# ON THE COHOMOLOGY OF LOCAL GROUPS

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**To Bernhard Hermann Neumann on his 60th birthday**

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## 1. Introduction

Most known homology theories (e.g. the homology of modules, rings, groups, sheaves, ...) have been found to be special cases of a general theory proposed by M. Barr and J. Beck [1], [2]. The aim of this paper is to show that the cohomology of a local group, as defined by W. T. van Est [4], also fits the scheme of Barr and Beck. At the same time it will be shown that local group cohomology is a relative derived functor in the sense of S. Eilenberg and J. C. Moore [3].

W. T. van Est's definition ([4], p. 396) runs as follows. Given are: a group $G$, a (left) $G$-module $X$ and $V \subset G$, a subset such that $1 \in V$ and $V = V^{-1}$ (i.e. $v^{-1} \in V$ whenever $v \in V$). A sequence $(v_1, \cdots, v_n)$ is called a $(V, n)$-tuple if $v_i v_{i+1} \cdots v_l \in V$ for every $1 \leqq i \leqq l \leqq n$. For $n = 1, 2, \cdots$ denote the set of $(V, n)$-tuples by $V^{(n)}$. A mapping $f: V^{(n)} \to X$ is called a $(V, n)$-cochain and its coboundary is the $(V, n+1)$-cochain $f\delta^n : V^{(n+1)} \to X$ defined by

$$(v_1, \cdots, v_{n+1})(f\delta^n) = v_1((v_2, \cdots, v_{n+1})f) + \sum_{1 \leqq i \leqq n} (-1)^i (v_1, \cdots, v_i v_{i+1}, \cdots, v_{n+1})f + (-1)^{n+1}(v_1, \cdots, v_n)f.$$

The $(V, 0)$-cochains are, by definition, the elements of $X$, and $(v_1)(x\delta^0) = v_1 x - x$ for each $x \in X$. For $n = 0, 1, 2, \cdots$ the group of $(V, n)$-cochains (under value-addition in $X$) contains the subgroup of $(V, n)$-cocycles (cochains satisfying $f\delta^n = 0$) and the latter contains the subgroup of $(V, n)$-coboundaries (cochains of the form $g\delta^{n-1}$). $H^n(V, X)$ is defined to be the quotient of these two subgroups (i.e. cocycles modulo coboundaries).

## 2. Generalizations of van Est's definition

The first generalization is immediate: it is not necessary to have a subset of a group, a local group $V$ will suffice. Recall that by a *local group* is meant [7] a set $V$ such that for certain pairs $(v, w) \in V \times V$ there is defined a product $vw \in V$ and the following axioms hold:



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LG1. *There is a* $1 \in V$ *such that* $1v = v1 = v$ *for every* $v$.

LG2. *For every* $v$ *there exists a* $v^{-1}$ *such that* $vv^{-1} = v^{-1}v = 1$.

LG3. *If* $vw$ *is defined, then so is* $w^{-1}v^{-1}$.

LG4. *If* $vw$ *and* $wz$ *are defined, then any of the products* $(vw)z$, $v(wz)$ *is defined iff the other is defined, and they are equal, when defined.*

Call an abelian group $X$ a $V$-module if $V$ operates from the left on the elements of $X$ so that $1x = x$, $v0 = 0$, $v(x+y) = vx+vy$ and $(wv)x = w(vx)$ whenever $wv$ is defined in $V$.

One can repeat now van Est's definition of $H^n(V, X)$, introducing the slight modification that $(V, n)$-tuples should be those sequences $(v_1, \cdots, v_n)$ for which all products $v_i v_{i+1} \cdots v_l$; $(1 \leqq i \leqq l \leqq n)$, for all possible ways of inserting the brackets, are defined.

But in the sequel neither the existence of inverses, nor an associativity law as stated in LG4 will be needed. This leads to the final generalization.

By a *partial monoid* we shall mean a set $V$ such that for certain pairs $(v, w) \in V \times V$ there is defined a product $vw \in V$ with the properties:

PM1. *There is a* $1 \in V$ *such that* $1v = v1 = v$ *for every* $v$.

PM2. *If* $vw$, $wz$ *and* $(vw)z$ *are defined, then* $v(wz)$ *is defined and equal to* $(vw)z$.

