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A NOTE ON ASSOCIATIVE POLYVERBAL OPERATIONS ON GROUPS

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Introduction. In his paper [**2**] O. N. Golovin introduced the notion of a neutral polyverbal operation on groups, of which Moran's verbal operation [**6**], and Gruenberg's and Šmel'kin's operations [**3**; **7**] are special cases. (Bronštein [**1**] proved, more generally, that every regular operation for which MacLane's postulate (see [**2**]) holds and which is invariant under addition of trivial factors, is a neutral polyverbal operation.)

Every neutral polyverbal operation is determined by some set of polywords (see Section 2). We are concerned here with neutral polyverbal operations determined by a set of binary polywords. In the present paper we show that if such a polyverbal operation is associative then the corresponding polyverbal subgroup cannot be contained in the third term of the lower central series of the base group (see Section 2). Theorem 10 of [**2**] and Theorem 2 of [**4**] follow immediately from our result. The nonassociativity of some of the examples of operations given by R. R. Struik in [**8**], can also be deduced from our theorem. We give most of the relevant definitions and the precise statement of our theorem in Section 2. Undefined terms can be found in [**2**].

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2. Definitions and statement of result. We first define a polyverbal product of groups. To this end let X_i ($i = 1, 2, \ldots$) be a free group, freely generated by the letters $x_{i1}, x_{i2}, \ldots$, and denote by $X = \prod_{i=1}^{\infty *} X_i$ the free product of the X_i ($i = 1, 2, \ldots$). The group X will be called the *base group* and its elements will be called *polywords*. The *Cartesian subgroup* of $X = \prod_{i=1}^{\infty *} X_i$ is defined to be the normal closure of the set $\{[a_i, a_j] | a_i \in X_i, a_j \in X_j, i \neq j\}$, where $[a_i, a_j] = a_i^{-1}a_j^{-1}a_ia_j$.

Let $\{G_\alpha | \alpha \in M\}$ be an arbitrary set of groups and $G = \prod_{\alpha \in M}{}^* G_\alpha$ their free product. A homomorphism from X to G is said to be *regular* if the image of each free factor X_i is contained in some G_α, $\alpha \in M$, and nontrivial images of different X_i are contained in different G_α.

The image of any polyword under a regular homomorphism from X to G is called the *value* of this polyword in G.

If VV is an arbitrary set of polywords from the Cartesian subgroup of X, then the normal subgroup generated by all the values in G of all polywords

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from VV, is called the *neutral polyverbal VV-subgroup* of $G = \prod_{\alpha \in M}^* G_\alpha$, denoted by $VV(G)$. (In the sequel we shall usually omit the word neutral: all our polyverbal subgroups will be neutral.) A familiar example of a polyverbal subgroup is the Cartesian subgroup of G, which is determined, for example, by the single polyword $[x_{11}, x_{21}]$.

Taking $G = X$ we obtain the polyverbal subgroup $VV(X)$ of the base group X determined by the set of polywords VV, for which we use the special symbol W. It is easy to check that for any group $G = \prod_{\alpha \in M}^* G_\alpha$ the equality $VV(G) = W(G)$ holds, where $W(G)$ is the polyverbal subgroup of G determined by the set W.

If W is any polyverbal subgroup of X, then the *W-product* (or polyverbal W-product) of the set $\{G_\alpha | \alpha \in M\}$ of groups is defined as

$$\prod_{\alpha \in M}^{W} G_\alpha = \left(\prod_{\alpha \in M} {}^* G_\alpha \right) \Big/ W(G).$$

We shall denote by

$$v(x_{i_1 1}, x_{i_1 2}, \ldots, x_{i_1 s_1}; \ldots ; x_{i_k 1}, x_{i_k 2}, \ldots, x_{i_k s_k})$$

an arbitrary polyword in X involving in its reduced form the letters from the k different factors $X_{i_1}, \ldots, X_{i_k}$. We shall then say that this polyword is *k-ary* or that its *arity* is k, and also that it has *absolute arity m* modulo $\gamma_{n+1}(X)$ if m is the smallest number of different letters involved in any polyword v satisfying $w \equiv v \bmod \gamma_{n+1}(X)$, where $\gamma_1(X) = X$, $\gamma_n(X) = [\gamma_{n-1}(X), X]$. Obviously $m \geqq k$. We shall further say that a polyword is *absolutely neutral* modulo $\gamma_{n+1}(X)$ if it becomes 1 mod $\gamma_{n+1}(X)$ when any one of the letters it involves is put equal to 1. The *weight* of a polyword $v \in X$ is defined as usual to be n if $v \in \gamma_n(X)$, $v \notin \gamma_{n+1}(X)$.

We are now able to state our result.

THEOREM. *If W is a neutral polyverbal subgroup determined by a set of binary polywords and the corresponding W-operation is associative, then $W \not\subseteq \gamma_3(X)$.*

Theorem 10 of [**2**] and Theorem 2 of [**4**] give sufficient conditions for non-associativity of W-operations determined by sets of binary polywords of weight more than 2. By our theorem all operations determined by such sets of binary polywords are non-associative.

