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WEIGHTED DIAMETERS OF COMPLETE SETS OF CONJUGATE ALGEBRAIC INTEGERS

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In this work, we generalise the study of Favard's Problems to the weighted diameters of a complete set of conjugate algebraic integers, that is, the roots of an irreducible monic polynomial with coefficients in $\mathbb{Z}$.

1. INTRODUCTION

We define the weighted diameter of a complete set $X_d = \{\alpha_i, 1 \leqslant i \leqslant d\}$ of conjugate algebraic integers of degree d with $d \geqslant n$ by:

$$t_n(X_d) = \sup_{(\alpha_i)\in X_d^n} \left(\prod_{i\neq j} |\alpha_i - \alpha_j|\right)^{1/n(n-1)}. \tag{1.1}$$

We denote by G the set of all complete sets of conjugate algebraic integers. We denote by G_d the subset of sets of conjugates of degree greater or equal to d, and by G'_d the subset of conjugates whose degree is equal to d.

The aim of this paper is to extend the study of minimal diameters to weighted diameters of complete sets of conjugate algebraic integers. That is to say, we generalise Favard's problems. An interesting application of these computations is to find the smaller discriminants of degree d because $t_d(X_d)^{d(d-1)}$ is the discriminant of the polynomial of degree d whose roots are in the set X_d.

We recall both Favard's Problems solved by Langevin, Reyssat and Rhin [**1**]:

$$(1)\quad \inf_{X\in G} t_2(X) \geqslant \sqrt{3} \qquad (2)\quad \lim_{d\to\infty} \inf_{X\in G_d} t_2(X) = 2.$$

The other results we know about t_2 are the following: Favard [**2**] has computed $\inf_{X\in G'_2} t_2(X)$, $\inf_{X\in G'_3} t_2(X)$, and Lloyd-Smith [**3**] $\inf_{X\in G'_4} t_2(X)$, $\inf_{X\in G'_5} t_2(X)$. We have computed $\inf_{X\in G'_d} t_2(X)$ for $d \in \{6,7,8,9\}$ [**4**]. In [**3**], Lloyd-Smith also gives the following result:

$$\forall d \in \mathbb{N} \quad t_3(X_d) \geqslant 3^{1/6} \tag{1.2}$$

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but he does not compute $t_3(X_d)$ for small values of d. This paper does it and suggests that (1.2) can be improved.

In Section 2, we give $\inf_{X\in G'_d} t_n(X)$ for small values of d and n. The values of $\inf_{X\in G'_d} t_2(X)$ enabled us to solve the first Favard problem (1). Hence it may also be possible to find $\inf_{X\in G_3} t_3(X)$ and $\inf_{X\in G_4} t_4(X)$. In Section 3 we give a few properties of weighted diameters. We explain now to obtain minimal weighted diameters in Section 4 and give tables of bounds for polynomial coefficients in Section 8 and proofs of bounds for Favard's generalised problems in Section 5 and 6 . In Section 7 we give a lower bound of the minimal distance between the roots X_d of a polynomial as a function of their diameter, that is, $t_2(X_d)$.

2. Numerical results

MINIMAL WEIGHTED DIAMETERS FOR SMALL VALUES OF d AND n.

We obtain the smallest discriminants when n is equal to d.

(1) The monic irreducible polynomials with $t_3(X_d)$ minimal are the following:

$d=3: x^3-x^2+1$, diam $=1.79423416\ldots$, $t_3=1.686376\ldots$, disc $=-23$

$d=3: x^3-x-1$, diam $=2.06509879\ldots$, $t_3=1.686376\ldots$, disc $=-23$

$d=3: x^3-x^2+2x-1$, diam $=2.61428256\ldots$, $t_3=1.686376\ldots$, disc $=-23$

$d=4: x^4-2x^3+2x^2-x+1$, diam $=1.89882892\ldots$, $t_3=1.487129\ldots$

$d=5: x^5-2x^4+2x^3-x^2+1$, diam $=2.06585026\ldots$, $t_3=1.623705\ldots$

$d=6: x^6-2x^5+3x^4-3x^3+2x^2-x+1$, diam $=2.37468455\ldots$, $t_3=1.658468\ldots$

$d=7: x^7-2x^6+2x^5-x^4-1$, diam $=1.99830811\ldots$, $t_3=1.697895\ldots$.