We generalize the notion of a $V$-module to that of a *partial* $V$-*module* by which we shall mean an abelian group $X$ such that for certain pairs $(v, x) \in V \times X$ a product $vx \in X$ is defined and the following axioms hold.

PVM1. $1x = x$ *for every* $x$ *and* $v0 = 0$ *for every* $v$.

PVM2. *If* $vx$, $vy$ *are defined, then* $v(x \pm y)$ *is defined and equal to* $vx \pm vy$.

PVM3. *If* $vw$, $wx$ *and* $(vw)x$ *are defined for* $v, w \in V$; $x \in X$, *then* $v(wx)$ *is defined and equal to* $(vw)x$.

If $vx$ is defined, we shall call $v$ a *multiplier* of $x$. We define now the set $V^{(n)}$ of $(V, n)$-tuples as before but limit the concept of a $(V, n)$-cochain to include precisely those mappings $f : V^{(n)} \to X$ which have the property that $v((v_1, \cdots, v_n)f)$ is defined (in $X$) whenever $(v, v_1, \cdots, v_n) \in V^{(n+1)}$. Thus the $(V, 0)$-cochains are those elements of $X$ which can be multiplied by each $v \in V$. Taking $X$ a partial $V$-module and using the cochain concept in the restricted sense we still may repeat van Est's definition. The resulting cohomology $H^n(V, X)$ is the one we wish to discuss below.

## 3. The theorem

We fix the partial monoid $V$. We denote by $\mathscr{C}$ the category of partial $V$-modules where by a morphism in $\mathscr{C}$ we mean a map $f : X \to Y$ which is additive (i.e.

such that $(x+x')f = xf+x'f)$ and multiplicative in the sense that $v(xf)$ is defined and equal to $(vx)f$ whenever $vx$ is defined.

LEMMA. *$\mathscr{C}$ is an additive category.*

Indeed, $\hom_{\mathscr{C}}(X, Y)$ is easily seen to be an abelian group under addition induced from $Y$. The zero object for $\mathscr{C}$ is the one-element group $\{0\}$ with $v0 = 0$ for all $v \in V$. If $X, Y \in \mathscr{C}$, denote by $X \oplus Y$ the sum of abelian groups and give it a partial $V$-module structure by postulating that $v(x, y)$ is defined iff $vx$, $vy$ are defined, and $v(x, y) = (vx, vy)$, when defined. Then $X \oplus Y$ is a product and coproduct in $\mathscr{C}$.

In the sequel the notation of M. Barr and J. Beck [2] will be adopted. We refer to [2] for the basic properties of the Barr-Beck cohomology functors

$$H^n(-, E)_{\mathbf{G}} : \mathscr{C} \to \mathfrak{A}; \qquad n = 0, 1, 2, \cdots$$

associated to any given contravariant functor $E : \mathscr{C} \to \mathfrak{A}$ (where $\mathfrak{A}$ is an abelian category), and any given cotriad $\mathbf{G}$ on $\mathscr{C}$. (We use the name 'triad' suggested by Saunders MacLane in preference to the name 'triple' used in [2]).

THEOREM. *There is a cotriad $\mathbf{G}$ on $\mathscr{C}$ such that for every $X \in \mathscr{C}$*

$$H^n(V, X) = H^n(Z, \hom_{\mathscr{C}}(-, X))_{\mathbf{G}}; \qquad n = 0, 1, 2, \cdots$$

*where $Z$ is the trivial partial $V$-module in which $vz = z$ for every $v \in V$ and every integer $z$.*

Recall that, given a cotriad $\mathbf{G}$ on $\mathscr{C}$, a sequence in $\mathscr{C}$

$$\cdots \to X_{-1} \to X_0 \to X_1 \to \cdots \tag{*}$$

is called $\mathbf{G}$-exact if the composite of any two consecutive morphisms is zero, and for every $Y \in \mathscr{C}$

$$\cdots \hom_{\mathscr{C}}(GY, X_{-1}) \to \hom_{\mathscr{C}}(GY, X_0) \to \hom_{\mathbf{G}}(GY, X_1) \to \cdots$$

is an exact sequence of abelian groups. Let $\mathscr{E}$ denote the class of $\mathbf{G}$-exact sequences. As shown in [2], § 4, the functor $H^n(-, E)_{\mathbf{G}}$ is the $n$th derived functor of $E$, relative to $\mathscr{E}$, in the sense of Eilenberg and Moore [3]. In particular, if $E = \hom_{\mathscr{C}}(-, X)$, then our theorem implies the

COROLLARY $H^n(V, X) = \operatorname{Ext}^n_{\mathscr{E}}(Z, X)$; $n = 0, 1, 2, \cdots$.

