Appendix F
Angular correlations

Consider the basic coincidence reaction

$$\mathrm{A}(S_1^{\pi_1})\,[\mathrm{e},\mathrm{e}'\;\mathrm{X}(S_\mathrm{X}^{\pi_\mathrm{X}})]\;\mathrm{A}'(S_2^{\pi_2}) \tag{F.1}$$

The angular distribution of particle X in the C-M system can be analyzed in more detail using some basic results from [Ja59]. If particle X is massive, so that all helicity states are present, one can make a change of basis to L–S coupling states for the final two-particle system.

$$\begin{aligned}
|JM\lambda_2\lambda_\mathrm{X}\rangle &= \sum_{LS}\langle J;LS|J;\lambda_2\lambda_\mathrm{X}\rangle\;|JM;LS\rangle \\
\langle J;LS|J;\lambda_2\lambda_\mathrm{X}\rangle &= \sqrt{(2L+1)(2S+1)}(-1)^{S-S_2+S_\mathrm{X}-L-2\lambda} \\
&\quad\times\begin{pmatrix} L & S & J \\ 0 & \lambda & -\lambda \end{pmatrix}\begin{pmatrix} S_2 & S_\mathrm{X} & S \\ \lambda_2 & -\lambda_\mathrm{X} & -\lambda \end{pmatrix}
\end{aligned} \tag{F.2}$$

Here $\lambda = \lambda_2 - \lambda_\mathrm{X}$. This transformation reproduces the usual non-relativistic L–S coupling wave functions [Ja59]; however, it is also a completely *general* unitary transformation, for with some algebra [Wa84], one establishes the relations

$$\begin{aligned}
\sum_{LS}\langle LS|\lambda_1\lambda_2\rangle\langle LS|\lambda_1'\lambda_2'\rangle &= \delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'} \\
\sum_{\lambda_1\lambda_2}\langle LS|\lambda_1\lambda_2\rangle\langle L'S'|\lambda_1\lambda_2\rangle &= \delta_{LL'}\delta_{SS'}
\end{aligned} \tag{F.3}$$

Here J is suppressed. The transformation thus remains valid for arbitrary relativistic motion of the final two particles. The transformation in Eq. (F.2) is real, and the coefficients are independent of M just as in the proof of the Wigner–Eckart theorem.

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The L–S basis states have two advantages. First, since they reduce to the usual non-relativistic L–S wave functions, one can use angular-momentum barrier arguments in this case to classify the contributions. Second, they produce *eigenstates of parity*, for again with some algebra [Wa84], one establishes the relation

$$P|JM;LS\rangle \quad = \quad \eta_2\eta_{\rm X}(-1)^L|JM;LS\rangle \tag{F.4}$$

The change of basis in Eq. (F.2) can now be substituted in the expression for the bilinear product of current matrix elements appropriately summed and averaged over the final and initial helicities in Eq. (13.68). The result is, again after some algebra [Wa84]

$$\begin{aligned}
\overline{\left(\mathscr{J}^{\lambda_k}\right)^{\star}_{\lambda_f,\lambda_i}\left(\mathscr{J}^{\lambda'_k}\right)_{\lambda_f,\lambda'_i}} \quad = \quad & \frac{1}{4k^{\star}q}\frac{1}{2S_1+1}\sum_{\lambda_1}\sum_{J}\sum_{J'}\sum_{L}\sum_{L'}\sum_{S}\sum_{l} \\
& \times(2J+1)(2J'+1)\sqrt{(2L+1)(2L'+1)}(-1)^{l+J+J'-S+\lambda_i} \\
& \times\begin{pmatrix} L & l & L' \\ 0 & 0 & 0 \end{pmatrix}\begin{Bmatrix} J & J' & l \\ L' & L & S \end{Bmatrix}\sqrt{4\pi(2l+1)}Y_{l,\lambda'_k-\lambda_k}(\theta_q,\phi_q) \\
& \times\begin{pmatrix} J & J' & l \\ \lambda_i & -\lambda'_i & \lambda_k-\lambda'_k \end{pmatrix}\langle LS|T^J|\lambda_1\lambda_k\rangle^{\star}\langle L'S|T^{J'}|\lambda_1\lambda'_k\rangle
\end{aligned} \tag{F.5}$$

Here $\lambda_i = \lambda_1 - \lambda_k$ and $\lambda'_i = \lambda_1 - \lambda'_k$, and a 6-j coefficient has been introduced [Ed74].

