SPACES OF HOLOMORPHIC FUNCTIONS AND HILBERT–SCHMIDT SUBSPACES

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In this note we construct certain Hilbert subspaces with Hilbert–Schmidt imbedding, for an arbitrary proper functional Hilbert space which consists of holomorphic functions. This work extends results of Chapter III in [**1**] and has applications in the regularity problem for generalised eigenfunctions (in particular to Theorem 2 in [**2**]). For an exposition of reproducing kernels and Bergman's kernel function we refer to [**4**].

Let P denote a polydisc $P=\{z\in\mathbf{C}:|z_j-a_j|<r_j\ (j=1,\dots,n)\}$ and $\mathscr{B}=\mathscr{B}(P)$ the space of all functions that are holomorphic in P and square integrable over P with respect to Lebesgue measure on $\mathbf{R}^{2n}$. $\mathscr{B}$, endowed with the L^2-norm, has a reproducing kernel $B(z,\zeta)$ which is Bergman's kernel; the function $B:P\to\mathscr{B}$ given by $z\mapsto B(\cdot,z)$ is (strongly) conjugate-holomorphic (i.e. $\bar{z}\mapsto B(\cdot,z)$ is holomorphic from $\bar{P}$ to $\mathscr{B}$). The following result extends Theorem III.1 of [**1**] (which corresponds to the case $\alpha=1$ below).

THEOREM 1. *There is a positive selfadjoint operator T of Hilbert–Schmidt type in the space $\mathscr{B}$ with Bergman's kernel $B(z,\zeta)$, with the following properties. For every $\alpha\geqq 0$, the Hilbert subspace $T^\alpha\mathscr{B}\equiv\mathscr{B}_\alpha$ of $\mathscr{B}$ with norm $\|\varphi\|_\alpha=\|T^{-\alpha}\varphi\|_{\mathscr{B}}$ contains all functions $B(\cdot,\zeta)$ $(\zeta\in P)$, and so does the nuclear countably-Hilbert space $\mathscr{B}_\infty\equiv\bigcap_{\alpha\in\mathbf{R}}\mathscr{B}_\alpha=\bigcap_{k=0}^{\infty}\mathscr{B}_k$*

$$\left(\textit{with metric}\quad \rho(0,u)=\sum_{k=0}^{\infty}2^{-k}\frac{\|u\|_k}{1+\|u\|_k}\right);$$

moreover the mapping $\zeta\mapsto B(\cdot,\zeta)$ is conjugate-holomorphic in the norm of every $\mathscr{B}_\alpha$ and in the metric of $\mathscr{B}_\infty$.

Proof. We construct the operator T. Let m denote a multi-index $(m_1,\dots,m_n)$ of *positive* integers, and write $\prod_{j=1}^{n}m_j=\bar{m}$. The monomials $\chi_m(z)=\prod_{j=1}^{n}(z_j-a_j)^{m_j-1}$ are orthogonal in $\mathscr{B}$; their norms are $\|\chi_m\|=\pi^{\frac{1}{2}n}\bar{m}^{-\frac{1}{2}}r_1^{m_1}\dots r_n^{m_n}$. Set $\varphi_m=\|\chi_m\|^{-1}\chi_m$; the φ_m form a complete orthonormal system ("CONS") for $\mathscr{B}$. The reproducing kernel $B(z,\zeta)$ is represented by $B(\cdot,\zeta)=\sum_m\overline{\varphi_m(\zeta)}\varphi_m$, the series converging in the norm of $\mathscr{B}$. We define the operator T by

$$T\varphi_m=\bar{m}^{-1}\varphi_m;$$

T is Hilbert–Schmidt, since $\sum\|T\varphi_m\|^2=\sum\bar{m}^{-2}<\infty$. For every real number α and every

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$\zeta \in P$ we have

$$\sum_m |\bar{m}^\alpha \varphi_m(\zeta)|^2 < \infty$$

and consequently

$$B(\cdot, \zeta) = \sum [\overline{\bar{m}^\alpha \varphi_m(\zeta)}][T^\alpha \varphi_m] \in T^\alpha \mathscr{B} = \mathscr{B}_\alpha \quad \text{for} \quad \alpha \geqq 0.$$

Since each $\mathscr{B}_\alpha$ is dense in $\mathscr{B}_\beta$ for any $0 \leqq \beta \leqq \alpha$, we may extend the chain of spaces $\mathscr{B}_\alpha$ ($\alpha \geqq 0$) by duality. For every $\alpha > 0$, regard $\mathscr{B}$ as a dense subspace of the anti-dual $(\mathscr{B}_\alpha)^*$, the imbedding given by

$$f(\varphi) = (f, \varphi)_0 \quad \text{for every} \quad \varphi \in \mathscr{B}_\alpha \quad \text{and} \quad f \in \mathscr{B} = \mathscr{B}_0.$$

