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Cost–Benefit Analysis of Bus Fare Subsidies under Financial Constraints: The Case of Asunción, Paraguay

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Abstract

We develop a model to perform a cost–benefit analysis of bus fare subsidies under financial constraints that preclude the purchase of additional buses. Our model considers users' costs in the provision of bus services and the lack of road pricing to internalize urban transport externalities in a context where financial constraints severely limit the institutional ability to plan and design the bus system. Because of financial constraints, the bus system can hardly accommodate demand during peak times: buses travel overcrowded, passengers cannot board the first bus to arrive at the bus stop, and they cannot arrive at their destination at their desired time. Another salient aspect of our model is the inclusion of motorcycles as a second private transport mode. Motorcycles are typical in many urban agglomerations in the emerging world and engender many negative transport externalities. According to our results, fare subsidies provide social benefits in Metropolitan Asunción. During peak hours, a higher subsidy is justified as the reduction of the unpriced external costs of substitute modes compensates for the increased cost created by an additional bus passenger. In the off-peak, a higher subsidy is justified (i) as the higher frequencies induced by the new bus ridership reduce waiting times and (ii) because of the reduction of the unpriced external costs of substitute modes. Although our model does not explicitly include inequality aversion, we discuss the distributional aspects of subsidies in the context of middle-income countries.

1. Introduction

Public transport systems like buses, urban trains, and undergrounds have been subsidized in the developed world for several decades. In South American cities, on the contrary, passengers used to pay the total cost (TC) of bus services, while metropolitan trains and underground fares were subsidized. Since the turn of the century, however, bus fares have

started to be subsidized in several metropolitan areas, such as Asunción (2011), Bogota (2013), Buenos Aires (2005), Montevideo (2006), San Paulo (2003), and Santiago (2005). A common argument in favor of subsidies was addressing distributive issues.

Microeconomic justifications for public transport subsidies include two prominent reasons (Small and Verhoef, 2007; Parry and Small, 2009; Litman, 2022a): decreasing average users' cost in bus provision and the absence of road pricing. The first reason relates to considering users' travel time as an input in producing public transport services (Mohring, 1972). As the number of passengers increases, bus frequencies also increase, reducing waiting times for each passenger. This positive externality, the Mohring effect, means that the total average costs – average users' cost plus average operators' cost – decrease as ridership increases. Consequently, marginal cost pricing would not be sufficient to cover costs, calling for subsidies.¹ Crowding would be a counterbalancing element: to the extent that bus operators respond to increased demand by permitting a higher occupancy factor within buses, the discomfort of riding a bus will increase.

The second reason deals with the absence of road pricing. If drivers do not pay the marginal external costs they create in terms of congestion, pollution, noise, and/or crashes, excessive driving will result. Public transport subsidies help mitigate this situation by encouraging some drivers to opt for public transport instead, thereby alleviating the negative impacts of driving externalities. These two justifications are related to economic efficiency.

Equity issues per se are a third reason to subsidize public transport (Basso and Silva, 2014, 2023; Pavón and Rizzi, 2019; Horcher and Tirachinni, 2021).² When public transport users are less well-off and without access to other means of transport, subsidies will diminish the economic burden of traveling for these users and increase social welfare. This third reason is debated in the literature. The argument is about how well-targeted the provision of this subsidy is. Typically, subsidies consist of direct transfers to bus operators as a function of total ridership. Therefore, the indiscriminate nature of the subsidy benefits not only those who most need it but also the well-off who can afford higher prices.

A reason to refrain from subsidizing public transport is the government's financial constraint.³ Jara-Díaz and Gschwender (2009) show how this affects the design of optimal bus transport systems. The government financial constraint has the effect of diminishing the value of travel time and the value of waiting time, mitigating or even neutralizing benefits due to decreasing average users' costs: bus companies will operate buses with greater capacity and reduce frequencies compared to a first best solution. In a country like Paraguay, where tax collection is low by international standards, the government faces significant challenges in funding different services, including transport-related expenditures.

There is a well-known theoretical body of knowledge on how to perform a cost–benefit analysis of public transport subsidies. Parry and Small (2009) is among the most relevant and accomplished studies about the social worth of public transport subsidies. These authors estimate the net social benefit of public transport subsidies for London, Los Angeles, and Washington metropolitan areas. They conclude that fare subsidies of 50 % or more of operating costs are welfare improving at the margin, providing the intellectual ground for increasing them. Another landmark study is Basso and Silva (2014). They analyze the effect of several urban transport policies, including bus fare subsidies, with particular

¹ Two transport economic textbooks explaining this result are Jara-Díaz (2007) and Small and Verhoef (2007).

