyapunov exponents and the Oseledets splitting of the cocycle $\mathcal{A}^*$. We note that this theorem is well known for invertible matrix cocycles. (See [**22**] and [**3**, Theorem 5.1.1]; for semi-invertible cocycles, this result follows implicitly from the contents of the proof of Corollary 17 [**16**]. We present a separate argument here for completeness.)

THEOREM 4. *The Lyapunov exponents of the cocycle $\mathcal{A}^*$ are given by (2). Furthermore, the Oseledets subspace that corresponds to λ_i is given by*

$$\bigg(\bigoplus_{j\neq i} E_j(x)\bigg)^{\perp}. \tag{40}$$

Proof. Let $F_i(x)$ be a subspace of $\mathbb{R}^d$ given by (40). It follows, from (4), that

$$A(x)\bigg(\bigoplus_{j\neq i} E_j(x)\bigg) \subset \bigoplus_{j\neq i} E_j(f(x)),$$

which readily implies that $A^*(f^{-1}(x))F_i(x) \subset F_i(f^{-1}(x))$ for each $i \in \{1, \ldots, k\}$. Furthermore, the subspaces $F_i(x)$ form a direct sum. Indeed, assume that

$$v \in F_i(x) \cap (F_1(x) + \cdots F_{i-1}(x) + F_{i+1}(x) + \cdots + F_k(x))$$

and write v in the form $v = v_1 + \cdots + v_k$, where $v_j \in E_j(x)$ for $j = 1, \ldots, k$. Since

$$v \in F_i(x) \quad \text{and} \quad v_1 + \cdots + v_{i-1} + v_{i+1} + \cdots + v_k \in \bigoplus_{j\neq i} E_j(x),$$

we conclude that $\langle v, v_1 + \cdots + v_{i-1} + v_{i+1} + \cdots + v_k\rangle = 0$. On the other hand, since

$$v \in F_1(x) + \cdots + F_{i-1}(x) + F_{i+1}(x) + \cdots + F_k(x) \subset E_i(x)^{\perp},$$

$\langle v, v_i\rangle = 0$. Hence $\langle v, v\rangle = 0$ and $v = 0$. We now want to prove that

$$\mathbb{R}^d = \bigoplus_{i=1}^{k} F_i(x).$$

In order to prove the above equality we are going to show that $(\bigoplus_{i=1}^k F_i(x))^{\perp} = \{0\}$. Take $v \in (\bigoplus_{i=1}^k F_i(x))^{\perp} = \bigcap_{i=1}^k F_i(x)^{\perp}$ and write it in a form $v = v_1 + \cdots + v_k$, where $v_i \in E_i(x)$, $i = 1, \ldots, k$. Since v and $v_2 + \cdots + v_k$ both belong to $F_1(x)^{\perp}$, we conclude that $v_1 \in F_1(x)^{\perp}$. However, since the subspaces $E_j(x)$ form a direct sum of $\mathbb{R}^d$, this implies that $v_1 = 0$. Similarly, we obtain that $v_j = 0$ for $j = 2, \ldots, k$ and thus $v = 0$. In order to complete the proof of the theorem, we are going to show that for μ-almost every $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}^*(x, n)u\| = \lambda_i \quad \text{for } u \in F_i(x)\backslash\{0\} \text{ and } i \in \{1, \ldots, k\}. \tag{41}$$

Take $i \geq 2$ and $u \in F_i(x)\backslash\{0\}$. We first note that $\mathcal{A}^*(x, n) = \mathcal{A}(f^{-n}(x), n)^*$. Hence

$$\|\mathcal{A}^*(x, n)u\| = \max_{\|v\|=1} |\langle \mathcal{A}^*(x, n)u, v\rangle| = \max_{\|v\|=1} |\langle u, \mathcal{A}(f^{-n}(x), n)v\rangle|. \tag{42}$$

https://doi.org/10.1017/etds.2016.55 Published online by Cambridge University Press

For $v \in E_i(f^{-n}(x))$, $\|v\| = 1$, one has $\mathcal{A}(f^{-n}(x), n)v \in E_i(x)$. Since $u \in F_i(x)\setminus\{0\}$, setting

$$c = c(x) = \tfrac{1}{2} \sup\{|\langle w_1, w_2\rangle| : w_1 \in F_i(x),\, w_2 \in E_i(x),\, \|w_1\| = \|w_2\| = 1\},$$

$$\|\mathcal{A}^*(x, n)u\| = \max_{\|v\|=1} |\langle u, \mathcal{A}(f^{-n}(x), n)v\rangle| \geq c\|u\| \cdot \|\mathcal{A}(f^{-n}(x), n)v\|. \tag{43}$$