Setting $(\mathscr{B}_\alpha)^* = \mathscr{B}_{-\alpha}$, we thus obtain a continuous scale of Hilbert spaces in the sense of S. G. Kreĭn and others (cf. [3] for instance), and this scale is nuclear, since the imbedding $\mathscr{B}_\alpha \subset \mathscr{B}_\beta$ is nuclear whenever $\alpha - \beta > 2$. In this chain we can define any (real) power of T by $T^\beta \varphi_m = \bar{m}^{-\beta} \varphi_m$ for arbitrary real β, and then $T^\beta \mathscr{B} = \mathscr{B}_\beta$. Moreover, all the spaces $\mathscr{B}_\beta$ consist of holomorphic functions on P, and the space $\mathscr{B}_\beta$ has a reproducing kernel given by $B_\beta(\cdot, \zeta) = T^{2\beta} B(\cdot, \zeta)$ ($\beta \in \mathbf{R}$). Also note that, for all $\alpha, \beta \in \mathbf{R}$, we have

$$T^\beta B(\cdot, \zeta) \in \mathscr{B}_\alpha \quad \text{for every} \quad \zeta \in P.$$

At the very end of [1] it was proved by direct calculation that $z \mapsto B(\cdot, z)$, regarded as a function from P into $\mathscr{B}_1$, is (strongly) conjugate holomorphic, i.e., it was shown that

$$\partial B(\cdot, \zeta)/\partial \bar{\zeta}_j = \lim_{h \to 0} \bar{h}^{-1}[B(\cdot, \zeta + h\varepsilon_j) - B(\cdot, \zeta)]$$

exists in the norm of $\mathscr{B}_1$. (Note that the derivative on the left can always be defined pointwise by

$$\frac{\partial B(\cdot, \zeta)}{\partial \bar{\zeta}_j}(z) = \frac{\partial B(z, \zeta)}{\partial \bar{\zeta}_j},$$

which is known to exist.) In a completely analogous way one may calculate that, for any real numbers α and β, the function $B_\beta : P \to \mathscr{B}_\alpha$ given by $z \mapsto B_\beta(\cdot, z)$ is conjugate-holomorphic in the norm of $\mathscr{B}_\alpha$.

Finally, set $\mathscr{B}_\infty = \bigcap_{\alpha \in \mathbf{R}} \mathscr{B}_\alpha$ and $\mathscr{B}_{-\infty} = \bigcup_{\alpha \in \mathbf{R}} \mathscr{B}_\alpha$, wit hthe corresponding projective and inductive limit topology, respectively. $\mathscr{B}_\infty$ is also equal to the nuclear countably-Hilbert space $\bigcap_{k \geqq 0} \mathscr{B}_k$ (k an integer); it is a Fréchet space in the metric (" quasi-norm ")

$$\sum 2^{-k} \frac{\| u \|_k}{1 + \| u \|_k}.$$

Then $z \mapsto B_\beta(\cdot, z)$ (for every fixed $\beta \in \mathbf{R}$) is also conjugate-holomorphic as a function from P into $\mathscr{B}_\infty$. The nuclear space $\mathscr{B}_{-\infty}$ is the strong anti-dual of $\mathscr{B}_\infty$ (and vice-versa), and it is continuously imbedded in the space $H(P)$ of all holomorphic functions in P with the topology of uniform convergence on compacts. The proof is complete.

REMARK. The space $\mathscr{B}_\beta$ has reproducing kernel $T^{2\beta}B(\cdot,\zeta)$. In any proper functional Hilbert space $\mathscr{H}$ on a set E, with reproducing kernel K, all bounded and certain unbounded operators L have a representation of the following kind:

$$(Lf)(x)=(Lf,K(\cdot,x))=(f,L^*K(\cdot,x))\quad\text{for}\quad x\in E,$$

i.e.,

$$(Lf)(x)=(f,\Lambda(\cdot,x)),$$

where $\Lambda(\cdot,x)=L^*K(\cdot,x)$ is the " kernel of L ". In the present situation, for any $\alpha,\beta\in\mathbf{R}$, the operator T^α in $\mathscr{B}_\beta$ is represented by a kernel $S_{\alpha,\beta}$, say, namely

$$S_{\alpha,\beta}(\cdot,z)=T^\alpha B_\beta(\cdot,z)=T^{\alpha+2\beta}B(z)=\sum\bar{m}^{-(\alpha+2\beta)}\overline{\varphi_m(\zeta)}\varphi_m;$$

$$(T^\alpha\varphi)(z)=(\varphi,T^\alpha B_\beta(\cdot,z))_\beta.$$

(Note that all φ_m lie in $\mathscr{B}_\infty$.)