It can be seen from our construction that $\mathscr{E}$ is the class of those sequences (*) which are exact, as sequences of abelian groups and have the property that for every $x \in X_n$ which is in the image of $X_{n-1}$ there is a $y \in X_{n-1}$ which is mapped in $x$ and has precisely the same multipliers as $x$.

## 4. Proof of the theorem: construction of a cotriad

Recall ([2], § 4) that if $\mathbf{G} = (G, \varepsilon, \delta)$ is a cotriad on $\mathscr{C}$ with natural transformations $\varepsilon : G \to \mathscr{C}$, $\delta : G \to GG$ then an object $P \in \mathscr{C}$ is termed ***G-projective*** if there is a morphism $s : P \to PG$ such that $s \cdot P\varepsilon = P$. A $\boldsymbol{G}$-exact sequence $0 \leftarrow X \leftarrow X_0 \leftarrow X_1 \leftarrow \cdots$ in which $X_0, X_1, \cdots$ are $\boldsymbol{G}$-projective is called a ***G-resolution*** of $X$ and the cohomology of $0 \to X_0 E \to H_1 E \to \cdots$ is $H^n(X, E)_{\boldsymbol{G}}$.

In view of this, the theorem will be proved if we find a cotriad $\boldsymbol{G}$ on $\mathscr{C}$ and a $\boldsymbol{G}$-resolution $0 \leftarrow Z \leftarrow X_0 \leftarrow \cdots$ such that the cohomology groups of

$$0 \to \hom_{\mathscr{C}}(X_0, X) \to \hom_{\mathscr{C}}(X_1, X) \to \cdots$$

coincide with $H^n(V, X)$.

The cotriad will be obtained from an adjoint pair of functors. $U : \mathscr{C} \to \mathscr{A}$ will be a functor that, when applied to an $X \in \mathscr{C}$, 'forgets' everything but the underlying set of $X$ and, for each $x \in X$, its multipliers. Accordingly define $\mathscr{A}$ to be the category such that $A$ is an object in $\mathscr{A}$ if $A$ is a set together with a mapping $A \to 2^V$ which assigns to each $a \in A$ a subset $V_a \subset V$ containing 1. A morphism $f : A \to A'$ in $\mathscr{A}$ is a set map such that $V_a \subset V_{f(a)}$ for every $a \in A$. Then we have the forgetful functor $U : \mathscr{C} \to \mathscr{A}$.

Given $A \in \mathscr{A}$, let $AF$ be the abelian group freely generated by all the formal products $v\langle a \rangle$ where $a \in A$, $v \in V_a$. This is made into a partial $V$-module by postuating that $v0 = 0$ for all $v \in V$ and if $0 \neq x = \sum_{1 \leqq i \leqq n} m_i v_i \langle a_i \rangle$ (reduced sum; $m_i \in Z$) then $vx$ is defined iff $vv_i$ is defined and belongs to $V_{a_i}$, for $i = 1, \cdots, n$. When defined,

$$vx = \sum_{1 \leqq i \leqq n} m_i (vv_i) \langle a_i \rangle.$$

The PVM axioms in § 2 are readily checked. A morphism $f : A \to A'$ in $\mathscr{A}$ induces the unique morphism $fF : AF \to A'F$ for which $\langle a \rangle (fF) = \langle af \rangle$ (we identify $1\langle a \rangle$ with $\langle a \rangle$). This defines a functor $F : \mathscr{A} \to \mathscr{C}$.