On the other hand, it follows, from Theorem 2, that, for each $\varepsilon > 0$, there exists a measurable function $C : \Lambda \to (0, \infty)$ defined on a Borel set $\Lambda \subset X$ of full μ-measure such that

$$C(f^n(x)) \leq C(x)e^{\varepsilon|n|} \quad \text{for } x \in \Lambda \text{ and } n \in \mathbb{Z} \tag{44}$$

and

$$\|\mathcal{A}(x, n)w\| \geq \frac{1}{C(x)} e^{(\lambda_i - \varepsilon)n} \|w\| \quad \text{for } x \in \Lambda,\, w \in E_i(x) \text{ and } n \geq 0. \tag{45}$$

It follows, from (44) and (45), that

$$\|\mathcal{A}(f^{-n}(x), n)v\| \geq \frac{1}{C(x)} e^{(\lambda_i - 2\varepsilon)n} \quad \text{for } n \geq 0. \tag{46}$$

By (42), (43) and (46),

$$\liminf_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}^*(x, n)u\| \geq \lambda_i - 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\liminf_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}^*(x, n)u\| \geq \lambda_i. \tag{47}$$

Now take an arbitrary $v \in \mathbb{R}^d$, $\|v\| = 1$ and write it as $v = v_1 + \cdots + v_k$, where $v_j \in E_j(f^{-n}(x))$ for $j = 1, \ldots, k$. Since $u \in F_i(x)$, $\langle u, \mathcal{A}(f^{-n}(x), n)v_j\rangle = 0$ for $j \neq i$. Hence

$$|\langle u, \mathcal{A}(f^{-n}(x), n)v\rangle| = |\langle u, \mathcal{A}(f^{-n}(x), n)v_i\rangle| \leq \|u\| \cdot \|\mathcal{A}(f^{-n}(x), n)v_i\|. \tag{48}$$

Furthermore, it follows, from Theorem 2, that, for each $\varepsilon > 0$, there exist measurable functions $C, K : \Lambda \to (0, \infty)$ defined on a Borel set Λ of full μ-measure such that, for every $x \in \Lambda$ and $n \in \mathbb{Z}$,

$$\|\mathcal{A}(x, n)w\| \leq C(x)e^{(\lambda_i+\varepsilon)n}\|w\| \quad \text{for } w \in E_i(x),\, n \geq 0, \tag{49}$$

$$\|w_i\| \leq K(x)\|w_1 + w_2\| \quad \text{for } i \in \{1, 2\},\, w_1 \in E_i(x) \text{ and } w_2 \in \bigoplus_{j\neq i} E_j(x), \tag{50}$$

$$C(f^n(x)) \leq C(x)e^{\varepsilon|n|} \quad \text{and} \quad K(f^n(x)) \leq K(x)e^{\varepsilon|n|}. \tag{51}$$

We note that the existence of the function K can be easily deduced from the appropriate bounds for the angles between $\bigoplus_{j=1}^{i} E_j(x)$ and $\bigoplus_{j=i+1}^{k} E_j(x)$ as well as $\bigoplus_{j=1}^{i-1} E_j(x)$ and $E_i(x)$ (see Lemma 1 and the proof of Theorem 2). Thus

$$\begin{aligned}\|\mathcal{A}(f^{-n}(x), n)v_i\| &\leq C(f^{-n}(x))e^{(\lambda_i+\varepsilon)n}\|v_i\| \\ &\leq C(f^{-n}(x))K(f^{-n}(x))e^{(\lambda_i+\varepsilon)n}\|v\| \\ &\leq C(x)K(x)e^{(\lambda_i+3\varepsilon)n}.\end{aligned}$$

Hence, using (42) and (48), we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}^*(x, n)u\| \leq \lambda_i + 3\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}^*(x, n)u\| \leq \lambda_i. \tag{52}$$

Obviously, (47) and (52) imply (41). Now we discuss the case $i = 1$. If $\lambda_1 > -\infty$, then one can repeat the above arguments and establish (41) for $i = 1$ also. If $\lambda_1 = -\infty$, then one can repeat the second estimates and obtain that

$$\limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}^*(x, n)u\| \leq L,$$

for $u \in F_1(x)$, where L is arbitrary real number. By letting $L \to -\infty$, we establish (41) in this situation also. □

