REPRESENTATION OF FUNCTIONS AS WEIERSTRASS-TRANSFORMS

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1. Introduction. The Weierstrass-respectively Weierstrass-Stieltjes transform of a function $F(t)$ or $\mu(t)$ is defined by

$$(1.1)\qquad f(x) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{\frac{-(t-x)^2}{4}} F(t)dt$$

and

$$(1.2)\qquad f(x) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{\frac{-(t-x)^2}{4}} d\mu(t)$$

for all x for which these integrals converge. In what follows we shall always assume that $F(t)$ is Lebesgue integrable in every finite interval and that $\mu(t)$ is a function of bounded variation.

It is easily seen that

$$(1.3)\qquad e^{-x^2} f(-2x) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-xt} [e^{-t^2/4} F(t)]\, dt$$

and hence $\sqrt{4\pi}\, e^{-x^2} f(-2x)$ is the two-sided Laplace transform

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of $e^{\frac{-t^2}{4}} F(t)$. It follows that the region of convergence is an interval. Replacing the real variable x by $s = x + iy$, then the region of convergence of (1.1) is a vertical strip [5; p. 238] Moreover, every Weierstrass transform which converges for $a < x < b$ is also holomorphic in $a < \operatorname{Re} s < b$.

Cooper [2] has obtained representation theories for the Laplace transform in connection with the integral transform:

$$F(\lambda, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} k(y, \lambda)\, f(c+iy)\, e^{t(c+iy)} dy$$

where $k(y, \lambda)$ is a summation kernel of the type usual in the theory of the Fourier integrals. Because of the relationship between the Laplace- and Weierstrass-transform one may expect to obtain a similar theory for the latter transform. This we shall establish here. To this end, we shall study the integral transforms

$$(1.4) \qquad F_\lambda(t, x) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} k(y, \lambda) e^{(x-t+iy)^2/4} f(x+iy)\, dy$$

where $a < x < b$ and $k(y, \lambda)$ is a Fourier summation kernel satisfying the following conditions:

The Fourier-transform of $k(y, \lambda)$ denoted by $K(t, \lambda)$ exists, $k(y, \lambda)$ is regular

$$(1.5) \qquad k(y, \lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyt} K(t, \lambda)\, dt$$

and

$$(1.6) \qquad \int_{-\infty}^{\infty} |K(t,\lambda)|\, dt \leq M,$$

where M is independent of λ.

The following result can be found in [2, p. 225].

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LEMMA 1. If (1.6) holds, then the set of transformations T_λ defined by

$$(T_\lambda g)(x) = \int_{-\infty}^{\infty} K(\frac{x-y}{2}, \lambda) g(y) dy$$

forms a bounded set of transformations from $L_p(-\infty, \infty)$ $(1 \leq p \leq \infty)$ to itself.

As in [3] we define the class of functions $A(a,b)$ to be those functions $f(x)$ defined on (a,b) such that $f(x)$ can be extended analytically into the complex plane satisfying: $f(x+iy)$ is holomorphic in the strip $a < x < b$ and

(1.7) $f(x+iy) = O(e^{\frac{y^2}{4}})$ $|y| \to \infty$, uniformly in every closed subinterval of $a < x < b$.

If $f(x)$ satisfies the above conditions with "O" in (1.7) replaced by "o" we say $f(x) \varepsilon B(a,b)$.

It is convenient to denote the Weierstrass- and and Weierstrass Stieltjes transform (1.1) and (1.2) by $f(x) = W(F,x)$ and $f(x) = WS(\mu, x)$. Throughout $||.||_p$, $p \geq 1$ denotes the L_p-norm and $p' = {}^{p}/_{p-1}$,

2. Necessary conditions for Weierstrass transforms

THEOREM 1. If $f(x) = W(F;x)$ with $F(t)\, e^{\frac{-(t-x)^2}{4}} \varepsilon L_p$ $(-\infty, \infty)$ $1 < p \leq 2$, $a < x < b$, and if for each $\lambda > 0$, $k(y,\lambda) \varepsilon L_p$ $(-\infty, \infty)$ then $f(x) \varepsilon B(a,b)$ and

(2.1) $$||e^{-(t-x)^2/4} F_\lambda(t,x)||_p \leq M \quad (1 < p \leq 2)$$

where M is independent of λ and $x \varepsilon (a,b)$.

