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LARGE DEVIATIONS FOR THE LONGEST GAP IN POISSON PROCESSES

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Abstract

The longest gap $L(t)$ up to time t in a homogeneous Poisson process is the maximal time subinterval between epochs of arrival times up to time t; it has applications in the theory of reliability. We study the Laplace transform asymptotics for $L(t)$ as $t \to \infty$ and derive two natural and different large-deviation principles for $L(t)$ with two distinct rate functions and speeds.

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1. Introduction

Suppose that $(N(t))_{t\geq 0}$ is a homogeneous Poisson process (for simplicity, assume that the intensity $\lambda = 1$). Let $0 =: T_0 < T_1 < T_2 < \cdots$ be the epochs of arrival times and $L(t)$ be the *longest gap* (or *largest gap*) between the epochs up to time t, that is,

$$L(t) = \max\{T_{n+1} \wedge t - T_n : T_n < t\}.$$

As a special and important family of counting processes, Poisson processes have been used as mathematical models in various disciplines such as biology, image processing and telecommunications. The longest gap $L(t)$ can be regarded as a natural continuous analogue of the Erdős–Rényi law for the longest run of heads or tails in coin tossing (see [8, 14]). There are many studies of longest runs (see, for example, [9, 11, 12]) and fundamental implications in reliability theory (see [3]).

The main motivation for studying $L(t)$ is its close relationship with the occurrence time $D(\ell)$ of the first gap with length ℓ in the Poisson process $(N(t))_{t\geq 0}$. Here, $D(\ell) = \min\{T_n : T_{n+1} - T_n \geq \ell\}$, so that

$$\{D(\ell) + \ell > t\} = \{L(t) < \ell\}.$$

In the theory of computer reliability, if one considers a job which usually takes a time ℓ to be executed on some system and assumes that a restart is required once a

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failure occurs, then $\{D(\ell) + \ell > t\}$ represents the event that the actual execution time $D(\ell) + \ell$ of the job is more than t. In [1] and [2] asymptotic results as $t \to \infty$ of the tail $\mathbb{P}(D(\ell) > t)$ have been obtained for a fixed time length ℓ (which is independent of t). Our goal in this paper is to investigate such asymptotic results allowing time-dependent length $\ell = \ell(t)$. This can happen naturally, for example, when there are more and more jobs assigned to the system during the execution of the current job (known as *computer multitasking*), resulting in an increased execution time $\ell(t)$. Another motivation to study such a time-dependent length $\ell = \ell(t)$ is the close relationship between $L(t)$ and the *maximal spacing* with applications in statistical inference. More precisely, if $\{U_k\}_{1\le k\le n}$ are independent, identically and uniformly distributed random variables on $(0, 1)$ and $0 =: U_{0,n} < U_{1,n} < \cdots < U_{n,n} < U_{n+1,n} := 1$ are the order statistics, then the *maximal spacing* is $\Delta_n := \max_{1\le k\le n+1} S_k^{(n)}$, where the *spacings* $S_k^{(n)}$ are defined as $S_k^{(n)} = U_{k,n} - U_{k-1,n}$ with $1 \le k \le n + 1$ (see [6, 7]). If $\Pi(n)$ is a Poisson random variable (with mean n) independent of $\{U_k\}_{1\le k\le n}$, then $\Delta_{\Pi(n)}$ has the same distribution as $L(n)/n$ [4, Section 5]. We aim to study large-deviation probabilities in the form $\mathbb{P}(L(t)/a(t) \ge c)$ as $t \to \infty$ for suitably chosen $a(t)$ and c.

Note that there is an exact formula for the distribution function of $L(t)$ [10, (2.3)]:

$$\mathbb{P}(L(t) > u) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^j e^{-ju}}{j!} \Big(1 - \frac{ju}{t}\Big)_+^j + \frac{1}{t} \sum_{j=1}^{\infty} \frac{(-1)^{j-1} t^j e^{-ju}}{(j-1)!} \Big(1 - \frac{ju}{t}\Big)_+^{j-1},$$

where $(x)_+^j = x^j$ for $x > 0$ and 0 for $x \le 0$, with the convention $(x)_+^0 = 1$ for $x > 0$ and 0 for $x \le 0$. However, this complicated form hardly provides useful asymptotic information. We first establish asymptotics of the Laplace transform (moment generating function) of $L(t)$, from which we can choose suitable $a(t)$ and c. Throughout, $a(t) \sim b(t)$ as $t \to \infty$ stands for $\lim_{t\to\infty} a(t)/b(t) = 1$, also written as $a(t) = (1 + o(1))b(t)$.