4. *Hölder continuity of Oseledets splitting*

In this section, we prove that the Oseledets subspaces of a Hölder continuous cocycle A are Hölder continuous on a set of arbitrarily large measure in X. For a subspace $A \subset \mathbb{R}^d$ and a vector $v \in \mathbb{R}^d$, we define

$$d(v, A) = \inf\{\|v - w\| : w \in A\}.$$

Furthermore, for two subspaces V and W of $\mathbb{R}^d$, we define the distance between them by

$$d(V, W) = \max\left\{ \sup_{w\in W, \|w\|=1} d(w, V), \sup_{v\in V, \|v\|=1} d(v, W)\right\}.$$

It turns out that the quantity $d(V, W)$ can be expressed in an equivalent form, which will be more suitable for our purposes. Let $P_V : \mathbb{R}^d \to \mathbb{R}^d$ and $P_W : \mathbb{R}^d \to \mathbb{R}^d$ be orthogonal projections onto V and W, respectively. The following lemma is well known (see, e.g., [**1**, p. 111]).

LEMMA 5. *For any two subspaces V and W of $\mathbb{R}^d$,*

$$d(V, W) = \|P_V - P_W\|.$$

Now let X be a metric space with a metric ρ and let $\Lambda \subset X$. We say that the family $E(x)$, $x \in \Lambda$ of subspaces of $\mathbb{R}^d$ is *Hölder continuous* on Λ if there exist $L, \varepsilon_0 > 0$ and $\beta \in (0, 1]$ such that

$$d(E(x), E(y)) \leq L\rho(x, y)^\beta \quad \text{for every } x, y \in \Lambda \text{ satisfying } \rho(x, y) \leq \varepsilon_0.$$

We now introduce the notion of a Hölder continuous cocycle. We say that the cocycle $\mathcal{A}$ is *Hölder continuous* if there exist $C, \nu > 0$ such that

$$\|A(x) - A(y)\| \leq C\rho(x, y)^\nu \quad \text{for } x, y \in X.$$

The following two simple lemmas will be particularly useful.

https://doi.org/10.1017/etds.2016.55 Published online by Cambridge University Press

LEMMA 6. *Let $\Lambda \subset X$. A family $E(x)$, $x \in \Lambda$ of subspaces of $\mathbb{R}^d$ is Hölder continuous if and only if the family $E(x)^\perp$, $x \in \Lambda$ is Hölder continuous.*

Proof. Let $P(x)$ denote the orthogonal projection onto $E(x)$ for $x \in \Lambda$. Then $\mathrm{Id} - P(x)$ is an orthogonal projection onto $E(x)^\perp$. Hence it follows, from Lemma 5, that

$$\begin{aligned} d(E(x), E(y)) = \|P(x) - P(y)\| &= \|(\mathrm{Id} - P(x)) - (\mathrm{Id} - P(y))\| \\ &= d(E(x)^\perp, E(y)^\perp) \end{aligned}$$

for every $x, y \in \Lambda$. The conclusion of the lemma now follows directly from the definition of Hölder continuity. □

LEMMA 7. *Let $\Lambda \subset X$ and assume that $E(x)$, $F(x)$, $x \in \Lambda$ are two families of subspaces of $\mathbb{R}^d$ such that:*
(1) *the subspaces $E(x)$ and $F(x)$ are orthogonal for each $x \in \Lambda$;*
(2) *the family $E(x)$, $x \in \Lambda$ is Hölder continuous; and*
(3) *the family $E(x) \oplus F(x)$, $x \in \Lambda$ is Hölder continuous.*
Then the family $F(x)$, $x \in \Lambda$ is also Hölder continuous.

Proof. Let $P(x)$ be an orthogonal projection onto $E(x)$ and let $Q(x)$ be an orthogonal projection onto $F(x)$ for $x \in \Lambda$. Since $E(x)$ and $F(x)$ are orthogonal, $P(x) + Q(x)$ is an orthogonal projection onto $E(x) \oplus F(x)$. Hence it follows, from Lemma 5, that

$$\begin{aligned} d(F(x), F(y)) = \|Q(x) - Q(y)\| &\le \|(P(x) + Q(x)) - (P(y) + Q(y))\| \\ &\quad + \|P(x) - P(y)\| \\ &= d(E(x) \oplus F(x), E(y) \oplus F(y)) \\ &\quad + d(E(x), E(y)) \end{aligned}$$

for every $x, y \in \Lambda$. The Hölder continuity of the family $F(x)$ follows directly from the Hölder continuity of families $E(x) \oplus F(x)$ and $E(x)$. □

We will use the following two auxiliary results from [**2**], which are slight generalizations of the original work of Brin [**9**] (see also [**5**, Lemmas 5.3.4. and 5.3.5]).