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Proof. As in the proof of [3, Theorem 2] $f(x) = W(F;x)$ exists and $f(x) \varepsilon B(a,b)$. To show that (2.1) is satisfied, we note that by Parsevals theorem ([4, Theorem 76])

$$e^{-(t-x)^2/4} F_\lambda(t,x) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} K(\frac{t-u}{2}, \lambda)\, [e^{-(u-x)^2/4} F(u)]\, du.$$

(2.1) follows now from Lemma 1.

The next theorem deals with the Weierstrass-Stieltjes transform.

THEOREM 2. If $f(x) = WS(\mu, x)$ with

$$(2.2) \qquad \int_{-\infty}^{\infty} e^{-(t-x)^2/4} |d\mu(t)| < \infty \quad (a < x < b)$$

and if $k(y,\lambda) \varepsilon L_1(-\infty, \infty)$ for each $\lambda > 0$, then $f(x) \varepsilon A(a,b)$ and

$$(2.3) \qquad || e^{-(t-x)^2/4} F_\lambda(t,x) ||_1 \leq M$$

Proof. By (2.2) $f(x) = WS(\mu, x)$ exists for all $a < x < b$ and as before $f(x) \varepsilon A(a,b)$. Also

$$F_\lambda(t,x) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} k(y,\lambda) e^{(x-t+iy)^2/4} f(x+iy) dy$$

$$= \frac{e^{(x-t)^2/4}}{4\pi} \int_{-\infty}^{\infty} k(y,\lambda) dy \int_{-\infty}^{\infty} e^{iy(u-t)/2} e^{-(u-x)^2/4} d\mu(u)$$

$$= \frac{e^{(x-t)^2/4}}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} K(\frac{t-u}{2}, \lambda) e^{-(u-x)^2/4} d\mu(u)$$

where the interchange of the order of integration is justified by Fubini's theorem. Hence by (1.6)

$$||e^{-(x-t)^2/4}F_\lambda(t,x)||_1 \leqslant \frac{M}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(u-x)^2/4}|d\mu(u)|.$$

3. Sufficient conditions.

THEOREM 3. If $f(x) \varepsilon B(a,b)$, (2.1) and $k(y,\lambda)e^{-y^2/4} f(x+iy)$ $\varepsilon L_1(-\infty,\infty)$ where $k(y,\lambda) \to 1 (\lambda \to \infty)$ uniformly in y for every finite interval, then there exists a function F such that $f(x) = W(F;x)$, where

$$e^{-(x-t)^2/4} F(t) \;\varepsilon\; L_p(-\infty,\infty), \qquad 1 < p \leqslant 2, \quad x \;\varepsilon\; (a,b).$$

Proof. Choose $x_o \varepsilon (a,b)$; then by (2.1) the family of functions

$$\{F_\lambda(t,x_o)e^{-(x_o-t)^2/4}\}$$

is bounded in $L_p(-\infty,\infty)$. By [5, Chapter I, Theorem 17a] there exists a subsequence $\{\lambda_k\}_{k=1}^{\infty}$ with $\lim_{k\to 0} \lambda_k = \infty$ and a function $F(t,x_o)$ with

$$e^{-(x_o-t)^2/4} F(t,x_o) \;\varepsilon\; L_p(-\infty,\infty)$$

such that

$$(3.1) \qquad \lim_{k\to\infty} \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-(x_o-t)^2/4} F_{\lambda_k}(t,x_o)\,\bar{\phi}(t)dt$$

$$= \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-(x_o-t)^2/4} F(t,x_o)\,\bar{\phi}(t)dt$$

for all $\phi \;\varepsilon\; L_{p'}$. In particular for all $\phi \;\varepsilon\; L_{p'} \cap L_1$ whose