Theorem 1.1. *As $t \to \infty$, the Laplace transform of $L(t)$ has the asymptotic behaviour:*

(i) $\mathbb{E}\exp\{yL(t)\} = t^{(1+o(1))y}$, *if $y < 1$;*
(ii) $\mathbb{E}\exp\{yL(t)\} = t^{(1+o(1))\cdot 2}$, *if $y = 1$;*
(iii) $\mathbb{E}\exp\{yL(t)\} = \exp\{(1 + o(1))(y - 1)t\}$, *if $y > 1$.*

With the help of Theorem 1.1, it is now possible to choose $a(t)$. The Gärtner–Ellis theorem (see [5, Section 2.3]) gives two different ways to do so. The first is $a(t) = \ln t$. In this case, one can define

$$\Lambda_t(y) = \ln \mathbb{E}\exp\{yL(t)/\ln t\}, \qquad y \in \mathbb{R},$$

and it is straightforward to compute (the so-called) *cumulant*

$$\Lambda(y) := \lim_{t\to\infty} (\ln t)^{-1} \Lambda_t(y \ln t) = \begin{cases} +\infty & \text{if } y > 1, \\ 2 & \text{if } y = 1, \\ y & \text{if } y < 1. \end{cases}$$

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Now the Fenchel–Legendre transform of $\Lambda(y)$, defined as $\Lambda^*(x) = \sup_{y\in\mathbb{R}}[yx - \Lambda(y)]$, is

$$\Lambda^*(x) = \begin{cases} +\infty & \text{if } x < 1, \\ x - 1 & \text{if } x \geq 1. \end{cases} \tag{1.1}$$

Corollary 1.2. *The normalised longest gap $L(t)/\ln t$ satisfies a large-deviation principle with a good rate function $\Lambda^*(x)$ given by (1.1) and a speed $\ln t$. That is,*

(i) *for any open set $O \subseteq \mathbb{R}$,*

$$\liminf_{t\to\infty} (\ln t)^{-1} \ln \mathbb{P}(L(t)/\ln t \in O) \geq -\inf_{x\in O} \Lambda^*(x); \tag{1.2}$$

(ii) *for any closed set $F \subseteq \mathbb{R}$,*

$$\limsup_{t\to\infty} (\ln t)^{-1} \ln \mathbb{P}(L(t)/\ln t \in F) \leq -\inf_{x\in F} \Lambda^*(x). \tag{1.3}$$

Taking $O = (1 + x, \infty)$ and $F = [1 + x, \infty)$ with any $x > 0$ in Corollary 1.2, one easily obtains $\lim_{t\to\infty}(\ln t)^{-1} \ln \mathbb{P}(L(t)/\ln t \geq 1 + x) = -x$.

The second choice is $a(t) = t$. In this case, define

$$\widetilde{\Lambda}_t(y) = \ln \mathbb{E} \exp\{yL(t)/t\}, \quad y \in \mathbb{R}.$$

Again it is straightforward to compute the cumulant

$$\widetilde{\Lambda}(y) := \lim_{t\to\infty} t^{-1} \Lambda_t(y \cdot t) = \begin{cases} 0 & \text{if } y < 1, \\ y - 1 & \text{if } y \geq 1. \end{cases}$$

The Fenchel–Legendre transform $\widetilde{\Lambda}^*(x)$ of $\widetilde{\Lambda}(y)$ is

$$\widetilde{\Lambda}^*(x) = \begin{cases} +\infty & \text{if } x < 0 \text{ or } x > 1, \\ x & \text{if } 0 \leq x \leq 1. \end{cases} \tag{1.4}$$

Corollary 1.3. *The normalised longest gap $L(t)/t$ satisfies a large-deviation principle with a good rate function $\widetilde{\Lambda}^*(x)$ given by (1.4) and a speed t. That is,*

(i) *for any open set $O \subseteq \mathbb{R}$,*

$$\liminf_{t\to\infty} t^{-1} \ln \mathbb{P}(L(t)/t \in O) \geq -\inf_{x\in O} \widetilde{\Lambda}^*(x);$$

(ii) *for any closed set $F \subseteq \mathbb{R}$,*

$$\limsup_{t\to\infty} t^{-1} \ln \mathbb{P}(L(t)/t \in F) \leq -\inf_{x\in F} \widetilde{\Lambda}^*(x).$$

Taking $O = (x, 1)$ and $F = [x, 1]$ with any $0 < x < 1$ in Corollary 1.3 easily gives $\lim_{t\to\infty} t^{-1} \ln \mathbb{P}(L(t)/t \geq x) = -x$.