LEMMA 8. *Let $(A_n)_{n\ge 1}$, $(B_n)_{n\ge 1}$ be two sequences of real matrices of order $d > 0$ such that, for some $0 < \lambda < \mu$ and $C \ge 1$, there exist subspaces E, E', F, F' of $\mathbb{R}^d$ satisfying $\mathbb{R}^d = E \oplus E' = F \oplus F'$ such that:*
(1) *$\|A_n u\| \le C\lambda^n \|u\|$ for $u \in E$ and $C^{-1}\mu^n\|v\| \le \|A_n v\|$ for $v \in E'$;*
(2) *$\|B_n u\| \le C\lambda^n \|u\|$ for $u \in F$ and $C^{-1}\mu^n\|v\| \le \|B_n v\|$ for $v \in F'$; and*
(3) *$\max\{\|v\|, \|w\|\} \le d\|v + w\|$ for $v \in E$, $w \in E'$ or $v \in F$, $w \in F'$.*
Then, for each pair $(\delta, a) \in (0, 1] \times [\lambda, +\infty)$ satisfying

$$\left(\frac{\lambda}{a}\right)^{n+1} < \delta \le \left(\frac{\lambda}{a}\right)^n \quad \textit{and} \quad \|A_n - B_n\| \le \delta a^n,$$

$d(E, F) \le (2 + d)C^2(\mu/\lambda)\delta^{\log(\mu/\lambda)/\log(a/\lambda)}$.

https://doi.org/10.1017/etds.2016.55 Published online by Cambridge University Press

LEMMA 9. *Assume that $\mathcal{A}$ is a Hölder continuous cocycle and that there exists $L > 0$ such that f is Lipschitz with constant L and such that $\|\mathcal{A}(x, n)\| \leq L^n$ for $n \geq 0$ and x in some fixed compact set $\Lambda \subset X$. Then there exist $C, \nu > 0$ such that $\|\mathcal{A}(x, n) - \mathcal{A}(y, n)\| \leq C^n d(x, y)^\nu$ for $x, y \in \Lambda$ and $n \geq 0$.*

We now arrive at our main result.

THEOREM 5. *Let $\mathcal{A}$ be a Hölder continuous cocycle satisfying (1) whose Lyapunov exponents and the corresponding Oseledets splitting are given by (2) and (3). Then, for each $i \in \{1, \ldots, k\}$ and $\delta > 0$, there exists a compact set $\Lambda \subset X$ of measure $\mu(\Lambda) > 1 - \delta$ such that the map $x \mapsto E_i(x)$ is Hölder continuous on Λ.*

Proof. We will divide the proof into several parts.

Step 1. Hölder continuity of $x \mapsto E_1(x) \oplus \cdots \oplus E_i(x)$. We will first show that, for each $i \in \{1, \ldots, k\}$ and $\delta > 0$, there exists a compact set $\Lambda \subset X$ satisfying $\mu(\Lambda) > 1 - \delta$ and such that the map $x \mapsto E_1(x) \oplus \cdots \oplus E_i(x)$ is Hölder continuous on Λ. We note that this part is essentially already contained in [**2**, **5**, **9**]; we include it for the sake of completeness.

Choosing $\varepsilon > 0$ such that $\lambda_i + \varepsilon < \lambda_{i+1} - \varepsilon$, it follows, from Theorem 2, that there exists a Borel set $\Lambda \subset X$ satisfying $\mu(\Lambda) = 1$ and Borel measurable functions $C, K : \Lambda \to (0, \infty)$ such that (20), (21), (22) and (23) hold (with the same convention as in the statement of Theorem 2 for λ_1 in (20), if $i = 1$). Moreover, there exists a measurable function $\tilde{C} : \Lambda \to (0, \infty)$ such that

$$\|\mathcal{A}(x, n)\| \leq \tilde{C}(x) e^{(\lambda_k + \varepsilon)n} \quad \text{for } x \in \Lambda \text{ and } n \geq 0. \tag{53}$$

For $l \in \mathbb{N}$, let

$$\Lambda_l = \{x \in \Lambda : C(x) \leq l,\ \tilde{C}(x) \leq l,\ K(x) \leq l\}.$$

It is easy to verify (see [**5**, p. 121]) that each set Λ_l is compact. Furthermore, we obviously have that

$$\Lambda_l \subset \Lambda_{l+1} \quad \text{and} \quad \bigcup_{l=1}^{\infty} \Lambda_l = \Lambda.$$