Fourier transforms φ are in L_p. Thus

$$\frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-(x-t)^2/4} F_{\lambda_k}(t, x)\, \bar{\phi}(t)dt$$

$$= \frac{1}{4\pi} \int_{-\infty}^{\infty} \bar{\phi}(t)dt \int_{-\infty}^{\infty} k(y, \lambda_k)e^{-y^2/4} e^{iy(x_o - t)/2} f(x_o + iy)dy$$

$$= \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-y^2/4} e^{iyx_o/2} k(y, \lambda_k) f(x_o + iy)dy \int_{-\infty}^{\infty} \bar{\phi}(t)e^{iyt/2}dt$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{\varphi}\left(\frac{y}{2}\right) e^{-y^2/4} e^{iyx_o/2} k(y, \lambda_k) f(x_o + iy)dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{\varphi}(y) e^{-y^2} e^{iyx_o} k(2y, \lambda_k) f(x_o + 2iy)dy$$

where the interchange of integration is justified by Fubini's theorem.

Now the functions

$$e^{-y^2} e^{iyx_o} K(2y, \lambda_k) f(x_o + 2iy) \tag{3.2}$$

are the Fourier transforms of $e^{-(x_o - t)^2/4} F_{\lambda_k}(t, x_o)$ and this family of functions is bounded in L_p. Therefore, by [4, Theorem 74] the family (3.2) is bounded in $L_{p'}$. By the weak compactness of the $L_{p'}$-space, there exists a subsequence $\{\lambda_{k_j}\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} \lambda_{k_j} = \infty$ such that for any $g \in L_p$

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$$\lim_{j\to\infty} \int_{-\infty}^{\infty} e^{-y^2} e^{iyx_o} k(2y,\lambda_{k_j}) f(x_o+2iy) g(y) dy$$

$$= \int_{-\infty}^{\infty} e^{-y^2} e^{iyx_o} f(x_o+2iy) g(y) dy$$

where $e^{-y^2} e^{iyx_o} f(x_o+2iy)$ is the limiting point of $e^{-y^2} e^{iyx_o} k(2y,\lambda_k) - f(x_o+2iy)$. Now the functions $\phi \varepsilon L_{p'} \cap L_1$ are dense in $L_{p'}$ so that in particular for $g(y) = \frac{1}{\sqrt{2\pi}} \bar{\varphi}(y)$

$$\lim_{j\to\infty} \int_{-\infty}^{\infty} e^{-(x_o-t)^2/4} F_{\lambda_{k_j}}(t, x_o) \bar{\phi}(t) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{\varphi}(y) e^{-y^2} e^{iyx_o} f(x_o+2iy) dy.$$

By [1, Theorem 4] and the fact that the functions (3.2) are the Fourier transforms of $e^{-(x_o-t)^2/4} F_{\lambda_k}(t, x_o)$ we obtain

$$\sqrt{2}\, e^{-y^2} e^{iyx_o} f(x_o+2iy) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyt} e^{-(x_o-t)^2/4} F(t, x_o) dt,$$

that is

$$(3.3) \qquad f(x_o+iy) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-(t-x_o-iy)^2/4} F(t, x_o) dt$$

It remains to show that $F(t, x_o)$ is independent of the choice of $x_o \varepsilon (a, b)$.