(2) The monic irreducible polynomials with $t_4(X_d)$ minimal are the following:

$d=4: x^4-2x^3+2x^2-x+1$, diam $=1.898828\ldots$, $t_4=1.487129\ldots$, disc $=117$

$d=5: x^5-x^4-x^3+x^2-1$, diam $=2.297221\ldots$, $t_4=1.473965\ldots$

$d=6: x^6-x^3+1$, diam $=1.969615\ldots$, $t_4=1.473478\ldots$.

(3) The monic irreducible polynomial with $t_5(X_5)$ and $t_2(X_5)$ minimal is the following:

$d=5: x^5-x^3+x^2+x-1$, diam $=2.127431\ldots$, $t_5=1.446531\ldots$, disc $=1609$.

(4) The monic irreducible polynomial with $t_6(X_6)$ and $t_2(X_6)$ minimal is the following:

$d=6: x^6-3x^5+4x^4-4x^3+4x^2-2x+1$, diam $=2.19017506\ldots$, $t_6=1.358195\ldots$, disc $=-9747$.

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where diam denotes the diameter of the set of roots of the and disc their discriminant.

REMARKS.

(1) For $t_5(X_5)$ minimal, we have several sets of conjugates with different diameters giving this minimum, for example:

$$x^5 - 2x^4 + 3x^3 - 3x^2 + 3x - 1,\ \text{diam} = 2.269315\ldots,\ t_5 = 1.446531\ldots$$
$$x^5 - x^4 - x^3 + x^2 - 1,\ \text{diam} = 2.297221\ldots,\ t_5 = 1.446531\ldots$$
$$x^5 - 2x^4 + 2x^3 + x^2 - 2x + 1,\ \text{diam} = 2.3413370\ldots,\ t_5 = 1.446531\ldots$$
$$x^5 - 2x^4 + x^3 + 2x^2 - 2x + 1,\ \text{diam} = 2.373647\ldots,\ t_5 = 1.446531\ldots$$
$$x^5 - 2x^4 + 3x^3 - 3x^2 + x - 1,\ \text{diam} = 2.527832\ldots,\ t_5 = 1.446531\ldots$$
$$x^5 - 3x^3 + 2x - 1,\ \text{diam} = 2.873670\ldots,\ t_5 = 1.446531\ldots.$$

(2) For $t_6(X_6)$ minimal, we have several sets of conjugates with different diameters giving this minimum, for example:

$$x^6 - x^5 + x^4 - 2x^3 + 4x^2 - 3x + 1,\ \text{diam} = 2.470480\ldots,\ t_6 = 1.358195\ldots$$
$$x^6 + x^4 + x^3 - 2x^2 - x + 1,\ \text{diam} = 2.558790\ldots,\ t_6 = 1.358195\ldots$$
$$x^6 - 2x^5 + 4x^4 - 4x^3 + 4x^2 - 3x + 1,\ \text{diam} = 2.577222\ldots,\ t_6 = 1.358195\ldots$$
$$x^6 + x^4 - x^3 - 2x^2 + x + 1,\ \text{diam} = 2.998363\ldots,\ t_6 = 1.358195\ldots.$$

(3) The polynomials with $t_2(X_d)$, $2 \leqslant d \leqslant 10$ minimal are given in [**4**].

(4) The minimum of $t_3(X_d)$ seems to be obtained for $d = 4$ and is equal to $1.487129\ldots$.

(5) For greater values of d and n, the search can not be completely exhaustive, but it is a way to find small discriminants.

(6) These results give an idea of how rapidly $\inf\limits_{X \in G'_d} t_n(X)$, $d \geqslant n$ decreases as a function of n. In fact, we know that $t_n(X)$ decreases as a function of n (this is proved in Section 3).

(7) To obtain lower bounds for $\inf t_3(X_d)$ we can use the table of lower bounds of discriminants obtained by Diaz [**5**]. These lower bounds complete the results of Section 2.1, they are better than (2.1) but they are not interesting if $d > 11$, indeed:

$$t_d(X_d) = D_d^{1/d(d-1)} \leqslant t_3(X_d) \tag{2.1}$$

where D_d is the discriminant of the polynomial P whose roots are in X_d. Hence:

$$\inf t_3(X_8) > 1.280969$$
$$\inf t_3(X_9) > 1.265460$$
$$\inf t_3(X_{10}) > 1.233284.$$

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If X_d is a totally real set, we have:

$$\inf t_3(X_8) > 1.401059$$
$$\inf t_3(X_9) > 1.361749$$
$$\inf t_3(X_{10}) > 1.329588.$$

(8) The minimal discriminants are found in [**6**].