Since $\mu(\Lambda) = 1$, there exists $l \in \mathbb{N}$ such that $\mu(\Lambda_l) > 1 - \delta$. On the other hand, it follows, from (20), (21), (22) and (23), that, for every $x \in \Lambda_l$,

$$\|\mathcal{A}(x, n)v\| \leq l e^{(\lambda_i + \varepsilon)n} \|v\| \quad \text{for } v \in E_1(x) \oplus \cdots \oplus E_i(x) \text{ and } n \geq 0; \tag{54}$$

$$\frac{1}{l} e^{(\lambda_{i+1} - \varepsilon)n} \|v\| \leq \|\mathcal{A}(x, n)v\| \quad \text{for } v \in E_{i+1}(x) \oplus \cdots \oplus E_k(x) \text{ and } n \geq 0; \tag{55}$$

and

$$\|u\| \leq l\|u + v\| \quad \text{and} \quad \|v\| \leq l\|u + v\| \tag{56}$$

for $u \in E_1(x) \oplus \cdots \oplus E_i(x)$ and $v \in E_{i+1}(x) \oplus \cdots \oplus E_k(x)$. Moreover, it follows, from (53), that

$$\|\mathcal{A}(x, n)\| \leq l e^{(\lambda_k + \varepsilon)n} \quad \text{for every } x \in \Lambda_l \text{ and } n \geq 0. \tag{57}$$

Hence, applying Lemmas 8 and 9 to sequences $A_n = \mathcal{A}(x, n)$ and $B_n = \mathcal{A}(y, n)$ with $x, y \in \Lambda_l$, we easily obtain the Hölder continuity of the map $x \mapsto E_1(x) \oplus \cdots \oplus E_i(x)$ on Λ_l.

https://doi.org/10.1017/etds.2016.55 Published online by Cambridge University Press

Step 2. Hölder continuity of $x \mapsto E_{i+1}(x) \oplus \cdots \oplus E_k(x)$. We now prove that, for each $i \in \{1, \ldots, k-1\}$ and $\delta > 0$, there exists a compact set $\Lambda \subset X$ satisfying $\mu(\Lambda) > 1 - \delta$ and such that the map $x \mapsto E_{i+1}(x) \oplus \cdots \oplus E_k(x)$ is Hölder continuous on Λ. We note that

$$(E_{i+1} \oplus \cdots \oplus E_k(x))^{\perp} = \bigg(\bigoplus_{j \neq 1} E_j(x)\bigg)^{\perp} \oplus \cdots \oplus \bigg(\bigoplus_{j \neq i} E_j(x)\bigg)^{\perp} = F_1 \oplus \cdots \oplus F_i,$$

where F_j, $j = 1, \ldots, i$ are the Oseledets spaces of the adjoint cocycle and the second equality follows from Theorem 4. We now apply Step 1 to the adjoint cocycle to find a measurable set $\Lambda \subset X$ satisfying $\mu(\Lambda) > 1 - \delta$ and such that the map

$$x \mapsto F_1 \oplus \cdots \oplus F_i = (E_{i+1} \oplus \cdots \oplus E_k(x))^{\perp}$$

is Hölder continuous on Λ. The desired conclusion now follows directly from Lemma 6. We note that this part is also contained in [**9**], [**5**] and [**2**], but is obtained by applying the previous step to the inverse of $\mathcal{A}$. Such an approach is unavailable to us because the inverse of $\mathcal{A}$ may not exist.

Step 3. Hölder continuity of $x \mapsto E_i(x)$. In the final part, we prove that, for each $i \in \{1, \ldots, k\}$ and $\delta > 0$, there exists a compact set $\Lambda \subset X$ satisfying $\mu(\Lambda) > 1 - \delta$ and such that the map $x \mapsto E_i(x)$ is Hölder continuous on Λ. We note that our arguments related to this part of the proof differ significantly from those in [**2**] and are arguably simpler. For $i = 1$ or $i = k$, there is nothing to prove since the conclusion follows directly from the previous two steps. Now take an arbitrary $i \in \{2, \ldots, k-1\}$ and $\delta > 0$ and let Λ_l be the set of measure greater than $1 - \delta$ constructed in the first step of the proof. Without loss of generality, we can assume that the mapping $x \mapsto E_i(x) \oplus \cdots \oplus E_k(x)$ is Hölder continuous on Λ_l since, otherwise (using Step 2 of the proof), we can pass to a compact subset of Λ_l of measure greater than $1 - 2\delta$ on which this holds. Let $P(x)$ denote the orthogonal projection onto $F(x) := E_i(x) \oplus \cdots \oplus E_k(x)$. For a point $x \in \Lambda_l$, we define a sequence of matrices $(A_n)_n$ by $A_n = \mathcal{A}(x, n)P(x)$. Choose any $v \in F(x)^{\perp} \oplus E_i(x)$ and write it in the form $v = v_1 + v_2$, where $v_1 \in F(x)^{\perp}$ and $v_2 \in E_i(x)$. Using (54) and the orthogonality of $F(x)^{\perp}$ and $E_i(x)$, we conclude that