Since $f(x) \varepsilon B(a, b)$, $f(S) e^{(s-t)^2/4}$ is holomorphic for

$a < \operatorname{Re} s < b$ then for $a < x_0 \le \operatorname{Re} s \le x_1 < b$ Cauchy's integral theorem yields

$$\int_{\mathcal{C}} f(x) e^{(s-t)^2/4} ds = 0,$$

where the contour $\mathcal{C}$ is determined by the vertices $x_0 \pm iT$, $x_1 \pm iT$ with $a < x_0 < x_1 < b$. Therefore,

$$0 = \left\{ \int_{x_0 - iT}^{x_0 + iT} + \int_{x_1 + iT}^{x_1 + iT} + \int_{x_1 + iT}^{x_1 - iT} + \int_{x_1 - iT}^{x_0 - iT} \right\} f(s) e^{(s-t)^2/4} ds$$

$$= I_1 + I_2 + I_3 + I_4$$

where as in the proof of [3, Theorem 2] $|I_2| \to \infty$ and $|I_4| \to \infty$ as $T \to \infty$. It follows that

$$(3.4) \quad (P) \int_{-\infty}^{\infty} [f(x_0 + iy) e^{(x_0 + iy - t)^2/4} - f(x_1 + iy) e^{(x_1 + iy - t)^2/4}] dy = 0$$

Now, by the regularity of $k(y, \lambda)$, hypotheses and (3.4)

$$\lim_{\lambda \to \infty} \int_{-\infty}^{\infty} k(y, \lambda) e^{(x_0 + iy - t)^2/4} f(x_0 + iy) dy$$

$$= \lim_{\lambda \to \infty} \int_{-\infty}^{\infty} k(y, \lambda) e^{(x_1 + iy - t)^2/4} f(x_1 + iy) dy.$$

From the uniqueness of weak limit

$$F(t, x_0) = F(t, x_1) \equiv F(t).$$

so that, by (3.4)

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$$f(x+iy) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-(t-x-iy)^2/4} F(t)dt,$$

which proves the theorem.

For the case $p = 1$ we have the following result.

THEOREM 4. If $f(x) \in A(a, b)$, (2.3) is satisfied and $k(y,\lambda)e^{-y^2/4} f(x+iy) \in L_1(-\infty, \infty)$ where $k(y,\lambda) \to 1$ as $\lambda \to \infty$, uniformly in y for any finite interval, then there exists a function μ with

$$\int_{-\infty}^{\infty} e^{-(x-t)^2/4} |d\mu(t)| < \infty, \quad \text{for each } x \in (a, b)$$

such that $f(x) = WS(\mu, x)$.

Proof. By hypotheses $F_\lambda(t, x)$ is well defined for all $x \in (a, b)$. Let

$$\mu_\lambda(t, x) \equiv \int_0^t F_\lambda(u, x)du,$$

then for an arbitrary finite interval $[\alpha, \beta]$ and fixed $x_0 \in (a, b)$

$$\int_\alpha^\beta |d\mu_\lambda(t,x_0)| \le \int_\alpha^\beta |F_\lambda(t,x_0)|dt \le \max_{\alpha \le t \le \beta} e^{(x_0-t)^2/4} \int_\alpha^\beta e^{-(x_0-u)^2/4} |F_\lambda(u,x_0)|du$$

uniformly in λ. Thus $\{\mu_\lambda(t, x_0)\}$ is of uniformly bounded variation in $[\alpha, \beta]$ and

$$|\mu_\lambda(\alpha,x_0)| \le \int_0^\alpha |F_\lambda(u,x_0)| du \le \max_{0<t<\alpha} e^{(x_0-t)^2/4} \int_0^\alpha e^{-(x_0-u)^2/4} |F_\lambda(u,x_0)| du$$

$< \infty$. Hence by [5, Theorem 16.3] there exists an increasing unbounded subsequence $\{\lambda_k\}$ and a function $\mu(t, x_0)$ of bounded variation in $\alpha \le t \le \beta$ such that

$$(3.5)\qquad \lim_{k\to\infty} \mu_{\lambda_k}(t, x_o) = \mu(t, x_o).$$

Moreover, by [5, Theorem 16.4 and Corollary] for any continuous function $h(t)$ in $[\alpha, \beta]$,

$$\lim_{k\to\infty} \int_\alpha^\beta h(t)\, du_{\lambda_k}(t, x_o) = \int_\alpha^\beta h(t)\, d\mu(t, x_o).$$