Let $A \in \mathscr{A}$, $X \in \mathscr{C}$ and let $f : A \to XU$ be in $\mathscr{A}$. Assign to $f$ the morphism $AF \to X$ given by

$$\sum_{i=1}^{n} m_i v_i \langle a_i \rangle \mapsto \sum_{i=1}^{n} m_i v_i (a_i f).$$

It is easy to see that this defines a natural equivalence of functors

$$\phi : \hom_{\mathscr{C}}(AF, X) \rightleftarrows \hom_{\mathscr{A}}(A, XU),$$

i.e. an adjunction $\phi$ of $F$ to $U$.

Let $\eta : \mathscr{A} \to FU$, $\varepsilon : UF \to \mathscr{C}$ be the unit and counit of the adjunction. The required cotriad $\boldsymbol{G} = (G, \varepsilon, \delta)$ is now given by $G = UF$ and $\delta = U\eta F$ [(8], Introduction).

## 5. Proof continued: a standard resolution

Call $f : X \to X'$ in $\mathscr{C}$ a *full morphism* if for every element $x'$ belonging to the image $Xf$ there is an $x \in X$ which has precisely the same multipliers as $x'$ and such that $x' = xf$.

LEMMA 1. *A sequence* $\cdots \to X_{-1} \xrightarrow{f} X_0 \xrightarrow{g} X_1 \to \cdots$ *is G-exact if and only if every morphism is full and the underlying sequence of abelian groups is exact.*

We shall prove the 'if' part only, since only this part will be needed below. The non-trivial portion of the argument consists in showing that if a given $h : XG \to X_0$ satisfies $hg = 0$, then there exists an $h' : XG \to X_{-1}$ such that $h'f = h$. Let $\bar{h} : XU \to X_0 U$ correspond to $h$ by the adjointness of $(F, U)$. Then the diagrams

$$\begin{tikzcd} & XG = XUF \arrow[dl, dashed, "k"'] \arrow[d, "h"] \\ X_{-1} \arrow[r, "f"'] & X_0 \end{tikzcd} \qquad \begin{tikzcd} & XU \arrow[dl, dashed, "\bar{k}"'] \arrow[d, "\bar{h}"] \\ X_{-1}U \arrow[r, "fU"'] & X_0 U \end{tikzcd}$$

are equivalent to each other, whence it will suffice to find a morphism $XU \to X_{-1} U$ which makes the second diagram commute. By assumption $a\bar{h}(gU) = 0$ for every $a \in XU$, whence $a\bar{h} = \xi_a(fU)$ for some $\xi_a \in X_{-1} U$ due to the exactness of the underlying sequence of abelian groups. Since $f$ is full, $\xi_a$ may be chosen so that $V_{\xi_a} = V_{a\bar{h}}$. Thus $a \to \xi_a$ defines a morphism which makes the second triangle commute.

Call an object in $\mathscr{C}$ *free* if it is of the form $AF$, where $A \in \mathscr{A}$.

LEMMA 2. *Every free object is G-projective.*

Indeed, we have $A\eta F : AF \to AFUF = AFG$ and the standard identity $A\eta F$. $AF\varepsilon = AF$ ([6], p. 60).

Let $V^{(0)}$ be a one-element set $\{e\}$, made into an object of $\mathscr{A}$ by $V_e = V$. Make the set $V^{(n)}$; $(n \geqq 1)$ of all $(V, n)$-tuples into an object of $\mathscr{A}$ by

$$V_{(v_1, \cdots, v_n)} = \{v \in V : (v, v_1, \cdots, v_n) \in V^{(n+1)}\}.$$

If $n \geqq 1$, define a morphism $\bar{d}_{n+1} : V^{(n+1)} \to V^{(n)}FU$ by

$$\begin{aligned}(v_1, \cdots, v_{n+1})\bar{d}_{n+1} &= v_1\langle v_2, \cdots, v_{n+1}\rangle \\ &\quad + \sum_{1 \leqq i \leqq n} (-1)^i \langle v_1, \cdots, v_i v_{i+1}, \cdots, v_{n+1}\rangle + (-1)^{n+1}\langle v_1, \cdots, v_n\rangle\end{aligned}$$

(where $\langle \cdots \rangle$ stands for $\langle(\cdots)\rangle$ and addition is in $V^{(n)}F$).