We now turn to the general case which corresponds to the preceding theorem. Let D be a connected domain in $\mathbf{C}^n$, and $\{\mathscr{F},D\}$ an arbitrary proper functional Hilbert space consisting of functions that are holomorphic in D; denote the reproducing kernel of $\mathscr{F}$ by $K(z,\zeta)$. Let P be any polydisc whose closure is contained in D. Then we have the following result.

THEOREM 2. *There exists a Hilbert subspace Φ of $\mathscr{F}$ containing all functions $K(\cdot,\zeta)$ ($\zeta\in P$), such that the imbedding of Φ into $\mathscr{F}$ is Hilbert–Schmidt, and the function $P\to\Phi$ given by $\zeta\mapsto K(\cdot,\zeta)$ is conjugate-holomorphic in the norm of Φ.*

Proof. Let $\mathscr{F}|_P$ be the space of restrictions to P of functions in $\mathscr{F}$. Since D is connected and the functions in $\mathscr{F}$ are holomorphic, the subspace $N(P)$ of $\mathscr{F}$ of functions vanishing identically on P is zero. Thus $\mathscr{F}|_P$ is " the same " as $\mathscr{F}\ominus N(P)=\mathscr{F}$. All functions in $\mathscr{F}|_P$ are holomorphic in a neighbourhood of the closure of P, hence square integrable on P, and so $\mathscr{F}|_P$ $(=\mathscr{F})$ is a Hilbert subspace of $\mathscr{B}(P)=\mathscr{B}$ with Bergman kernel B. Let G be the " kernel of $\mathscr{F}$ in $\mathscr{B}$ ", i.e., $(f,h)_{\mathscr{B}}=(f,Gh)_{\mathscr{F}}$ for $f\in\mathscr{F}$, $h\in\mathscr{B}$; its square root (taken in $\mathscr{B}$) is the canonical partial isometry of $\mathscr{B}$ onto $\mathscr{F}$. Then the reproducing kernel of $\mathscr{F}|_P$ is the restriction of $K(z,\zeta)$ to $P\times P$, and $K(\cdot,\zeta)|_P=GB(\cdot,\zeta)$ for $\zeta\in P$. Let $\{\varphi_m\}$ and T in $\mathscr{B}$ be as described in Theorem 1. Then

$$K(\cdot,\zeta)|_P=GB(\cdot,\zeta)=\sum_m\bar{m}^\alpha\overline{\varphi_m(\zeta)}GT^\alpha\varphi_m$$

for all $\alpha\geqq 0$, and

$$\sum_m\|GT^\alpha\varphi_m\|^2_{\mathscr{F}}=\sum_m\|G^{\frac12}T^\alpha\varphi_m\|^2_{\mathscr{B}}\leqq\|G\|_{\mathscr{B}}\sum_m\|T^\alpha\varphi_m\|^2_{\mathscr{B}}\qquad(\|G^{\frac12}\|^2=\|G\|\quad\text{in}\quad\mathscr{B}).$$

Now let $\{\psi_m\}$, where m varies over all multi-indices of positive integers, be a CONS in

$\mathscr{F} = \mathscr{F}|_P$, and put

$$L\psi_m = GT^\alpha\varphi_m = \bar{m}^{-\alpha}G\varphi_m$$

for some fixed α. Then $K(\cdot, \zeta) \in L\mathscr{F}$ for all $\zeta \in P$, and, if $\alpha \geqq 1$, then L is Hilbert–Schmidt in $\mathscr{F}$.

Let $\alpha \geqq 1$ (fixed) from now on, and make $L\mathscr{F}$ into a Hilbert–Schmidt subspace Φ of $\mathscr{F}$ by defining the following norm on it:

$$\| v \|_\Phi^2 = \inf\{\sum |\xi_m|^2 : v = \sum \xi_m L\psi_m \text{ in } \mathscr{F}\}.$$