² Other reasons exist to justify public transport subsidies (Litman (2022a)).

³ This financial constraint applies when subsidized public or tendered bus companies run the services.

consideration of inequality in Santiago, Chile, and London. Among several findings, they conclude that, without road tolls, a 100 % fare subsidy for buses in Santiago would be welfare improving. These two studies elaborate detailed models to consider a wide range of elements that are key for a thorough cost–benefit analysis of fare subsidies.⁴

Considering Latin American case studies, there are only a few cost–benefit analyses of public transport subsidies other than Basso and Silva (2014). Parry and Timilsina (2010) analyze different transport policies for México City. When considering optimal fares for rail and buses as an isolated policy, they conclude that subsidy fares were near optimal.⁵ Pavón and Rizzi (2019) consider different urban transport policies in the contexts of dense city centers, as typical in many South American cities. They obtain a similar result to Basso and Silva: the lack of road pricing calls for a 100 % bus fare subsidy. Gomez Gélvez and Mojica (2022) estimate optimal subsidies for buses for Bogota, distinguishing between the Bus Rapid Transit system and the conventional bus system without dedicated lanes, applying the Parry and Small model. They found that optimal subsidies for 2019 traffic conditions should cover between 11 % and 34 % of the fare cost, depending on the time of the day and the public transport system. As in Parry and Small, these authors find that crowding contributes to a negative externality among bus passengers in peak periods, subtracting from the benefits of decreasing average users' costs. Tirachini and Proost (2021) propose a marginal tax reform model for Santiago, Chile. They conclude that increasing the cost of using the car and raising the bus fare subsidies in both peak and off-peak periods increase welfare. In an update of their original work, Basso and Silva (2023) apply a new version of their model to address the cases of Bogota, Sao Paulo, and Santiago, concluding once again about the welfare-improving effects of subsidies on public transport. In these studies, Basso and Silva (2014, 2023), Pavón and Rizzi (2019), and Tirachini and Proost (2021) consider equity explicitly. The latter study shows how relevant this topic is: in case distributional impacts are ignored, their model suggests reducing bus fare subsidies in the peak while increasing them in the off-peak, reversing their previous result.

This article proceeds with a cost–benefit analysis of public transport subsidies for the Metropolitan Area of Asunción, Paraguay. We apply a methodology very close, albeit simpler, to that developed by Parry and Small. Our methodology is simpler because of two reasons. First, in Asunción, the only means of public transport is the bus; there is no urban train or underground. Second, no public transport agency is responsible for planning and designing the bus system. Public companies provide bus services under limited governmental regulatory supervision. Hence, as analysts, we cannot assume that some relevant variables such as bus capacity, number of routes, and/or frequencies are chosen at their social optimum.

Our study has two innovations. First, we consider a strict financial constraint. On the one hand, the government cannot increase the subsidies paid to bus operators; on the other hand, private companies cannot raise their debt levels by securing additional loans in the market. The combination of these factors precludes the purchase or acquisition of additional buses. This state of affairs severely limits the ability to increase frequencies at peak periods of the day. As expertly described by Parry and Small (2009), decreasing average users' costs

⁴ These two studies refer to many other relevant studies in the field since the 80s. We will not review them.

⁵ At the time of this study, Mexico City had two types of public bus services. One type was provided by private operators, who charged fares to recover TCs. The other type of service was provided by a subsidized public company.

depends on how public transport supply responds to increases in demand. At many bus stops, passengers cannot board the first bus to arrive and have to wait for a second or third bus, increasing total travel times and creating schedule delay costs as many patrons cannot travel at their most preferred travel window. In other words, average users' costs – the Mohring effect – become increasing. We propose a simple model to estimate these negative external effects for the peak.

The second innovation of our study is the explicit treatment of motorcycles as a second private transport alternative in our cost–benefit analysis of bus fare subsidies. As in many other South American cities, motorcycles are pervasive, and most motorcyclists are users who otherwise would be most likely traveling by bus. Motorcycles not only detract demand from buses but also create several externalities, significantly increasing road accidents to the point that they become a most serious public health issue.⁶

Our methodology will serve for future analysis of the economic efficiency of bus subsidies for cities from the developing world that share similar characteristics to those of Metropolitan Asunción. The main drawback of our cost–benefit analysis is the absence of an explicit consideration of inequality. As travel information for metropolitan Asunción is very scant, we could not develop a detailed transport demand model as in Basso and Silva (2014) or Pavón and Rizzi (2019). For a clear understanding of ignoring inequality in a transport cost–benefit analysis, we refer the reader to Gálvez and Jara-Díaz (1998).