2. Proof of Theorem 1.1

Theorem 1.1 will be proved through a series of lemmas.

LEMMA 2.1. *For all* $y \in \mathbb{R}$,

$$\liminf_{t\to\infty}(\ln t)^{-1}\ln \mathbb{E}\exp\{yL(t)\} \geq y.$$

PROOF. The case when $y = 0$ is trivial. If $y > 0$, then

$$\begin{aligned}(\ln t)^{-1}\ln \mathbb{E}\exp\{yL(t)\} &\geq (\ln t)^{-1}\ln \mathbb{E}(\exp\{yL(t)\}, |L(t)/\ln t - 1| \leq \varepsilon)\\ &\geq (\ln t)^{-1}\ln \exp\{y(1-\varepsilon)\ln t\}\cdot \mathbb{P}(|L(t)/\ln t - 1| \leq \varepsilon)\\ &= y(1-\varepsilon) + (\ln t)^{-1}\ln \mathbb{P}(|L(t)/\ln t - 1| \leq \varepsilon).\end{aligned}$$

Since $L(t)/\ln t$ converges to 1 almost surely (see [10]),

$$\lim_{\varepsilon\to 0^+}\liminf_{t\to\infty}(\ln t)^{-1}\ln \mathbb{E}\exp\{yL(t)\} \geq \lim_{\varepsilon\to 0^+} y(1-\varepsilon) = y.$$

If $y < 0$, a similar argument yields

$$\lim_{\varepsilon\to 0^+}\liminf_{t\to\infty}(\ln t)^{-1}\ln \mathbb{E}\exp\{yL(t)\} \geq \lim_{\varepsilon\to 0^+} y(1+\varepsilon) = y. \qquad \square$$

We make use of the following global estimation of the distribution function of $L(t)$.

LEMMA 2.2 [13]. *For* $1 \leq \ell \leq t$,

$$(1 - ae^{-\lfloor \ell \rfloor})^{\lceil t \rceil - \lfloor \ell \rfloor} \leq \mathbb{P}(L(t) < \ell) \leq (1 - e^{-\lceil \ell \rceil})^{\lfloor t \rfloor - \lceil \ell \rceil}, \tag{2.1}$$

where $a = (1 - e^{-1})^{-2}$, $\lfloor x \rfloor$ *denotes the largest integer* $n \leq x$ *and* $\lceil x \rceil$ *denotes the smallest integer* $n \geq x$.

Since the values of ℓ and t are usually large, for simplicity from now on we will write all the integer parts in (2.1) as just ℓ and t.

LEMMA 2.3. *For* $y < 1$,

$$\limsup_{t\to\infty}(\ln t)^{-1}\ln \mathbb{E}\exp\{yL(t)\} \leq y.$$

PROOF. We first rewrite

$$(\ln t)^{-1}\ln \mathbb{E}\exp\{yL(t)\} = (\ln t)^{-1}\ln \mathbb{E}(\exp\{yL(t)\}, |L(t)/\ln t - 1| \leq \varepsilon \cup |L(t)/\ln t - 1| > \varepsilon).$$

Therefore,

$$\begin{aligned}&\limsup_{t\to\infty}(\ln t)^{-1}\ln \mathbb{E}\exp\{yL(t)\}\\ &\qquad = \max\Big\{\limsup_{t\to\infty}(\ln t)^{-1}\ln \mathbb{E}(\exp\{yL(t)\}, |L(t)/\ln t - 1| \leq \varepsilon),\\ &\qquad\qquad \limsup_{t\to\infty}(\ln t)^{-1}\ln \mathbb{E}(\exp\{yL(t)\}, |L(t)/\ln t - 1| > \varepsilon)\Big\}.\end{aligned} \tag{2.2}$$

As in the proof of Lemma 2.1, the first limit can be estimated by

$$\limsup_{t\to\infty}(\ln t)^{-1}\ln\mathbb{E}(\exp\{yL(t)\},|L(t)/\ln t-1|\le\varepsilon)\le\begin{cases}y(1+\varepsilon) & \text{if } y>0,\\ y(1-\varepsilon) & \text{if } y<0.\end{cases}\tag{2.3}$$

The second limit is more complicated and the assumption $y<1$ is needed. We rewrite

$$\begin{aligned}&\limsup_{t\to\infty}(\ln t)^{-1}\ln\mathbb{E}(\exp\{yL(t)\},|L(t)/\ln t-1|>\varepsilon)\\&\qquad=\limsup_{t\to\infty}(\ln t)^{-1}\ln\mathbb{E}(\exp\{yL(t)\},\{L(t)/\ln t-1>\varepsilon\}\cup\{L(t)/\ln t-1<-\varepsilon\}).\end{aligned}$$