Hence in particular for $h(t) = e^{-(x_o - t)^2/4}$

$$\int_\alpha^\beta e^{-(x_o-t)^2/4} |d\mu(t, x_o)| \le \underline{\lim}_{k\to\infty} \int_\alpha^\beta e^{-(x_o-t)^2/4} |d\mu_{\lambda_k}(t, x_o)|$$

$$\le \underline{\lim}_{k\to\infty} \int_{-\infty}^{\infty} e^{-(x_o-t)^2/4} |F_\lambda(t, x_o)| dt < \infty.$$

so that

$$\int_{-\infty}^{\infty} e^{-(x_o-t)^2/4} |d\mu(t, x_o)| < \infty.$$

Now, by (3.5)

$$(3.6)\quad \mu(t, x_o) = \lim_{k\to\infty} \int_0^t F_{\lambda_k}(u, x_o) du$$

$$= \lim_{k\to\infty} \frac{1}{\sqrt{4\pi}} \int_0^t du \int_{-\infty}^{\infty} k(y, \lambda_k) e^{(x_o+iy-u)^2/4} f(x_o+iy) dy$$

$$= \lim_{k\to\infty} \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} k(y, \lambda_k) f(x_o+iy) dy \int_0^t e^{(x_o+iy-u)^2/4} du$$

where the interchange of order of integration is justified by Fubini's theorem.

Now we show that $\mu(t, x_o)$ is independent of the choice of $x_o \varepsilon (a, b)$. Since $f \varepsilon A(a, b)$,

$$f(s) \int_o^t e^{(s-u)^2/4} du$$

is holomorphic in the strip $a < \text{Re}\, s < b$ then in exactly the same way as in the proof of [3, Theorem 1] we find that

$$(P) \int_{-\infty}^{\infty} \{f(x_o+iy) [\int_o^t e^{(x_o+iy)^2/4} du] - f(x_1+iy)[\int_o^t e^{(x_1+iy-u)^2/4} du]\} dy = 0$$

where $a < x_o < x_1 < b$. By the regularity of $k(y,\lambda)$ and the fact that $k(y\lambda)f(x+iy)\, e^{-y^2/4} \varepsilon L_1(-\infty, \infty)$ it follows from (3.6) that

$$\mu(t, x_o) = \mu(t, x_1) = \mu(t).$$

Now let $G(t)$ be a continuous function in $[-T, T]$ and zero outside the interval. If g denotes the Fourier transform of G then

$$\int_{-\infty}^{\infty} G(t)e^{-(x-t)^2/4} F_{\lambda_k}(t, x)dt = \int_{-\infty}^{\infty} G(t)e^{-(x-t)^2/4} d\mu_k(t, x)$$

$$= \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} G(t)e^{-(x-t)^2/4} dt \int_{-\infty}^{\infty} k(y,\lambda_k)e^{(x+iy-t)^2/4} f(x+iy)\, dy$$

$$= \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} k(y,\lambda_k)e^{iyx/2} f(x+iy)e^{-y^2/4} dy \int_{-\infty}^{\infty} G(t)e^{-iyt/2} dt$$

$$= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} g(\tfrac{y}{2})k(y,\lambda_k)\, e^{iyx/2} f(x+iy)e^{-y^2/4}\, dy$$

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and

$$\sqrt{2}\ k(2y,\lambda_k)e^{ixy}e^{-y^2}f(x+2iy)$$

is the Fourier transform of $e^{-(x-t)^2/4}F_{\lambda_k}(t,x)$. That is

$$k(2y,\lambda_k)e^{ixy}f(x+2iy)e^{-y^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(t-x)^2/4}\, d\mu_k(t,x).$$

Let $k \to \infty$, then a change of variable yields

$$f(x+iy) = \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} e^{-(t-x-iy)^2/4}\, d\mu(t),$$

which is the result.

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