Transition amplitudes into states of definite parity can be defined by

$$\begin{aligned}
c(LS;J;\lambda_1) \quad &\equiv \quad \frac{\kappa^{\star}}{\omega^{\star}}\langle LS|T^J|\lambda_1,0\rangle \\
t(LS;J;\lambda_1) \quad &\equiv \quad \langle LS|T^J|\lambda_1,+1\rangle
\end{aligned} \tag{F.6}$$

Recall these are functions of (W,k^2) and still contain all the dynamics. Parity invariance then implies that

$$\langle LS|T^J|\lambda_1,\lambda_k\rangle \quad = \quad \eta(-1)^{L+J-S_1}\langle LS|T^J|-\lambda_1,-\lambda_k\rangle \tag{F.7}$$

Again $\eta \equiv \eta_1\eta_2^{\star}\eta_{\rm X}^{\star}$. This relation allows one to eliminate $\langle LS|T^J|-\lambda_1,-1\rangle$ and leads to the selection rule

$$c(LS;J;\lambda_1) \quad = \quad \eta(-1)^{L+J-S_1}c(LS;J;-\lambda_1) \tag{F.8}$$

Upon substitution of the appropriate values of λ_k, one can identify the

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coefficients appearing in Eqs. (13.71) as

$$\begin{aligned}
A_l &= \sum K^l_{JJ'}(LL'S\lambda_1) \\
&\quad \times \begin{pmatrix} J & J' & l \\ \lambda_1 & -\lambda_1 & 0 \end{pmatrix} c(LS;J;\lambda_1)^\star c(L'S;J';\lambda_1) \\
B_l &= -2\sum K^l_{JJ'}(LL'S\lambda_1) \\
&\quad \times \begin{pmatrix} J & J' & l \\ \lambda_1-1 & -\lambda_1+1 & 0 \end{pmatrix} t(LS;J;\lambda_1)^\star t(L'S;J';\lambda_1) \\
C_l &= \frac{-2}{\sqrt{l(l+1)}}\sum K^l_{JJ'}(LL'S\lambda_1) \\
&\quad \times \begin{pmatrix} J & J' & l \\ \lambda_1 & -\lambda_1+1 & -1 \end{pmatrix} \mathrm{Re}\, c(LS;J;\lambda_1)^\star t(L'S;J';\lambda_1) \\
D_l &= \frac{-1}{\sqrt{(l-1)l(l+1)(l+2)}}\sum K^l_{JJ'}(LL'S\lambda_1)(-1)^{L'+J'-S_1} \\
&\quad \times \begin{pmatrix} J & J' & l \\ \lambda_1-1 & -\lambda_1-1 & 2 \end{pmatrix} t(LS;J;\lambda_1)^\star t(L'S;J';-\lambda_1)
\end{aligned} \tag{F.9}$$

Here $\sum \equiv \sum_J \sum_{J'} \sum_S \sum_L \sum_{L'} \sum_{\lambda_1}$ and the common summand factor is defined by

$$\begin{aligned}
K^l_{JJ'}(LL'S\lambda_1) &\equiv \frac{2l+1}{2S_1+1}(2J+1)(2J'+1)\sqrt{(2L+1)(2L'+1)} \\
&\quad \times(-1)^{J+J'+l-S+\lambda_1} \begin{Bmatrix} J & J' & l \\ L' & L & S \end{Bmatrix} \begin{pmatrix} L & L' & l \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{F.10}$$

Thus we have derived a general expression for the angular distribution in the C-M system for the coincidence reaction in Eq. (F.1). The derivation is completely relativistic, as long as particle X has non-zero rest mass so that all helicity amplitudes are present in the reaction.

For a 0^+ nuclear target, these angular correlation coefficients are discussed and tabulated in [Kl83, Wa84]. We give one other application here.

Consider pion electroproduction from the nucleon so that particle X is a pion and the initial and final target states are the nucleon with $J^\pi = 1/2^+$. For the pseudoscalar pion $S_X = 0$ and $\eta_X = -1$. For the nucleon $S_1 = S_2 = 1/2$ and $\eta_1 = \eta_2 = +1$. It follows from Eq. (F.2) that only one value of the total spin $S = 1/2$ enters the analysis, and this quantum number will subsequently be suppressed. The parity of the final π–N states follows from Eq. (F.4)

$$P|JM;L\rangle = (-1)^{L+1}|JM;L\rangle \tag{F.11}$$

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There are now only two values of the initial nucleon helicity $\lambda_1 = \pm 1/2$, and the sum over this quantity can be immediately performed. Introduce the notation

$$
\begin{aligned}
c(LJ\lambda_1) &\equiv \frac{k^\star}{\omega^\star}\langle L|T^J(W,k^2)|\lambda_1,0\rangle \\
t(LJ\lambda_1) &\equiv \langle L|T^J(W,k^2)|\lambda_1,+1\rangle
\end{aligned}
\tag{F.12}
$$