For the first part with $\{L(t)/\ln t-1>\varepsilon\}$, if $y<0$ then an estimate similar to that for the first limit can be made. But if $y>0$ then we need to make the separation

$$\begin{aligned}&\limsup_{t\to\infty}(\ln t)^{-1}\ln\mathbb{E}(\exp\{yL(t)\},\{L(t)/\ln t-1>\varepsilon\})\\&\qquad=\limsup_{t\to\infty}(\ln t)^{-1}\ln\mathbb{E}\Big(\exp\{yL(t)\},\bigcup_{k=1}^{\infty}\{1+k\varepsilon<L(t)/\ln t\le 1+(k+1)\varepsilon\}\Big)\\&\qquad\le\limsup_{t\to\infty}(\ln t)^{-1}\ln\Big(\sum_{k=1}^{\infty}e^{y[1+(1+k)\varepsilon]\ln t}\cdot\mathbb{P}(1+k\varepsilon<L(t)/\ln t)\Big)\\&\qquad=y(1+\varepsilon)+\limsup_{t\to\infty}(\ln t)^{-1}\ln\Big(\sum_{k=1}^{\infty}e^{yk\varepsilon\ln t}\cdot\mathbb{P}(1+k\varepsilon<L(t)/\ln t)\Big).\end{aligned}$$

In order to analyse further, we use the global estimation in Lemma 2.2. From the mean value theorem in the form $a^x=a^0+a^\theta\cdot\ln(a)\cdot x$ with $a>0$, $x>0$ and $\theta\in[0,x]$, we can derive the estimate

$$\begin{aligned}\mathbb{P}(1+k\varepsilon<L(t)/\ln t)&=1-\mathbb{P}(L(t)/\ln t\le 1+k\varepsilon)\\&\le 1-[(1-ae^{-(1+k\varepsilon)\ln t})^{1/(ae^{-(1+k\varepsilon)\ln t})}]^{ae^{-(1+k\varepsilon)\ln t}(t-(1+k\varepsilon)\ln t)}\\&=-[(1-ae^{-(1+k\varepsilon)\ln t})^{1/(ae^{(1+k\varepsilon)\ln t})}]^{\theta_t}\\&\qquad\cdot\ln((1-ae^{-(1+k\varepsilon)\ln t})^{1/(ae^{-(1+k\varepsilon)\ln t})})\cdot at^{-(1+k\varepsilon)}\cdot(t-(1+k\varepsilon)\ln t)\\&\le at^{-k\varepsilon}\end{aligned}$$

where $\theta_t\in[0,\ at^{-(1+k\varepsilon)}(t-(1+k\varepsilon)\ln t)]$. If $y<1$, then we put this estimate back into the previous estimate and get

$$\begin{aligned}&\limsup_{t\to\infty}(\ln t)^{-1}\ln\mathbb{E}(\exp\{yL(t)\},\{L(t)/\ln t-1>\varepsilon\})\\&\qquad=y(1+\varepsilon)+\limsup_{t\to\infty}(\ln t)^{-1}\ln\Big(\sum_{k=1}^{\infty}e^{yk\varepsilon\ln t}\cdot\mathbb{P}(1+k\varepsilon<L(t)/\ln t)\Big)\\&\qquad\le y(1+\varepsilon)+\limsup_{t\to\infty}(\ln t)^{-1}\ln\Big(\sum_{k=1}^{\infty}e^{yk\varepsilon\ln t}\cdot at^{-k\varepsilon}\Big)\end{aligned}$$

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$$= y(1+\varepsilon) + \limsup_{t\to\infty} (\ln t)^{-1} \ln\bigg(\sum_{k=1}^{\infty} a t^{-(1-y)k\varepsilon}\bigg)$$
$$= y(1+\varepsilon)$$

where the last step follows from the fact that $y < 1$. We have proved that

$$\limsup_{t\to\infty} (\ln t)^{-1} \ln \mathbb{E}(\exp\{yL(t)\}, \{L(t)/\ln t - 1 > \varepsilon\}) \le y(1+\varepsilon). \tag{2.4}$$