$$\|A_n v\| = \|A_n v_2\| = \|\mathcal{A}(x, n)v_2\| \leq l e^{(\lambda_i + \varepsilon)n} \|v_2\| \leq l e^{(\lambda_i + \varepsilon)n} \|v\|.$$

Hence

$$\|A_n v\| \leq l e^{(\lambda_i + \varepsilon)n} \|v\| \quad \text{for every } v \in F(x)^{\perp} \oplus E_i(x) \text{ and } n \geq 0. \tag{58}$$

On the other hand, it follows readily, from (55), that

$$\frac{1}{l} e^{(\lambda_{i+1} - \varepsilon)n} \|v\| \leq \|A_n v\| \quad \text{for every } v \in E_{i+1}(x) \oplus \cdots \oplus E_k(x) \text{ and } n \geq 0. \tag{59}$$

We now want to establish appropriate bounds for the angles between $F(x)^{\perp} \oplus E_i(x)$ and $E_{i+1}(x) \oplus \cdots \oplus E_k(x)$. Select an arbitrary $u = u_1 + u_2 \in F(x)^{\perp} \oplus E_i(x)$ with $u_1 \in F(x)^{\perp}$, $u_2 \in E_i(x)$ and $v \in E_{i+1}(x) \oplus \cdots \oplus E_k(x)$. By (56),

$$\|u + v\|^2 = \|u_1 + u_2 + v\|^2 = \|u_1\|^2 + \|u_2 + v\|^2 \geq \|u_2 + v\|^2 \geq \frac{1}{l^2} \|v\|^2,$$

which implies that

$$l\|u+v\| \geq \|v\|.$$

Similarly,

$$\|u+v\|^2 = \|u_1\|^2 + \|u_2+v\|^2 \geq \frac{1}{l^2}(\|u_1\|^2 + \|u_2\|^2) = \frac{1}{l^2}\|u\|^2,$$

which implies that

$$l\|u+v\| \geq \|u\|.$$

Hence

$$\max\{\|u\|, \|v\|\} \leq l\|u+v\| \quad \text{for } u \in F(x)^{\perp} \oplus E_i(x) \text{ and } v \in \bigoplus_{j=i+1}^{k} E_j(x). \tag{60}$$

In order to apply Lemma 8, we need to bound the quantity $\|A_n - B_n\|$, where $B_n = \mathcal{A}(y,n)P(y)$ and y is some other point in Λ_l.

$$\begin{aligned}\|A_n - B_n\| &= \|\mathcal{A}(x,n)P(x) - \mathcal{A}(y,n)P(y)\| \\ &\leq \|\mathcal{A}(x,n)P(x) - \mathcal{A}(x,n)P(y)\| + \|\mathcal{A}(x,n)P(y) - \mathcal{A}(y,n)P(y)\| \\ &\leq \|\mathcal{A}(x,n)\| \cdot d(F(x), F(y)) + \|\mathcal{A}(x,n) - \mathcal{A}(y,n)\|.\end{aligned}$$

By applying Lemma 9, (57) and the Hölder continuity of the map $x \mapsto F(x)$, we conclude that there exists $C, \nu > 0$ (independent of x and y) such that

$$\|A_n - B_n\| \leq C^n d(x,y)^{\nu}.$$

It now follows, from Lemma 8, that the map $x \mapsto F(x)^{\perp} \oplus E_i(x)$ is Hölder continuous on a set Λ_l, which, in view of Lemma 7, implies the Hölder continuity of the map $x \mapsto E_i(x)$ on the same set. □

5. *Cocycles on Hilbert spaces*

In this section, we state a generalization of our results to cocycles on Hilbert spaces.