We check that $K(\cdot, \zeta)$ is conjugate-holomorphic in the norm of Φ. It is known that the two limits

$$\partial K(z, \zeta)/\partial\bar{\zeta}_j = \lim_{h\to 0} \bar{h}^{-1}(K(\cdot, z), K(\cdot, \zeta + h\varepsilon_j) - K(\cdot, \zeta))_{\mathscr{F}}$$

and

$$\lim_{h\to 0} \bar{h}^{-1}[K(\cdot, \zeta + h\varepsilon_j) - K(\cdot, \zeta)] \quad \text{in the norm of } \mathscr{F}$$

exist, and the value of the second of these at z is just the first limit. Write

$$\bar{h}^{-1}[K(\cdot, \zeta + h\varepsilon_j) - K(\cdot, \zeta)] = K_{\zeta,h}.$$

Because of the uniqueness of limits we only have to show now that the $K_{\zeta,h}$ converge, as $h \to 0$, in the norm of Φ; then their limit must lie in Φ and equal $\partial K(z, \zeta)/\partial\bar{\zeta}_j$. If $\zeta \in P$, then

$$K(\cdot, \zeta) = \sum_m \overline{m^\alpha\varphi_m(\zeta)} L\psi_m$$

and

$$\| K(\cdot, \zeta) \|_\Phi^2 \leqq \sum_m |\bar{m}^\alpha\varphi_m(\zeta)|^2.$$

Due to our constructions, G restricted to $\mathscr{B}_\alpha$ is a bounded operator of $\mathscr{B}_\alpha$ into Φ; moreover $K_{\zeta,h} = G\bar{h}^{-1}[B_\alpha(\cdot, \zeta + h\varepsilon_j) - B_\alpha(\cdot, \zeta)] \equiv GB_{\alpha,\zeta,h}$, and the $B_{\alpha,\zeta,h}$ converge in $\mathscr{B}_\alpha$ by Theorem 1. Thus the $K_{\zeta,h}$ converge in the norm of Φ as $h \to 0$. The proof is complete.

Remark. One could cover D by a sequence of polydiscs P_k whose closures all lie in D, redefine L and Φ suitably, and obtain the conclusions of Theorem 2 simultaneously for all $\zeta \in D$.

It would be desirable to obtain a slightly better result than the space Φ in Theorem 2, namely a nuclear Fréchet space Ψ continuously imbedded in $\mathscr{F}$, containing all $K(\cdot, \zeta)$ ($\zeta \in D$), such that the map $D \to \Psi$ given by $\zeta \mapsto K(\cdot, \zeta)$ is conjugate holomorphic in the topology of Ψ. For the present we leave aside the problem of constructing such a nuclear subspace in a direct way (without recourse to the space $\mathscr{B}(P)$).

If we restrict ourselves to a fixed polydisc whose closure lies in D, and if the closure $\mathscr{M}$ of $\mathscr{F}|_P = \mathscr{F}$ in $\mathscr{B} = \mathscr{B}(P)$ reduces the operator T, we can proceed as follows. Let Q be the

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orthogonal projection onto $\mathcal{M}$ in $\mathcal{B}$. Then Q commutes with every T^α, and QT generates the nuclear Hilbert scale $\{Q\mathcal{B}_\alpha : \alpha \in \mathbf{R}\}$, $Q\mathcal{B}_\alpha = QT^\alpha\mathcal{B}$. In this case the nullspace of the kernel G (G defined in the proof of Theorem 2) equals the orthogonal complement of $\mathcal{M}$, and, as G is injective on every $Q\mathcal{B}_\alpha$, we simply transfer the norm $\| \ \|_\alpha$ from $Q\mathcal{B}_\alpha$ to $GQ\mathcal{B}_\alpha = G\mathcal{B}_\alpha$ and obtain the nuclear Fréchet space $\Psi = \bigcap \{G\mathcal{B}_\alpha, \| \ \|_\alpha\}$ ($\alpha \in \mathbf{R}$), which contains all $K(\cdot, \zeta)$ ($\zeta \in P$). The norm $\| \ \|_\alpha$ on $G\mathcal{B}_\alpha$ is the same as that given in the above definition of a norm on Φ (when we used $L\psi_m = \bar{m}^{-\alpha}G\varphi_m$). Since $\zeta \mapsto K(\cdot, \zeta)$ is conjugate-holomorphic in every norm $\| \ \|_\alpha$, it is conjugate-holomorphic in the metric

$$\sum 2^{-k} \frac{\| u \|_k}{1 + \| u \|_k} \quad \text{of} \quad \Psi.$$

Remark. If Φ^* and Ψ^* are the (strong) anti-duals of the Φ and Ψ above, then $\mathcal{F} \subset \Phi^* \subset H(D)$ or $\mathcal{F} \subset \Psi^* \subset H(D)$ with continuous linear imbeddings, where $H(D)$ is the space of all holomorphic functions on D with the topology of uniform convergence on compacts.

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