Our analysis concludes that, at the margin, bus subsidies in Metropolitan Asunción provide a positive welfare return both at peak and off-peak periods of the day. The highest return from subsidies occurs at nonpeak periods of the day because of decreasing average users' costs. At peak periods, average users' costs are increasing, becoming a source of welfare loss as an additional passenger increases waiting time and schedule delays for all other passengers, thereby creating negative externalities. This welfare loss is compensated by the corrective effect of subsidies regarding the underpricing of cars and motorcycles.

This article has six additional sections. The second section provides a brief description of our case study. The third section depicts our modeling strategy, and the fourth explains how we estimate all the relevant costs and benefits. The fifth section presents the main results, and an ensuing discussion follows in the sixth section. [Section 7](#) concludes the article.

2. A brief description of Metropolitan Asunción and salient urban transport facts

The Metropolitan Area of Asunción is home to 2.2 million people and comprises the city of Asunción, Paraguay's capital. As the destination of most commuting trips is the capital city, the structure of the bus system is radial, with most bus lines converging to the city center.

Private companies provide public transport services in Metropolitan Asunción. These companies own the buses and run the services according to a schedule under a permission agreement. These companies face severe financial restrictions and struggle to renew their fleet, and at any time, many buses cannot provide service as they need repair. On average, the total bus fleet operating at any time is around 1,150 units, with many additional buses being out of service. A typical bus has a capacity of 60 passengers (seating and standing passengers). The Ministry of Public Works and Communications establishes the technical

⁶ The reader could verify this statement by doing an internet search and writing the Spanish words “motos,” “accidentes viales,” and “Asunción.”

fare, the fare subsidy, and the frequencies and grants permission agreements to bus operators. These permissions are allocated – grandfathered – to those operators that have run the bus lines historically without calls for tenders. The precariousness of these permissions – accounting-wise, they are not an asset – is one of the reasons affecting these companies' creditworthiness.⁷

The Government of Paraguay started subsidizing public transport in Metro Asunción in 2011, overhauling the subsidy system in 2014. Currently, subsidies are paid by the Ministry of Public Works and Communications to the bus operators every month. These payments are based on the number of validated passengers transported by each bus operator. The subsidy amount differs because there are two types of services (low standard and high standard).

Like many other cities in the developing world, Asunción saw a vast increase in the number of users driving a motorcycle in the last 20 years. This transport mode constitutes a very convenient and cheap alternative for many people. The typical motorcycle is a two-stroke engine 50 CC model. Transport authorities recognize that motorcycles are a relevant transport issue because of the externalities they create in terms of congestion, noise, and road crashes. There are almost 290,000 registered motorcycles in Metropolitan Asunción and about 510,000 registered vehicles (light vehicles, Vans, and SUVs).

Transport demand data are scant for Metropolitan Asunción. A mobility survey was taken in 2021,⁸ but most of its results are still unavailable. Assuming 2.43 trips per vehicle per day⁹ and an occupancy rate of 1.4 for cars and 1.3 for motorcycles; there are more than 1.73 million daily trips by car and almost 910,000 daily trips by motorcycle. According to information from the Ministry of Public Works, total daily trips by bus are expected to be around 630 thousand by 2023.¹⁰ With this level of ridership, the fare subsidy would amount to USD 52 million for the whole year.¹¹

In Metropolitan Asunción, buses share the road with all other vehicles. No dedicated infrastructure for buses, bus lanes, or other bus traffic priority exists. Hence, public buses are subject to high congestion levels like other motorized vehicles, especially during peak hours. This high level of congestion increases cycle time, not permitting higher frequencies. Anecdotal evidence suggests that users mostly complain about waiting time and crowding, especially during peak hours.

In 2021, the per capita GDP at current values for Paraguay was USD 5,959 (World Bank, 2022); in comparison, Chile had a figure of USD 14,300. According to the World Bank (2023), Paraguay is considered an upper-middle-income country whose income is closer to the lower end of the range of values for these countries. Tax collection, including social security contributions, is at 14 % of GDP, with taxes on goods and services accounting for

⁷ As a possibility, bus fares could increase to finance the purchase of additional buses. These increases, however, will not improve the creditworthiness of bus companies. In addition, fare increases would counter the very reason by which subsidies were established. Bus fare increases in South America are a delicate issue as they could trigger political and social instability.