3. Properties of weighted diameters

Proposition 1. Basic Lemma. *If X is a bounded convex set of the complex plane, then*

$$t_n(X) \leqslant n^{2/(n^2-1)} t_{n+1}(X). \tag{3.1}$$

Proof: First of all, $\operatorname{disc}(F_n(z)(z - z_{n+1})) = F_n^2(z_{n+1}) \operatorname{disc}(F_n(z))$, so

$$t_n(X)^{n-1} f_n^2(X) \leqslant t_{n+1}^{n+1}(X) \tag{3.1.a}$$

with $f_n(X) = \sup_X \left|F_n(z)^{1/n}\right|$ where z_{n+1} is selected in such a way that

$$|F_n(z_{n+1})| = \sup_X |F_n(z)|.$$

Furthermore: $\operatorname{disc}(F_n(z)) = \operatorname{Res}\left(F_n, F_n'\right) = n^n \prod_{F_n'(z)=0} F_n(z)$ and hence

$$t_n^2(X) \leqslant n^{2/(n-1)} f_n^2(X). \tag{3.1.b}$$

By elimination of $f_n(X)$ between (3.1.a), (3.1.b), we obtain the inequality (3.1). ∎

Corollary.

$$t_2(X) \leqslant 2^{2/3} 3^{1/4} \ldots (n-1)^{2/[(n-1)^2-1]} t_n(X). \tag{3.2}$$

In particular:

$$t_2(X) \leqslant 2^{2/3} t_3(X).$$

Proof: This is evident by induction. ∎

Proposition 2. *Let X be a compact subset of $\mathbb{C}$ and let $z_1, z_2, \ldots, z_n$ be in X. We define*

$$V(z_1, z_2, \ldots, z_n) = \prod_{1 \leqslant i < j \leqslant n} |z_i - z_j|$$
$$V_n = V_n(X) = \sup_{(z_1, z_2, \ldots, z_n) \in X} V(z_1, z_2, \ldots, z_n)$$
$$t_n = t_n(X) = V_n^{2/n(n-1)}.$$

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Then $(t_n)_{n\in\mathbb{N}}$ is decreasing to a real number $t(X)$ called the transfinite diameter of X.

PROOF: Let $w_1, w_2, \ldots, w_{n+1}$ be the points of X which give the maximum of $V_{n+1}(X)$, let $V(w_1, w_2, \ldots, w_{n+1}) = \left(\prod_{1\leqslant j\leqslant n+1} |w_i - w_j|\right) \cdot V(w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_{n+1})$ with $j \neq i$. Hence $V_{n+1} \leqslant \left(\prod_{1\leqslant j\leqslant n+1} |w_i - w_j|\right) \cdot V_n$ with $j \neq i$. In particular:

$$V_{n+1} \leqslant \left(\prod_{1\leqslant j\leqslant n+1} |w_1 - w_j|\right) . V_n \text{ with } j \neq 1,$$

$$V_{n+1} \leqslant \left(\prod_{1\leqslant j\leqslant n+1} |w_2 - w_j|\right) . V_n \text{ with } j \neq 2,$$

$$\vdots$$

$$V_{n+1} \leqslant \left(\prod_{1\leqslant j\leqslant n+1} |w_{n+1} - w_j|\right) . V_n \text{ with } j \neq n+1.$$

By multiplying together these inequalities, we obtain

$$V_{n+1}^{n+1} \leqslant V_{n+1}^2 . V_n^{n+1}.$$

On dividing by V_{n+1}^2, we find that

$$V_{n+1}^{n-1} \leqslant V_n^{n+1}.$$

Since $t_{n+1} = V_{n+1}^{2/n(n+1)} = \left(V_{n+1}^{n-1}\right)^{2/(n+1)n(n-1)}$ we obtain the inequality $t_{n+1} \leqslant \left(V_n^{n+1}\right)^{2/(n+1)n(n-1)}$ and $t_{n+1} \leqslant t_n$. □

COROLLARY. $\inf_{X\in G'_d} t_n(X)$ *is decreasing as a sequence in* n.