3. Lemma and proof of the theorem.

LEMMA. *Let W be a polyverbal subgroup determined by a set VV of binary polywords and let the corresponding W-operation be associative. If w is a polyword in W which is absolutely neutral modulo $\gamma_{n+1}(X)$ and has minimal weight n in W, then the absolute arity modulo $\gamma_{n+1}(X)$ of w is 2.*

Proof. Let w be, as in the lemma, an absolutely neutral polyword of W of minimal weight n (i.e., $W \subseteq \gamma_n(X)$, $W \not\subseteq \gamma_{n+1}(X)$). Suppose that the absolute arity of w modulo $\gamma_{n+1}(X)$ is $l > 2$. It will be shown that this leads to contradiction.

To begin with, we change the first indices of letters x_{ij} involved in the polyword w so that different letters have different first indices, and change all the second indices to 1. The resulting polyword v say, has both arity and absolute arity l modulo $\gamma_{n+1}(X)$ and is of weight n. Since by hypothesis our W-operation is associative, it follows from [**2**, Theorem 7] that

$$v \in W \cap \prod_{i=1}^{\infty} {}^* \langle x_{i1} \rangle = W\left(\prod_{i=1}^{\infty} {}^* \langle x_{i1} \rangle \right).$$

By assumption the polyverbal subgroup W is determined by the set VV of binary polywords of weights $\geqq n$. Let us denote by S the set of all values in $\prod_{i=1}^{\infty}{}^* \langle x_{i1} \rangle$ of polywords from VV. Note that every value $s \in S$ is of weight $\geqq n$ and has absolute arity 2. The set S generates the subgroup $W(\prod_{i=1}^{\infty}{}^* \langle x_{i1} \rangle)$ as a normal subgroup. Thus v may be written as

$$v \equiv s_1 s_2 \ldots s_k \bmod \gamma_{n+1}(X),$$

where $s_j \in S$, $(j = 1, 2, \ldots, k)$. Note that the s_j $(j = 1, 2, \ldots, k)$ commute modulo $\gamma_{n+1}(X)$. Therefore those s_j involving the same pair of letters can be collected together to form words u_i $(i = 1, 2, \ldots, r)$ of absolute arity 2, such that distinct u_i involve distinct pairs of letters. Thus

$$v \equiv u_1 u_2 \ldots u_p \bmod \gamma_{n+1}(X).$$

Choose any u_j and set equal to 1 all x_{i1} not involved in u_j. Then as v is absolutely neutral modulo $\gamma_{n+1}(X)$ we shall have that $u_j \in \gamma_{n+1}(X)$. It follows that $v \in \gamma_{n+1}(X)$ contradicting the fact that v is of weight n.

Proof of the theorem. Let W be a polyverbal subgroup determined by a set of binary polywords, and suppose that the corresponding W-operation is associative, and that $W \subseteq \gamma_n(X)$, $W \not\subseteq \gamma_{n+1}(X)$ where, contrary to the theorem, $n > 2$. We shall find in the polyverbal subgroup W a polyword v of weight n which is absolutely neutral and has absolute arity 3 modulo $\gamma_{n+1}(X)$. Since this contradicts the lemma, the theorem will follow.

By assumption W contains some polyword $\hat{w}$ of weight n. Express $\hat{w}$ as a product $\prod_{\hat{c}} \hat{c}$ modulo $\gamma_{n+1}(X)$ of left-normed commutators of weight n with letters x_{ij} as entries (see [**5**, Theorem 5.4]). If we collect together those factors c with the same set of entries we obtain that

$$\hat{w} \equiv \prod_{d} d \quad \bmod \gamma_{n+1}(X),$$

where each factor d is a product of left-normed commutators of weight n with the same set of entries, and different d have different (but possibly intersecting) sets of entries.

Choose any $d \notin \gamma_{n+1}(X)$ with a minimal set of entries in its left-normed factors, and set equal to 1 all letters not involved in this d. Then we obtain a new polyword

$$w \equiv d \equiv \prod_{c} c \mod \gamma_{n+1}(X),$$

where the c are the aforementioned left-normed factors of d. Thus w is of weight n, lies in W by definition of W, and is absolutely neutral modulo $\gamma_{n+1}(X)$. According to the lemma, the absolute arity of w modulo $\gamma_{n+1}(X)$ is 2, which means that every factor c above has precisely two letters x, y say, as entries. To stress this we shall write

$$w = w(x, y) \equiv \prod_{c} c(x, y) \mod \gamma_{n+1}(X).$$

(Obviously the absolute arity of w is 2.)