For the second part with $\{L(t)/\ln t - 1 < -\varepsilon\}$, the case $y > 0$ can be handled similarly. For the case $y < 0$, we make a similar separation as in the proof of (2.4) and argue as follows:

$$\limsup_{t\to\infty} (\ln t)^{-1} \ln \mathbb{E}(\exp\{yL(t)\}, \{L(t)/\ln t - 1 < -\varepsilon\})$$
$$= \limsup_{t\to\infty} (\ln t)^{-1} \ln \mathbb{E}\bigg(\exp\{yL(t)\}, \bigcup_{k=1}^{[1/\varepsilon]-1} \{1-(k+1)\varepsilon < L(t)/\ln t \le 1-k\varepsilon\}\bigg)$$
$$\le \limsup_{t\to\infty} (\ln t)^{-1} \ln\bigg(\sum_{k=1}^{[1/\varepsilon]-1} e^{y[1-(k+1)\varepsilon]\ln t} \cdot \mathbb{P}(1-(k+1)\varepsilon < L(t)/\ln t \le 1-k\varepsilon)\bigg).$$

Since there are only finitely many terms in the summation, we have the simple estimate

$$\le \max_{1\le k\le[1/\varepsilon]-1} \Big\{y[1-(k+1)\varepsilon] + \limsup_{t\to\infty} (\ln t)^{-1} \ln \mathbb{P}(L(t)/\ln t < 1-k\varepsilon)\Big\}$$
$$= \max_{1\le k\le[1/\varepsilon]-1} \{y[1-(k+1)\varepsilon] - \infty\}$$
$$= -\infty,$$

where $-\infty$ appears because of Lemma 2.7(iii) below. Therefore,

$$\limsup_{t\to\infty} (\ln t)^{-1} \ln \mathbb{E}(\exp\{yL(t)\}, \{L(t)/\ln t - 1 < -\varepsilon\}) = -\infty. \tag{2.5}$$

The proof is completed by using the estimates (2.3), (2.4) and (2.5) in (2.2). □

LEMMA 2.4. *For* $y = 1$,

$$\lim_{t\to\infty} (\ln t)^{-1} \ln \mathbb{E} \exp\{yL(t)\} = 2.$$

PROOF. We use the estimates established in Lemma 2.7(i) below:

$$c_2 t^{-(1+k\varepsilon)}(t-(1+k\varepsilon)\ln t) \le \mathbb{P}(1+k\varepsilon < L(t)/\ln t) \le c_1 t^{-(1+k\varepsilon)}(t-(1+k\varepsilon)\ln t),$$

for $k = 1, \ldots, [(t/\ln t - 1)/\varepsilon]$. On the one hand, for every $\varepsilon > 0$,

$$(\ln t)^{-1} \ln \mathbb{E} \exp\{yL(t)\} \ge (\ln t)^{-1} \ln \mathbb{E} \exp\{y \cdot L(t), \{L(t)/\ln t > 1+\varepsilon\}\}$$
$$= (\ln t)^{-1} \ln \mathbb{E} \exp\Big\{yL(t), \bigcup_{k=1}^{[(t/\ln t-1)/\varepsilon]} \{1+k\varepsilon < L(t)/\ln t \le 1+(k+1)\varepsilon\}\Big\}$$

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$$\geq (\ln t)^{-1} \ln \sum_{k=1}^{[(t/\ln t-1)/\varepsilon]} \exp\{(1+k\varepsilon)\ln t\}$$
$$\cdot (\mathbb{P}\{L(t)/\ln t > 1+k\varepsilon\} - \mathbb{P}\{L(t)/\ln t > 1+(k+1)\varepsilon\})$$
$$\geq 1 + (\ln t)^{-1} \ln \sum_{k=1}^{[(t/\ln t-1)/\varepsilon]} t^{k\varepsilon}$$
$$\cdot (c_2 \cdot t^{-(1+k\varepsilon)}(t-(1+k\varepsilon)\ln t) - c_1 \cdot t^{-(1+(k+1)\varepsilon)}(t-(1+(k+1)\varepsilon)\ln t))$$
$$= 1 + (\ln t)^{-1} \ln \sum_{k=1}^{[(t/\ln t-1)/\varepsilon]} \Big(\frac{c_2}{t}(t-(1+k\varepsilon)\ln t) - \frac{c_1}{t^{1+\varepsilon}}(t-(1+(k+1)\varepsilon)\ln t)\Big)$$
$$\sim 1 + (\ln t)^{-1} \ln \Big[\frac{c_2}{t} \cdot \frac{t^2}{2\varepsilon \ln t} - \frac{c_1}{t^{1+\varepsilon}} \cdot \frac{t^2}{2\varepsilon \ln t}\Big]$$
$$\sim 1 + (\ln t)^{-1} \ln \Big[\frac{c_2}{t} \cdot \frac{t^2}{2\varepsilon \ln t}\Big] \sim 1 + 1 = 2.$$