⁸ The trip generation rate per person is 1.2, a very low figure that may be influenced by behavior associated with the COVID-19 pandemic.

⁹ This value is taken from Valparaíso, Chile.

¹⁰ This figure does not include trips in bus services offered by the counties that comprise the Metropolitan Area of Asunción. The Ministry of Public Works does not regulate these services. These trips are low in comparison and do not go beyond the bounds of the county.

¹¹ As a reference, the bus fare subsidy in Santiago, Chile, amounts to USD 520 million for 2021.

more than 50 % of the total collection. Regarding taxes on income, profit, and capital gains, corporate-level taxes represent 19 % of total tax collection, and personal-level taxes represent only 1 % (OECD, 2023).¹² As a percentage of GDP, Paraguay has the second-lowest total tax revenues in Latin America, thus severely limiting the government’s ability to finance different public expenditures.

3. The model

The model considers two typical efficiency-related arguments: external costs among bus users and car-use externalities such as congestion, road crashes, noise, local pollutants, and CO₂ emissions. We estimate the social welfare impact of fare subsidies at the margin starting from the following simple equation (Small & Verhoef, 2007).

$$SW = \int_0^{q_a} p_a(\rho_a, q_m, q_b) d\rho_a + \int_0^{q_m} p_m(0, \rho_m, q_b) d\rho_m + \int_0^{q_b} p_b(0, 0, \rho_b) d\rho_b - q_a c_a(q_a) - q_m c_m(q_m) - q_b c_b(q_b) - Ext(q_a, q_b, q_b) \tag{1}$$

SW: social welfare

p_i : inverse demand (generalized cost) of mode i (i = car (a), motorcycle (m), bus (b))

q_i : demand for mode i

c_i : average cost mode i

Ext : Externalities produced by mode

ρ : is a variable of integration.

The three integrals represent the area under the demand curve for each of the three modes under the assumption of a representative consumer. The following three terms comprise the total cost of operating the three modes of transport and are given by the total demand for mode i times the average cost per passenger. This average cost includes the user’s average cost (cost of time, crowding, etc.) and the average operator’s costs. The seventh term represents the amount of welfare loss created by transport externalities.

In Appendix A, we show that upon differentiation of SW with respect to q_b , we arrive at the following equation that enables us to estimate the monetary social welfare impact of public transport subsidies at the margin:

$$\frac{dWS}{dq_b} = -S_b - q_b \frac{\partial c_b}{\partial q_a} - \frac{\partial Ext}{\partial q_b} - \left(q_a \frac{\partial c_a}{\partial q_a} + \frac{\partial Ext}{\partial q_a} \right) \frac{\varepsilon_a^{p_b}}{\varepsilon_b^{p_b}} \frac{q_a}{q_b} - \left(q_m \frac{\partial c_m}{\partial q_m} + \frac{\partial Ext}{\partial q_m} \right) \frac{\varepsilon_m^{p_b}}{\varepsilon_b^{p_b}} \frac{q_m}{q_b} \tag{2}$$

S_b is the fare subsidy and $\varepsilon_i^{p_b}$ the price elasticity of the mode i demand with respect to the bus fare. The second term on the right-hand side of the equation is the benefit attributed to decreasing average users’ costs, the so-called “Mohring effect.” This benefit is observed in the off-peak periods; in the peak periods, the Mohring effect is negative in our setting. The third term reflects all the negative externalities that the circulation of buses creates, such as congestion, crashes, emissions of local and global pollutants, and noise. The fourth and fifth terms compute the benefit of users switching from the car and motorcycle to the bus. The parentheses account for the marginal external costs of both transport modes and $\frac{\varepsilon_i^{p_b}}{\varepsilon_b^{p_b}} \frac{q_i}{q_b}$ (i : car

¹² Engel et al. (1999) show that Chile’s highly right-skewed income distribution makes raising more personal income taxes difficult as most formal workers are exempted from this tax.

and motorcycle) give the proportion of new bus users switching from cars and motorcycles due to a marginal increase in the level of the subsidy.