THEOREM 6. *Let X be a Borel subset of a separable complete metric space, let $f : X \circlearrowleft$ be a bi-Lipschitz ergodic transformation, let $\mathcal{H}$ be a Hilbert space and let $A : X \to \mathcal{B}(\mathcal{H})$ take values in the space of all compact operators. Further, assume that A satisfies (1) and that $x \mapsto A(x)$ is Hölder continuous in the operator norm topology. Then either of the following hold.*

(1) *There is a finite sequence of numbers (note that below we order Lyapunov exponents and Oseledets spaces in reverse order to (2) and (3), starting with the largest one)*

$$\lambda_1 > \lambda_2 > \cdots > \lambda_k > \lambda_\infty = -\infty$$

and a decomposition

$$\mathcal{H} = E_1(x) \oplus \cdots \oplus E_k(x) \oplus E_\infty(x)$$

such that

$$A(x)E_i(x) = E_i(f(x)), \quad i = 1, \ldots, k \quad \textit{and} \quad A(x)E_\infty(x) \subset E_\infty(f(x))$$

https://doi.org/10.1017/etds.2016.55 Published online by Cambridge University Press

and

$$\lim_{n\to\infty}\frac{1}{n}\log\|\mathcal{A}(x,n)v\| = \lambda_i \quad \textit{for } v\in E_i(x)\backslash\{0\},\ i\in\{1,\ldots,k\}\cup\{\infty\}.$$

Moreover, each $E_i(x)$, $i=1,\ldots,k$ is a finite-dimensional subspace of $\mathcal{H}$. The maps $x\mapsto E_i(x)$, $i=1,\ldots,k$ are Hölder continuous on a compact set of arbitrarily large measure.

(2) *There exists an infinite sequence of numbers*

$$\lambda_1>\lambda_2>\cdots>\lambda_k>\cdots>\lambda_\infty=-\infty$$

and a decomposition

$$\mathcal{H}=E_1(x)\oplus\cdots\oplus E_k(x)\oplus\cdots\oplus E_\infty(x)$$

such that

$$A(x)E_i(x)=E_i(f(x)),\quad 1\le i<\infty \quad \textit{and} \quad A(x)E_\infty(x)\subset E_\infty(f(x))$$

and

$$\lim_{n\to\infty}\frac{1}{n}\log\|\mathcal{A}(x,n)v\| = \lambda_i \quad \textit{for } v\in E_i(x)\backslash\{0\},\ i\in\mathbb{N}\cup\{\infty\}.$$

Moreover, each $E_i(x)$, $i\neq\infty$ is a finite-dimensional subspace of $\mathcal{H}$. The maps $x\mapsto E_i(x)$, $i\neq\infty$ are Hölder continuous on a compact set of arbitrarily large measure.

Proof. In cases (1) and (2), all statements, except the Hölder continuity, follow from Theorem 17 in [**13**], which extends Theorem 1 to a semi-invertible version of Oseledets' theorem under weaker conditions than those in our hypotheses: namely, that f is an ergodic homeomorphism, $\mathcal{H}$ is a Banach space, A takes values in the space of all quasi-compact operators and $x\mapsto A(x)$ is μ-continuous; the resulting Oseledets splitting is μ-continuous.

Case 1. In this case one is able to repeat the arguments in the proof of Theorem 5 and establish the Hölder continuity. Indeed, we emphasize that all of the preparatory results that we used extend to this setting. In particular, Theorem 4 and Lemma 8 are valid for cocycles on Hilbert space (with the same proof) and the corresponding version of Theorem 2 also holds. More precisely, one is able to repeat all arguments in the proof Theorem 2, with the exception of Lemma 2.

The statement of Lemma 2 remains unchanged in our Hilbert space setting. The proof in the Hilbert space case is identical to the proof of Proposition 14 in [**13**], where Proposition 14 is now applied to the cocycle $\mathcal{A}(x,n)$ restricted to $E^1(x)$.

Case 2. Arguing as in case 1 above, one is able to show Hölder continuity of Oseledets subspaces $E_i(x)$ for $i\neq\infty$ on a compact set of arbitrarily large measure for any Hölder continuous cocycle. Unfortunately, we are not able to establish a similar property for $E_\infty(x)$. The major obstacle in repeating our arguments is the fact that one does not have a lower bound for the expansion on a subspace complementary to $E_\infty(x)$. □

https://doi.org/10.1017/etds.2016.55 Published online by Cambridge University Press

Acknowledgements. We thank Ian Morris for pointing us to the reference [**1**] and Anthony Quas for useful discussions. We also thank an anonymous referee for their careful reading of the manuscript. D.D. was supported by an Australian Research Council Discovery Project DP150100017 and, in part, by the Croatian Science Foundation under the project IP-2014-09-2285. G.F. is supported by an ARC Future Fellowship.