Denote by $u_k(x, y)$ the product of those left-normed factors $c(x, y)$ which contain the entry y k times (and x $n - k$ times). Then

$$w(x, y) \equiv \prod_{k=1}^{n-1} u_k(x, y) \mod \gamma_{n+1}(X).$$

Since $w(x, y) \notin \gamma_{n+1}(X)$, there exists k_0 $(1 \leqq k_0 \leqq n - 1)$ such that

$$u_{k_0}(x, y) \notin \gamma_{n+1}(X).$$

We can suppose that $k_0 > 1$ (by interchanging the roles of x and y if necessary).

If $w(x, yz)$ denotes the new polyword obtained from $w(x, y)$ by replacing y by yz where x, y, z are letters from different factors X_i, then

$$w(x, yz) \equiv \prod_{c} c(x, yz) \mod \gamma_{n+1}(X).$$

Since by assumption our W-operation is associative it follows from [**2**, Theorem 7] that $w(x, yz)$ lies in W.

If the left-normed commutator $c(x, y)$ contains the entry y k times, then $c(x, yz)$ can be expressed modulo $\gamma_{n+1}(X)$ as a product of 2^k left-normed commutators, two of which are binary and the rest ternary (by using distributivity modulo $\gamma_{n+1}(X)$ of multiplication through formation of commutators of weight n). If we denote the product of these ternary factors by $r(x, y, z)$, then $c(x, yz) \equiv c(x, y)c(x, z)r(x, y, z) \mod \gamma_{n+1}(X)$. Note that $r(x, y, y) = (c(x, y))^{2k-2}$.

By multiplying the above equalities for every c from u_k, we obtain

$$u_k(x, yz) \equiv u_k(x, y)u_k(x, z)s_k(x, y, z) \mod \gamma_{n+1}(X),$$

where $s_k(x, y, z) = \prod r(x, y, z)$. Note as before that $s_k(x, y, y) = (u_k(x, y))^{2k-2}$. Now

$$w(x, yz) \equiv \prod_{k=1}^{n-1} u_k(x, yz) \equiv \prod_{k=1}^{n-1} u_k(x, y) \prod_{k=1}^{n-1} u_k(x, z) \prod_{k=1}^{n-1} s_k \mod \gamma_{n+1}(X),$$

i.e., $w(x, yz) = w(x, y) \cdot w(x, z) \cdot v$, where $v = (\prod_{k=1}^{n-1} s_k) \cdot f, f \in \gamma_{n+1}(X)$.

It is enough now to show that $\prod_{k=1}^{n-1} s_k \notin \gamma_{n+1}(X)$, since then v will be an absolutely neutral polyword of weight n in W of absolute arity 3 modulo $\gamma_{n+1}(X)$. Hence we shall obtain the desired contradiction with the lemma.

Let Z denote the ring of integers. Denote (as in [**5**, pp. 299, 300]) by A_0 the free associative Z-algebra freely generated by x_1, x_2, x_3, and by Λ_0 the free Lie Z-algebra in A_0 freely generated by x_1, x_2, x_3, under bracket multiplication given by $[\alpha, \beta] = \alpha\beta - \beta\alpha$. Let F be a free group freely generated by x, y, z.

Denote by $\beta^n(x_1, x_2, x_3)$ any Lie monomial of degree n with entries x_1, x_2, x_3 (possibly repeated) in Λ_0 (see [**5**, p. 301]), and by $\beta^n(x, y, z)$ the corresponding commutator of weight n in F. Then by [**5**, Theorem 5.12], the mapping δ from $\gamma_n(F)/\gamma_{n+1}(F)$ to the abelian group (under addition) of all homogeneous Lie elements of degree n, defined by

$$\delta : \beta^n(x, y, z) \longrightarrow \beta^n(x_1, x_2, x_3),$$

is an isomorphism.

We can now easily prove that the element $\prod_{k=1}^{n-1} s_k$ of F is not in $\gamma_{n+1}(F)$. Recall that s_k is a product of left-normed commutators c, with letters x, y, z as entries, involving the entry x precisely $n - k$ times. Thus if $s_k \notin \gamma_{n+1}(F)$ then the element $s_k\delta$ of Λ_0 is a homogeneous polynomial of degree n, homogeneous of degree $n - k$ in x. Hence for $k_1 \neq k_2$, $s_{k_1}\delta$ and $s_{k_2}\delta$ have no common terms.

It is enough now to show that $s_{k_0} \notin \gamma_{n+1}(X)$ for k_0 chosen as above.

Denote by φ the endomorphism of X which maps z into y and maps the remaining letters identically. Then if $s_{k_0}(x, y, z)$ were in $\gamma_{n+1}(X)$, we should have

$$s_{k_0}(x, y, z)\varphi = u_{k_0}^{2k_0-2} \in \gamma_{n+1}(X)$$

which is impossible since $u_{k_0} \notin \gamma_{n+1}(X)$ and $\gamma_n(X)/\gamma_{n+1}(X)$ is torsion-free (see [**5**, Theorem 5.12]).

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