If Equation (2) is positive (negative), the subsidy will increase (decrease) welfare. If the value of this equation equals zero, then the current subsidy is at its optimal level.¹³

3.1. Limitations of the model

The main limitation of our model – Equation (1) – is that we use inverted demand curves instead of a model with explicit direct or indirect utility functions. Hence, we need to assume the values of the own-price and cross-price elasticities as given, imposing very stringent conditions on the shape of demand curves that must be isoelastic. This limitation restricts Equation (2) to be helpful in analyzing the efficiency of the subsidy at the margin. If we wanted to compute optimal values, we would need to assume that the elasticity values do not change, which would make these optimal values less reliable. The same applies if the modeler wanted to analyze the optimal value of some other level-of-service variable, such as frequencies.

In addition, our approach is static, as in Parry and Small, Basso and Silva, or Borjesson *et al.* (2017). The static model only permits considering different level-of-service variables – travel times and waiting times – at an aggregate level at peak and off-peak periods. In addition, capacity constraints impose crowding costs and schedule delay costs. Once again, a static model only allows us to analyze these costs coarsely. A more realistic approach that considers how crowding and schedule delays evolve during the peak and how these affect trip-timing would be like Fosgerau (2009) or De Palma *et al.* (2015), (2017).

Another relevant limitation of our model is the lack of consideration of distributional impacts, especially given that the main argument for establishing the subsidy to the fare of public buses was to make urban travel affordable for the whole population. Data scarcity does not permit us to develop a more detailed model considering different types of users. As typical in this type of analysis, our model consists of a representative agent or consumer who represents behavior at an aggregate level. Our model also ignores feedback effects between transport costs and the decision to participate in the labor force and/or the number of hours to work. Likewise, we ignored the opportunity cost of public funds to pay for the subsidies. The discussion section will address inequality and this topic together as they are inextricably associated.

We do not consider economies of scale to be associated with the spatial density of the bus system. The further away the bus lines are structured, the more significant access costs will be. We assume away these costs. As mentioned in Section 2, the system of bus line routes was not planned by a transport authority intending to maximize social welfare; it is the result of the evolution of the routes according to the acumen of bus operators over the years. This state of affairs will not change in the foreseeable future.

The last relevant limitation is the unavailability of local transport data, making it necessary to rely on the transference of key parameters – for instance, the value of travel time and the value of life and limb, to name just a few. The lack of local data takes us back to our first limitation, which is to start from a simple model assuming the elasticities of demand curves as given.

¹³ If Equation (2) is equated to zero, we would obtain the optimal fare. We do this just as a theoretical exercise in Section 5, but this result is less reliable because we need to assume isoelastic demand curves. See Section 3.1.

4. Computation of transport social external costs

To make Equation (2) operative, we must invoke different submodels to compute the necessary values. This section explains how we compute different social costs for the three modes we consider. This section also provides information on all relevant parameters included in the analysis.

Values in Equation (2) will be expressed in USD c (cents of a US dollar as of October 2022) per passenger-kilometer (USD c/pak-km).

4.1. External effects among bus users – peak period

Assume a bus company operating a fleet of B buses with capacity K . Buses travel a distance of $2L$ (L is the distance in any one direction) with total cycle time CT . The maximum frequency will be $f = B/CT$. Due to financial constraints, the fleet of buses cannot increase; therefore, frequencies cannot augment at peak times. Consequently, buses may run at overcapacity to accommodate a higher number of passengers during the peak. h is a factor to account for overcapacity ($h \geq 1$), and overcrowding sets in when this value exceeds one. If so, the total capacity is given by Kh . Even with $h > 1$, demand can still exceed supply capacity, and passengers will have to wait extra time to board a bus as many of them will not be able to board the first bus to arrive at the bus stop. As supply cannot accommodate demand, passengers will incur schedule delay costs because they cannot arrive at their destination at their desired time. Some will arrive early; others will arrive late. For ease of presentation, we will consider the morning peak, where the demand in the inbound direction y_i is higher than in the outbound direction y_o . We assume that the afternoon peak mirrors the morning peak. Riders travel a distance l .

There is a desired span of arrival times for which demand exceeds supply. If passengers travel a distance l , then $y_i l/L > Khf > Kf$.¹⁴ The following equation expresses TC:

$$\begin{aligned}
 TC = & f \left(T + t \left(Kh \frac{L}{l} + \frac{y_o}{f} \right) \right) C_{peak} + V_{WT} \left(\frac{r}{f} + \frac{y_i}{Khf^2 L} \frac{l}{f} - \frac{1}{f} \right) y_i + V_{WT} \frac{r}{f} y_o \\
 & + V_{TT} (1 + (h - 1)\gamma) \left(\tau + tKh \frac{l}{L} \right) y_i + V_{TT} \left(\tau + t \frac{y_o l}{f L} \right) y_o \\
 & + V_{SD} \frac{1}{2} \left(\frac{y_i}{Khf} \frac{l}{L} - 1 \right) \frac{d_{arrival}}{2} y_i
 \end{aligned} \tag{3}$$