Define further $\bar{d}_1 : V^{(1)} \to V^{(0)}FU$ by $(v_1)\bar{d}_1 = v_1\langle e\rangle - \langle e\rangle$ and $\bar{d}_0 : V^{(0)} \to ZU$ by $e\bar{d}_0 = 1$. By the adjointness of $(F, U)$ there corresponds to $\bar{d}_n$ a morphism $d_n : V^{(n)}F \to V^{(n-1)}F$ if $n \geqq 1$ and $d_0 : V^{(0)}F \to Z$ if $n = 0$.

LEMMA 3. $0 \leftarrow Z \xleftarrow[d_0]{} V^{(0)}F \xleftarrow[d_1]{} V^{(1)}F \xleftarrow[d_2]{} \cdots$ *is a G-resolution.*

By Lemma 2, $V^{(0)}F, V^{(1)}F, \cdots$ are $G$-projective. It remains to verify the conditions of Lemma 1. Define morphisms of abelian groups $h_{-1} : Z \to V^{(0)}F$, $h_n : V^{(n)}F \to V^{(n+1)}F$, $n = 0, 1, 2, \cdots$ by prescribing their values on the free generators:

$$1h_{-1} = \langle e\rangle, \quad v\langle e\rangle h_0 = \langle v\rangle, \quad v\langle v_1, \cdots, v_n\rangle h_n = \langle v, v_1, \cdots, v_n\rangle.$$

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A straightforward calculation gives

$$h_{-1}d_0 = Z \quad\text{and}\quad d_n h_{n-1} + h_n d_{n+1} = V^{(n)}F.$$

Thus $d_{n+1}d_n = 0$ ($n = 0, 1, 2, \cdots$) by induction on $n$ (see [5], p. 115). Moreover, the underlying sequence of abelian groups is exact. Finally, $d_n$ is a full morphism since $xh_{n-1}$ has the same multipliers as $x$, and $xh_{n-1}d_n = x$ whenever $x \in V^{(n)}Fd_n = \operatorname{Ker} d_{n-1}$.

Observe now that $\hom_{\mathscr{A}}(V^{(n)}, XU)$ is precisely the set of $(V, n)$-cochains; and if an additive structure is introduced in it from $X$ (the same one as in the beginning of this paper), then the adjointness

$$\phi : \hom_{\mathscr{C}}(V^{(n)}F, X) \to \hom_{\mathscr{A}}(V^{(n)}, XU)$$

is an isomorphism of abelian groups.

We assert now that $\phi$ is an isomorphism of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & \hom_{\mathscr{C}}(V^{(0)}F, X) \arrow[r, "\partial^0"'] \arrow[d, "\phi"] & \hom_{\mathscr{C}}(V^{(1)}F, X) \arrow[r, "\partial^1"'] \arrow[d, "\phi"] & \cdots \\
0 \arrow[r] & \hom_{\mathscr{A}}(V^{(0)}, XU) \arrow[r, "\delta^0"'] & \hom_{\mathscr{A}}(V^{(1)}, XU) \arrow[r, "\delta^1"'] & \cdots,
\end{tikzcd}$$

where the upper complex is obtained by applying the $\hom_{\mathscr{C}}(-, X)$ functor to the resolution in Lemma 3 and the lower one is that which was used in §§ 1, 2 to define $H^n(V, X)$.

Indeed, let $f : V^{(n)} \to XU$ be a $(V, n)$-cochain and suppose $f = g\phi$. Then $g\partial^n$ is given by the first of the diagrams

$$\begin{tikzcd}
V^{(n+1)}F \arrow[d, "d_{n+1}"'] \arrow[dr, "g\partial^n"] & \\
V^{(n)}F \arrow[r, "g"'] & X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
V^{(n+1)} \arrow[d, "d_{n+1}\phi = \bar{d}_{n+1}"'] \arrow[dr, "g\partial^n\phi"] & \\
V^{(n)}FU \arrow[r, "gU"'] & XU
\end{tikzcd}$$

and the second diagram (which follows from the first by the naturality of $\phi$) gives $g\partial^n\phi$. The definition of $\bar{d}_{n+1}$ implies now that $g\partial^n\phi$ coincides with the coboundary $f\delta^n$ of § 1. Thus the cohomology of the upper complex is $H^n(V, X)$. The aim proposed at the beginning of § 4 has been achieved.

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