On the other hand,

$$\limsup_{t\to\infty} (\ln t)^{-1} \ln \mathbb{E} \exp\{yL(t)\}$$
$$= \max\Big\{\limsup_{t\to\infty} (\ln t)^{-1} \ln \mathbb{E} \exp\{yL(t), \{L(t)/\ln t \leq 1+\varepsilon\}\},$$
$$\limsup_{t\to\infty} (\ln t)^{-1} \ln \mathbb{E} \exp\{yL(t), \{L(t)/\ln t > 1+\varepsilon\}\}\Big\}.$$

For the first limit,

$$\limsup_{t\to\infty} (\ln t)^{-1} \ln \mathbb{E} \exp\{yL(t), \{L(t)/\ln t \leq 1+\varepsilon\}\}$$
$$\leq \limsup_{t\to\infty} (\ln t)^{-1} \ln \exp\{(1+\varepsilon)\ln t\} \mathbb{P}\{L(t)/\ln t \leq 1+\varepsilon\}$$
$$= (1+\varepsilon).$$

For the second limit,

$$(\ln t)^{-1} \ln \mathbb{E} \exp\{yL(t), \{L(t)/\ln t > 1+\varepsilon\}\}$$
$$= (\ln t)^{-1} \ln \mathbb{E} \exp\Big\{yL(t), \bigcup_{k=1}^{[(t/\ln t-1)/\varepsilon]} \{1+k\varepsilon < L(t)/\ln t \leq 1+(k+1)\varepsilon\}\Big\}$$
$$\leq (\ln t)^{-1} \ln \sum_{k=1}^{[(t/\ln t-1)/\varepsilon]} \exp\{(1+(k+1)\varepsilon)\ln t\} \cdot \mathbb{P}\{L(t)/\ln t > 1+k\varepsilon\}$$
$$\leq (1+\varepsilon) + (\ln t)^{-1} \ln \sum_{k=1}^{[(t/\ln t-1)/\varepsilon]} t^{k\varepsilon} \cdot c_1 \cdot t^{-(1+k\varepsilon)}(t-(1+k\varepsilon)\ln t)$$
$$\sim (1+\varepsilon) + (\ln t)^{-1} \ln \frac{c_1}{t} \cdot \frac{t^2}{\varepsilon \ln t} \sim (1+\varepsilon) + 1.$$

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Therefore, when $y = 1$,

$$\limsup_{t\to\infty} (\ln t)^{-1} \ln \mathbb{E} \exp\{yL(t)\} \le (1+\varepsilon) + 1,$$

which completes the proof by sending $\varepsilon \to 0$. □

In order to study the asymptotic behaviour of $\mathbb{E}\exp\{yL(t)\}$ when $y > 1$, we need to study a large-deviation probability which may be of independent interest.

Lemma 2.5. *For a fixed x with $0 < x < 1$,*

$$\lim_{t\to\infty} t^{-1} \ln \mathbb{P}(L(t)/t \ge x) = -x.$$

Proof. We apply the global estimation in Lemma 2.2 for $\ell = tx$ and obtain

$$1 - (1 - ae^{-tx})^{t-tx} \le \mathbb{P}(L(t)/t \ge x) \le 1 - (1 - e^{-tx})^{t-tx}.$$

The lower bound can be treated as

$$\begin{aligned} 1 - (1 - ae^{-tx})^{t-tx} &= 1 - [(1 - ae^{-tx})^{1/(ae^{-tx})}]^{ae^{-tx}(t-tx)} \\ &= -[(1 - ae^{-tx})^{1/(ae^{-tx})}]^{\theta_t} \cdot \ln((1 - ae^{-tx})^{1/(ae^{-tx})}) \cdot ae^{-tx}(t - tx) \end{aligned}$$

where $\theta_t \in (0, ae^{-tx}(t - tx))$. Therefore, for t large enough, the lower bound satisfies

$$1 - (1 - ae^{-tx})^{t-tx} \ge a(1 - \delta)e^{-tx}(t - tx)$$

for some small $\delta > 0$. Similar arguments on the upper bound give

$$a(1 - \delta)e^{-tx}(t - tx) \le \mathbb{P}(L(t)/t \ge x) \le (1 + \delta)e^{-tx}(t - tx). \tag{2.6}$$