Equations (F.9), which give the angular distributions in the C-M system through Eqs. (13.71) then reduce to the form

$$
\begin{aligned}
A_l &= \sum K^l_{JJ'}(LL') \\
&\quad \times \begin{pmatrix} J & J' & l \\ 1/2 & -1/2 & 0 \end{pmatrix} c(LJ\tfrac{1}{2})^\star c(L'J'\tfrac{1}{2}) \\
B_l &= -\sum K^l_{JJ'}(LL') \\
&\quad \times \Bigg[\begin{pmatrix} J & J' & l \\ -1/2 & 1/2 & 0 \end{pmatrix} t(LJ\tfrac{1}{2})^\star t(L'J'\tfrac{1}{2}) \\
&\quad - \begin{pmatrix} J & J' & l \\ -3/2 & 3/2 & 0 \end{pmatrix} t(LJ,-\tfrac{1}{2})^\star t(L'J',-\tfrac{1}{2})\Bigg] \\
C_l &= \frac{-1}{\sqrt{l(l+1)}}\sum K^l_{JJ'}(LL')\,\mathrm{Re}\; c(LJ\tfrac{1}{2})^\star \\
&\quad \times \Bigg[\begin{pmatrix} J & J' & l \\ 1/2 & 1/2 & -1 \end{pmatrix} t(L'J'\tfrac{1}{2}) \\
&\quad -\eta(-1)^{L+J-1/2}\begin{pmatrix} J & J' & l \\ -1/2 & 3/2 & -1 \end{pmatrix} t(L'J',-\tfrac{1}{2})\Bigg] \\
D_l &= \frac{-1}{\sqrt{(l-1)l(l+1)(l+2)}}\sum K^l_{JJ'}(LL')(-1)^{L'+J'-1/2} \\
&\quad \times \begin{pmatrix} J & J' & l \\ -1/2 & -3/2 & 2 \end{pmatrix} \mathrm{Re}\; t(LJ\tfrac{1}{2})^\star t(L'J',-\tfrac{1}{2})
\end{aligned}
\tag{F.13}
$$

Here one is left with $\sum \equiv \sum_J \sum_{J'} \sum_L \sum_{L'}$, and the common summand is now

$$
\begin{aligned}
K^l_{JJ'}(LL') &\equiv (2l+1)(2J+1)(2J'+1)\sqrt{(2L+1)(2L'+1)} \\
&\quad \times(-1)^{J+J'+l}\begin{Bmatrix} J & J' & l \\ L' & L & 1/2 \end{Bmatrix}\begin{pmatrix} L & L' & l \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
\tag{F.14}
$$

Also

$$
\eta \equiv \eta_1\eta_2\eta_X = -1 \tag{F.15}
$$

As one application, suppose the pion electroproduction proceeds entirely through the first excited state of the nucleon with $J^\pi = 3/2^+$. In this case

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only one total angular momentum contributes so that $J = J'$. Furthermore, since $L = J \mp 1/2$ the positive parity picks out $L = L' = 1$ from Eq. (F.11). The summand can be evaluated with the aid of [Ed74] to give $K^2_{3/2,3/2}(11) = 8\sqrt{5}$, and further evaluation of the required 3-j symbols leads to the explicit angular distributions

$$
\begin{aligned}
\overline{|\mathscr{J}_c|^2} &= \frac{1}{k^\star q}[1 + P_2(\cos\theta_q)]\left|c\left(1\frac{3}{2}\frac{1}{2}\right)\right|^2 \\
\overline{|\mathscr{J}^{+1}|^2} + \overline{|\mathscr{J}^{-1}|^2} &= \frac{1}{k^\star q}\left\{\left[\left|t\left(1\frac{3}{2}\frac{1}{2}\right)\right|^2 + \left|t\left(1\frac{3}{2},-\frac{1}{2}\right)\right|^2\right]\right. \\
&\quad \left. + P_2(\cos\theta_q)\left[\left|t\left(1\frac{3}{2}\frac{1}{2}\right)\right|^2 - \left|t\left(1\frac{3}{2},-\frac{1}{2}\right)\right|^2\right]\right\} \\
\mathrm{Im}\,\overline{\mathscr{J}_c^\star(\mathscr{J}^{+1} + \mathscr{J}^{-1})} &= \frac{1}{k^\star q}\sin\phi_q\, P_2^{(1)}(\cos\theta_q) \\
&\quad \times\left[-\frac{1}{\sqrt{3}}\mathrm{Re}\, c\left(1\frac{3}{2}\frac{1}{2}\right)^\star t\left(1\frac{3}{2},-\frac{1}{2}\right)\right] \\
\mathrm{Re}\,\overline{(\mathscr{J}^{+1})^\star(\mathscr{J}^{-1})} &= \frac{1}{k^\star q}\cos 2\phi_q\, P_2^{(2)}(\cos\theta_q) \\
&\quad \times\left[\frac{1}{2\sqrt{3}}\mathrm{Re}\, t\left(1\frac{3}{2}\frac{1}{2}\right)^\star t\left(1\frac{3}{2},-\frac{1}{2}\right)\right]
\end{aligned}
\tag{F.16}
$$

The integrals over the angle-dependent terms vanish when $\int d\Omega_q$ is performed, leaving just the angle-independent terms in the inclusive cross section.

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