PROPOSITION 3. *Let D be the unit disk. Then*

$$t_n(D) = n^{1/(n-1)}.$$

PROOF: See [7]. □

CONJECTURE.

$$\lim_{d\to\infty} \inf_{X\in G_d} t_n(X) = n^{1/(n-1)} \tag{3.3}$$

REMARK. This result is true for n=2. The cyclotomic polynomials must have the smallest weighted diameters for big values of d.

4. Tools giving minimal weighted diameters

We want to find the sets of conjugate algebraic integers X_d such that $t_n(X_d) < C_{d;n}$ where $C_{d;n}$ is fixed. We choose for example $C_{d;2} = 2$, $C_{d;3} = 1.69$. If we don't find any polynomial, we choose a greater value.

Proposition. *If $P(z) = z^d + b_1 z^{d-1} + \ldots + b_k z^{d-k} + \ldots + b_d$ is a polynomial whose roots satisfy: $t_n(X_d) < C_{d;n}$ then the coefficients B_i of the polynomial Q, obtained by a translation of the barycentre of the roots of P at the origin, satisfy*

$$|B_i| \leqslant \inf_{r \in]0;+\infty[} \frac{\left(r^2 + \delta^2/3\right)^{d/2}}{r^{d-i}} \tag{4.1}$$

with $2 \leqslant i \leqslant d$, $\delta = 2^{2/3} 3^{1/4} \ldots (d-1)^{2/[(d-1)^2-1]} C_{d;n}$.

Remark. $B_1 = 0$

Proof: The bound $C_{d;n}$ of $t_n(X_d)$ gives a bound for $t_2(X_d)$ thanks to formula (3.2) and the bounds given for the coefficients as a function of the diameter are proved in [**4**]. Furthermore [**4**] gives also good bounds for B_2, B_3, B_4 thanks to Newton's formulae, since we suppose the barycentre of the roots to be at the origin. Tables of B_2, B_3, B_4 are given in Section 8. $\square$

Algorithm. We fix $C_{d;n}$ and we find all polynomials with $t_n(X_d) < C_{d;n}$ thanks to the algorithm given in [**4**]. We use formulae which 'translate' the barycentre at the origin:

$$\begin{aligned}
B_1 &= 0 \\
B_2 &= \binom{d}{d-2}(-b_1/d)^2 + b_1\binom{d-1}{d-2}(-b_1/d) + b_2 \\
&\ \vdots \\
B_k &= \binom{d}{d-k}(-b_1/d)^k + b_1\binom{d-1}{d-k}(-b_1/d)^{k-1} + \ldots + b_j\binom{d-j}{d-k} \\
&\qquad (-b_1/d)^{k-j} + \ldots + b_k \\
&\ \vdots \\
B_d &= (-b_1/d)^d + b_1(-b_1/d)^{d-1} + \ldots + b_j(-b_1/d)^{k-j} + \ldots + b_d.
\end{aligned}$$

We iterate on the polynomials obtained and we get, using the formulae again, polynomials with integer coefficients whose barycentre can be supposed to be in the segment [0,0.5] thanks to an integral translation or a symmetry with respect to the origin. The algorithm to compute the smallest weighted diameters is recursive and similar to the algorithm given in [**4**], following a method given by Robinson [**8**]. We need only a procedure which computes $t_n(X_d)$. Using the barycentre of the roots at the origin is efficient for obtaining small bounds and a simple formula for the bounds.

5. Asymptotic property of t_3

Lemma 1. *If A,B,C are three points of the unit circle, the product $AB \times AC \times BC$ is maximal if and only if $AB = AC = BC = \sqrt{3}$.*

Proof: If we fix one side, the product AB × AC × BC is maximal if (ABC) is an isosceles triangle, hence it is equilateral. ∎

Corollary.

$$\lim_{d\to\infty} \inf_{X \in G_d} t_3(X) \leqslant 3^{1/2}. \tag{5.1}$$

Proof: If P is a cyclotomic polynomial of degree at least 3, then

$$t_3(P) \leqslant 3^{1/2}$$

and there are cyclotomic polynomials for each degree $d \geqslant 3$. ∎

Lemma 2.

$$\lim_{d\to\infty} \inf_{X_d} t_3(X_d) \geqslant 2^{1/3}. \tag{5.2}$$

Proof: We use $\lim_{d\to\infty} \inf_{X \in G_d} t_2(X) = 2$ and the basic lemma of Section 3. ∎

Proposition. *There is a monic polynomial P in $\mathbb{Z}[X]$ with 3 roots arbitrarily close to:* $1, e^{2i\pi/3}, e^{4i\pi/3}$.