The result follows directly from (2.6). □

Lemma 2.6. *For $y \ge 1$,*

$$\lim_{t\to\infty} t^{-1} \ln \mathbb{E} \exp\{yL(t)\} = y - 1.$$

Proof. On the one hand, from Lemma 2.5, for $0 < x < 1$,

$$\begin{aligned} \lim_{t\to\infty} t^{-1} \ln \mathbb{E} \exp\{yL(t)\} &\ge \lim_{t\to\infty} t^{-1} \ln \mathbb{E}[\exp\{yL(t)\}, L(t)/t > x] \\ &\ge yx + \lim_{t\to\infty} t^{-1} \ln \mathbb{P}(L(t)/t > x) \\ &= yx - x \to y - 1 \quad \text{as } x \to 1. \end{aligned}$$

On the other hand,

$$\begin{aligned} \lim_{t\to\infty} t^{-1} \ln \mathbb{E} \exp\{yL(t)\} &= \lim_{t\to\infty} t^{-1} \ln \mathbb{E}(\exp\{yL(t)\}, \{L(t)/t \le \varepsilon\} \cup \{L(t)/t > \varepsilon\}) \\ &= \max\Big\{\lim_{t\to\infty} t^{-1} \ln \mathbb{E}(\exp\{yL(t)\}, \{L(t)/t \le \varepsilon\}), \\ &\qquad \lim_{t\to\infty} t^{-1} \ln \mathbb{E}(\exp\{yL(t)\}, \{L(t)/t > \varepsilon\})\Big\}. \end{aligned}$$

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The first limit is

$$\lim_{t\to\infty} t^{-1} \ln \mathbb{E}(\exp\{yL(t)\}, \{L(t)/t \le \varepsilon\}) \le y\varepsilon.$$

The second limit can be handled as follows:

$$\begin{aligned}
&\lim_{t\to\infty} t^{-1} \ln \mathbb{E}(\exp\{yL(t)\}, \{L(t)/t > \varepsilon\}) \\
&\quad = \lim_{t\to\infty} \frac{1}{n} \ln \mathbb{E}\Big(\exp\{yL(t)\}, \bigcup_{k=1}^{[1/\varepsilon]-1} \{k\varepsilon < L(t)/t \le (k+1)\varepsilon\}\Big) \\
&\quad \le \max_{1\le k\le[1/\varepsilon]-1} \Big\{y(k+1)\varepsilon + \lim_{t\to\infty} \frac{1}{n} \ln \mathbb{P}(k\varepsilon < L(t)/t)\Big\} \\
&\quad = \max_{1\le k\le[1/\varepsilon]-1} \{y(k+1)\varepsilon - k\varepsilon\} \\
&\quad = \max_{1\le k\le[1/\varepsilon]-1} \{\varepsilon \cdot k \cdot (y-1) + \varepsilon y\} \\
&\quad \le (y-1) + \varepsilon y,
\end{aligned}$$

where the last inequality comes from the fact that $y \ge 1$. The proof now follows by taking $\varepsilon \to 0^+$. □

LEMMA 2.7.

(i) *For any small* $\varepsilon > 0$ *and* $k = 1, \ldots, [(t/\ln t - 1)/\varepsilon]$, *for large* t,

$$\begin{aligned}
c_2 \cdot t^{-(1+k\varepsilon)}(t - (1+k\varepsilon)\ln t) &\le \mathbb{P}(1 + k\varepsilon < L(t)/\ln t) \\
&\le c_1 \cdot t^{-(1+k\varepsilon)}(t - (1+k\varepsilon)\ln t),
\end{aligned} \tag{2.7}$$

where $c_i > 0, i = 1, 2$, *are two constants.*

(ii) *For each* $x > 0$,

$$\lim_{t\to\infty} (\ln t)^{-1} \ln \mathbb{P}(L(t)/\ln t \ge 1 + x) = -x.$$

(iii) *For* $0 < x < 1$,

$$\lim_{t\to\infty} (\ln t)^{-1} \ln[-\ln \mathbb{P}(L(t)/\ln t \le 1 - x)] = x.$$

PROOF. From (2.1) and the arguments between (2.3) and (2.4),

$$\begin{aligned}
\mathbb{P}(1 + k\varepsilon < L(t)/\ln t) &= 1 - \mathbb{P}(L(t) \le (1+k\varepsilon)\ln t) \\
&\le 1 - (1 - a \cdot e^{-(1+k\varepsilon)\ln t})^{t-(1+k\varepsilon)\ln t} \\
&\le c_1 \cdot t^{-(1+k\varepsilon)}(t - (1+k\varepsilon)\ln t)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}(1 + k\varepsilon < L(t)/\ln t) &= 1 - \mathbb{P}(L(t) \le (1+k\varepsilon)\ln t) \\
&\ge 1 - (1 - e^{-(1+k\varepsilon)\ln t})^{t-(1+k\varepsilon)\ln t} \\
&\ge c_2 \cdot t^{-(1+k\varepsilon)}(t - (1+k\varepsilon)\ln t),
\end{aligned}$$