REFERENCES

[**1**] N. Akhiezer and I. Glazman. *Theory of Linear Operators in Hilbert Space*. Dover, New York, 1993.

[**2**] V. Araújo, A. I. Bufetov and S. Filip. On Hölder-continuity of Oseledets subspaces. *J. Lond. Math. Soc.* **93** (2016), 194–218.

[**3**] L. Arnold. *Random Dynamical Systems (Springer Monographs in Mathematics)*. Springer, Berlin, 1998.

[**4**] V. Baladi. *Positive Transfer Operators and Decay of Correlations,* Vol. 16. World Scientific, Singapore, 2000.

[**5**] L. Barreira and Y. Pesin. *Nonuniform Hyperbolicity (Encyclopedia of Mathematics and its Applications, 115).* Cambridge University Press, New York, 2007.

[**6**] L. Barreira, Y. Pesin and J. Schmeling. Dimension and product structure of hyperbolic measures. *Ann. of Math.* (2) **149**(3) (1999), 755–783.

[**7**] L. Barreira and C. Valls. Noninvertible cocycles: robustness of exponential dichotomies. *Disc. Contin. Dynam. Syst.* **32**(12) (2012), 4111–4131.

[**8**] A. Blumenthal and L. S. Young. Entropy, volume growth and SRB measures for Banach space mappings. *Preprint*, 2015, http://arxiv.org/pdf/1510.04312.pdf.

[**9**] M. Brin. *Hölder Continuity of Invariant Distributions (Smooth Ergodic Theory and its Applications).* American Mathematical Society, Providence, RI, 2001, pp. 91–93.

[**10**] R. Cogburn. The ergodic theory of Markov chains in random environments. *Z. Wahrscheinlichkeit* **66**(1) (1984), 109–128.

[**11**] G. Froyland and C. González-Tokman. Stability and approximation of invariant measures of Markov chains in random environments. *Stoch. Dyn.* **16**(01) (2016), 1650003.

[**12**] G. Froyland, S. Lloyd and A. Quas. Coherent structures and isolated spectrum for Perron–Frobenius cocycles. *Ergod. Th. & Dynam. Sys.* **30**(3) (2010), 729–756.

[**13**] G. Froyland, S. Lloyd and A. Quas. A semi-invertible Oseledets theorem with applications to transfer operator cocycles. *Discr. Contin. Dynam. Syst.* **33**(9) (2013), 3835–3860.

[**14**] G. Froyland, S. Lloyd and N. Santitissadeekorn. Coherent sets for nonautonomous dynamical systems. *Physica* D **239**(16) (2010), 1527–1541.

[**15**] C. González-Tokman and A. Quas. A semi-invertible operator Oseledets theorem. *Ergod. Th. & Dynam. Sys.* **34**(5) (2014), 1230–1272.

[**16**] C. González-Tokman and A. Quas. A concise proof of the Multiplicative Ergodic Theorem on Banach spaces. *J. Mod. Dyn.* **9** (2015), 237–255.

[**17**] A. Katok. Lyapunov exponents, entropy and periodic orbits for diffeomorphisms. *Publ. Math. Inst. Hautes Études Sci.* **51** (1980.), 137–173.

[**18**] Z. Lian and K. Lu. *Lyapunov Exponents and Invariant Manifolds for Random Dynamical Systems in a Banach Space (Memoirs of the American Mathematical Society, 206).* American Mathematical Society, Providence, RI, 2010.

[**19**] R. Mañé. Lyapounov exponents and stable manifolds for compact transformations. *Geometric Dynamics (Rio de Janeiro, 1981) (Lecture Notes in Mathematics, 1007).* Springer, Berlin, 1983.

[**20**] V. I. Oseledets. A multiplicative ergodic theorem. Characteristic Lyapunov, exponents of dynamical systems. *Tr. Mosk. Mat. Obs.* **19** (1968), 179–210.

[**21**] Y. B. Pesin. Characteristic Lyapunov exponents and smooth ergodic theory. *Russian Math. Surveys* **32** (1977), 55–114.

[**22**] D. Ruelle. Characteristic exponents and invariant manifolds in Hilbert space. *Ann. of Math.* (2) (1982), 243–290.

[**23**] P. Thieullen. Fibres dynamiques asymptotiquement compacts exposants de Lyapounov. Entropie. Dimension. *Ann. Inst. Henri Poincaré* C **4**(1) (1987), 49–97.