Proof: Let p be a prime and k an integer such that $|k/p - 1/3|$ and $|2\pi/p|$ are sufficiently small. Then by the continuity of $x \to e^{ix}$ we get $\left|e^{2i\pi/p} - 1\right| \leqslant \varepsilon$, $\left|e^{2ik\pi/p} - e^{2i\pi/3}\right| \leqslant \varepsilon$, $\left|e^{4ik\pi/p} - e^{4i\pi/3}\right| \leqslant \varepsilon$ with ε arbitrary. ∎

Corollary. $3^{1/2}$ *is the greatest limit point of the function* $\inf_{X \in G_3} t_3(X)$.

6. Asymptotic property of t_4

Theorem.

$$1 \leqslant \lim_{d\to\infty} \inf_{X \in G_d} t_4(X) \leqslant 2^{2/3}. \tag{6.1}$$

PROOF: Let S be the set of the roots of a cyclotomic polynomial and $S' = \{M_1, M_2, M_3, M_4\}$ the points with affix $\alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*$ such that:

$$t_4(S') = \Big(\prod_{1\leqslant i\neq j\leqslant 4} |\alpha_i^* - \alpha_j^*|\Big)^{1/12} = \sup_{(\alpha_i)\in S^4}\Big(\prod_{i\neq j}|\alpha_i - \alpha_j|\Big)^{1/12}, \quad \text{hence:}$$

$$t_4(S')^6 = M_1M_2 \times M_1M_3 \times M_1M_4 \times M_2M_3 \times M_2M_4 \times M_3M_4.$$

Since the points M_1, M_2, M_3, M_4 are on a circle, thanks to Ptolomy Theorem we get:

$$M_1M_3 \times M_2M_4 = M_1M_2 \times M_3M_4 + M_2M_3 \times M_1M_4.$$

Hence:

$$\begin{aligned}(M_1M_3 \times M_2M_4)^2 \times M_1M_3 \times M_2M_4 \\ &= (M_1M_2 \times M_3M_4)^2 \times M_1M_3 \times M_2M_4 \\ &\quad + (M_2M_3 \times M_1M_4)^2 \times M_1M_3 \times M_2M_4 \\ &\quad + 2 \times M_1M_2 \times M_1M_3 \times M_1M_4 \times M_2M_3 \times M_2M_4 \times M_3M_4.\end{aligned}$$

In such a way, we obtain, using $A^2 + B^2 \geqslant 2AB$,

$$t_2(S')^6 \geqslant 4t_4(S')^6$$

and so

$$t_4(S') \leqslant \frac{t_2(S')}{2^{1/3}}.$$

Hence we have $t_4(S) = t_4(S') \leqslant (t_2(S'))/2^{1/3}$ and we get $t_4(S) \leqslant 2^{2/3}$ which leads to

$$\lim_{d\to\infty} \inf_{X\in G_d} t_4(X) \leqslant 2^{2/3}.$$

The lower bound is due to the following Fekete-Szegö's theorem and the fact that $t_n(X)$ is a decreasing function of n, hence $t_n(X) \geqslant 1$. ∎

FEKETE-SZEGÖ'S THEOREM. *For any complete set X of conjugate algebraic integers*

(6.2) $$t(X) \geqslant 1.$$

PROOF: See [**9**]. ∎

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7. Distance between the roots of a polynomial

Proposition 1. *Let $P(z) = \sum_{i=0}^{d} a_i z^i$ be a monic polynomial with real coefficients. Then, for all complex numbers z,*

$$|P(z)|^{2/d} \leqslant \left(\frac{\delta^2}{3} + |z-g|^2\right) \tag{7.1}$$

where $g = -a_{d-1}/d$ is the barycentre of the roots of P and δ is the diameter $t_2(X_d)$ of the set X_d of the roots of P. By computing the mean value of $|P(z)|^2$ on a circle of radius 1 centred at the origin, we deduce from (7.1) and Parseval's identity that:

$$\sum_{i=0}^{d} |a_i|^2 \leqslant \int_0^1 \left(\frac{\delta^2}{3} + 1 - 2g\cos(2\pi t) + g^2\right)^d dt \tag{7.2}$$

with $g{=}-a_{d-1}/d$, hence:

$$\sum_{i=0}^{d} |a_i|^2 \leqslant \left(\frac{\delta^2}{3} + (1+g)^2\right)^d. \tag{7.3}$$

Proof: See [**4**]. ∎

Remark. the bound of (7.2) can be computed numerically with the PARI system. It is better than the bound of (7.3).