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which proves (i). To show (ii), one just needs to think of $1 + k\varepsilon$ as $1 + x$ in (2.7) and (ii) follows directly from (i). To prove (iii), the global estimation (2.1) implies that

$$(1 - ae^{-(1-x)\ln t})^{t-(1-x)\ln t} \le \mathbb{P}(L(t) \le (1-x)\ln t) \le (1 - e^{-(1-x)\ln t})^{t-(1-x)\ln t}.$$

Taking logarithms on all sides of the above inequalities and applying the asymptotics $\ln(1-\alpha) = -(1+o(1))\alpha$ as $\alpha \to 0$, the inequalities become, for large t,

$$-at^{-(1-x)}(t - (1-x)\ln t) \le \ln \mathbb{P}(L(t) \le (1-x)\ln t) \le -t^{-(1-x)}(t - (1-x)\ln t).$$

Taking logarithms again completes the proof of (iii). □

3. Proofs of Corollaries 1.2 and 1.3

Here we only present the detailed proof of Corollary 1.2. The proof Corollary 1.3 is almost identical.

The large-deviation upper bound (1.3) follows directly from the Gärtner–Ellis theorem [5, Section 2.3]. For the large-deviation lower bound (1.2), it suffices to prove that for a fixed point $y > 1$,

$$\lim_{\delta\to 0}\liminf_{t\to\infty}(\ln t)^{-1}\ln \mathbb{P}(L(t)/\ln t \in B_{y,\delta}) \ge -(y-1), \tag{3.1}$$

where $B_{y,\delta}$ is the open ball centred at y with radius δ. To achieve (3.1), we write

$$\mathbb{P}(L(t)/\ln t \in B_{y,\delta}) = \mathbb{P}(L(t)/\ln t > y - \delta) - \mathbb{P}(L(t)/\ln t \ge y + \delta).$$

To analyse the logarithm, we apply the inequality $\ln(a-b) \ge \ln(a) - b/(a-b)$ for $a > b > 0$. Therefore,

$$\begin{aligned}
&\lim_{\delta\to 0}\liminf_{t\to\infty}(\ln t)^{-1}\ln \mathbb{P}(L(t)/\ln t \in B_{y,\delta})\\
&\quad\ge \lim_{\delta\to 0}\liminf_{t\to\infty}(\ln t)^{-1}\bigg(\ln[\mathbb{P}(L(t)/\ln t > y-\delta)]\\
&\qquad - \frac{\mathbb{P}(L(t)/\ln t \ge y+\delta)}{\mathbb{P}(L(t)/\ln t > y-\delta) - \mathbb{P}(L(t)/\ln t \ge y+\delta)}\bigg).
\end{aligned} \tag{3.2}$$

We can apply Lemma 2.7(ii) to handle the first limit as follows:

$$\lim_{\delta\to 0}\liminf_{t\to\infty}(\ln t)^{-1}\ln[\mathbb{P}(L(t)/\ln t > y-\delta)] = \lim_{\delta\to 0} -(y-1-\delta) = -(y-1). \tag{3.3}$$

For the second ratio term, it follows from applying Lemma 2.7(ii) twice that

$$\mathbb{P}(L(t)/\ln t \ge y+\delta) \le \exp\{[-(y-1+\delta) + \varepsilon_1]\ln t\}$$

and

$$\mathbb{P}(L(t)/\ln t > y-\delta) \ge \exp\{[-(y-1-\delta) - \varepsilon_2]\ln t\}$$

for sufficiently small $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. Thus, assuming $2\delta - \varepsilon_1 - \varepsilon_2 > 0$,

$$\begin{aligned}
&\frac{\mathbb{P}(L(t)/\ln t \ge y+\delta)}{\mathbb{P}(L(t)/\ln t > y-\delta) - \mathbb{P}(L(t)/\ln t \ge y+\delta)}\\
&\quad= \frac{1}{\mathbb{P}(L(t)/\ln t > y-\delta)/\mathbb{P}(L(t)/\ln t \ge y+\delta) - 1}\\
&\quad\le \frac{1}{e^{(2\delta-\varepsilon_1-\varepsilon_2)\ln t} - 1} \to 0 \quad \text{as } t \to \infty.
\end{aligned} \tag{3.4}$$

Then (3.1) follows by taking (3.3) and (3.4) back into (3.2).

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