Proposition 2. *Let α_i and α_j be two zeros of P, then*

$$|\alpha_i - \alpha_j| \geqslant \sqrt{6}\,|\Delta|^{1/2}\, d^{-d/2}((d-1)(2d-1))^{-1/2} L(P)^{-(d-1)}$$

with $L(P) = \left(\sum_{i=0}^{d} a_i^2\right)^{1/2}$, where Δ is the discriminant of P.

Proof: See [**10**]. ∎

Corollary 1. *We obtain a lower bound for the minimal distance between the roots of a polynomial as a function of the diameter of these roots, that is, the maximal distance between these roots. For a monic irreducible polynomial with coefficients in $\mathbb{Z}$, we can suppose that g is in $[0, 0.5]$.*

$$|\alpha_i - \alpha_j| \geqslant \sqrt{6}\,|\Delta|^{1/2}\, d^{-d/2}((d-1)(2d-1))^{-1/2}\left(\frac{\delta^2}{3} + (1+g)^2\right)^{-d(d-1)}.$$

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Corollary 2. *We obtain a better minoration of the minimal distance between the roots for a monic irreducible polynomial P with coefficients in $\mathbb{Z}$ if P is totally real.*

$$|\alpha_i - \alpha_j| \geqslant \sqrt{6}\Big(e^{d/2}/((Log 2 + 1/2)(n+1)!)\Big)^{1/2}((d-1)(2d-1))^{-1/2} \left(\frac{\delta^2}{3} + (1+g)^2\right)^{-d(d-1)}.$$

Proof: The lower bound for the discriminant of a totally real polynomial is given by Rogers. See [**11**]. □

8. Bounds for the coefficients

(1) The bounds for B_5, B_6, B_7, B_8, B_9 are obtained with formula of [**4**]:

$$|B_{d-i}| \leqslant \inf_{r \in [0,\infty[} \frac{\left(r^2 + \delta^2/3\right)^{d/2}}{r^i},$$

since the barycentre is supposed to be at the origin. The derivative of

$$r \to \frac{\left(r^2 + \delta^2/3\right)^{d/2}}{r^i},\ 1 \leqslant i \leqslant d-1$$

is easy to compute.

The minimum of $r \to \left(r^2 + \delta^2/3\right)^{d/2}/r^i$ is obtained for $r = \left(\delta/\sqrt{3}\right) \times \sqrt{i/(d-i)}$. For $i = 0$ it is obtained with $r = 0$.

(2) Bounds for B_2 as functions of the degree and the diameter:

$d \setminus$ diam	2.7	3.14	2.13
3	-3.24,1.83	-4.39,2.47	$-2.02, 1.14$
4	-4.86,3.65	-6.58,4.93	$-3.03, 2.27$
5	-5.84,3.65	-7.89,4.93	$-3.63, 2.27$
6	-7.29,5.47	-9.86,7.4	$-4.54, 3.41$
7	-8.34,5.47	-11.27,7.4	$-5.19, 3.41$
8	-9.72,7.29	-13.15,9.86	$-6.05, 4.54$
9	-10.8,7.29	-14.61,9.86	$-6.73, 4.54$

(3) Bounds for $|B_3|$ as functions of the degree and the diameter:

$d \setminus$ diam	2.7	3.14	2.13
3	6.04	9.49	2.97
4	9.66	14.90	4.65
5	11.67	18.36	5.73
6	14.14	22.24	6.94
7	15.93	25.05	7.82
8	18.95	29.80	9.30
9	21.36	33.60	10.49

(4) Bounds for B_4 as functions of the degree and the diameter:

$d \setminus$ diam	2.7	3.14	2.13
3	-5.41,17.06	-11.45,31.21	$-2.89, 6.61$
4	-9.57,29.53	-16.65,54.01	$-4.14, 11.44$
5	-12.11,38.27	-21.06,70	$-5.23, 14.82$
6	-14.22,53.15	-24.70,97.22	$-6.15, 20.59$
7	-16.28,65.08	-28.25,119.04	$-7.06, 25.21$
8	-19.15,82.67	-33.29,151.22	$-8.28, 32.02$
9	-21.80,97.69	-37.91,178.69	$-9.41, 37.84$

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