Total cost (TC) incorporate six terms. The first term represents the costs of operating the service. C_{peak} is the hourly cost of operating one bus in the peak period. Cycle time, CT , is given by $T + t(Kh \frac{L}{l} + y_o)$. T denotes the time the bus is moving. The following term represents the time the bus dwells in the bus stop with patrons boarding and alighting — the time a passenger takes to board and alight is t . In the inbound direction, the number of passengers per bus is $KhLl/l$, and in the outbound direction, it is y_o/f . The total number of buses is $f^* CT$.

The second and third terms represent the average passenger waiting time in the inbound and outbound directions, respectively. Inbound passengers will have to wait more than one bus on average if there is excess demand, even after accounting for overcrowding. V_{WT} is the

¹⁴ That is, demand per unit of time > capacity with overcrowding > capacity without overcrowding.

value of waiting time. In the inbound direction, passengers will wait $\left(\frac{r}{f} + \frac{y_i}{Khf^2} \frac{l}{L} - \frac{1}{f}\right)$. The parameter related to reliability is r . If this value equals $\frac{1}{2}$, then buses arrive at the bus stop at equal-spaced intervals, and passengers arrive uniformly distributed among bus arrivals. If the r value is higher than $\frac{1}{2}$, then intervals between bus arrivals are no longer equal.

If demand can be accommodated with the current supply, allowing for overcrowding, then $y_i/(KhL)$ equals f , and waiting time reduces to r/f . If demand exceeds supply and buses travel with excess capacity ($h > 1$), then $\frac{y_i}{Khf^2} \frac{l}{L} - \frac{1}{f} > 1$ and passengers, on average, cannot board the first bus to arrive at the bus stop. For example, assume that demands double supply (even at excess capacity) such that $y_i/(KhL) = 2f$ and that $r = 1$ arrival times between consecutive buses distribute exponentially; then average waiting time will be two times the average headway. In the outbound direction, passengers will wait, on average, the average arrival time between consecutive services. A passenger traveling in the opposite direction does not affect the frequency level; however, she does affect the total travel time for the passengers traveling in the same direction because of boarding and alighting.

Travel time in the direction of higher demand (fourth term) will be higher than in the opposite direction (fifth term). If h is greater than one, overcrowding will make the value of travel time more onerous. The factor $(1 + (h - 1) * \gamma)$ amplifies the value of travel time (V_{TT}) as long as demand exceeds capacity. γ is a proportionality factor accounting for the increase in the value of travel time with overcrowding. Passengers' travel time in the inbound direction is given by $\left(\tau + tKh\frac{l}{L}\right)$, where $\tau = \frac{l}{L} \frac{T}{2}$. The expression in parenthesis comprises the time the bus is moving plus the time at the bus stops without moving to enable boarding and alighting (dwell time). In the outbound direction, there is no overcrowding, so travel time equals $\left(\tau + t\frac{y_i}{f} \frac{l}{L}\right)$.

The sixth term accounts for schedule delay costs. The expression $\frac{y_i}{Khf} \frac{l}{L} - 1$ represents the excess demand as a proportion of capacity. Consider that passengers want to arrive at their destination during a period, the extension of which is $d_{arrival}$. Desired arrival times are uniformly distributed during this period. As supply cannot accommodate demand, some passengers will arrive at their destination before the beginning of the period of desired arrival times, and some other passengers after the end of the desired arrival times. Assume that passengers depart in the same order as their preferred arrival time and that those who depart first arrive first. Departures are such that half of the passengers will arrive early and the other half late, and only those who want to arrive at half the time of the desired arrival period will not experience schedule delay.¹⁵ Suppose $d_{arrival}$ spans 1 h and excess demand equals 0.5. In that case, the first passengers arrive 15 min earlier than desired, with early arrival decreasing linearly to 0 min for those who want to arrive at the mid-time of preferred arrival times. From this moment onward, late arrivals increase linearly up to 15 min of late arrival for those arriving at the latest time, at the end of the peak. The average schedule delay will be

¹⁵ We make an analogy with the bottleneck model (Arnott *et al.*, 1993). To do so, we make a continuous approximation of the arrival times of bus passengers as bus passengers arrive in batches at discrete times at their destination. Without this approximation, calculations would become intractable. Parry and Small (2009) also use this approximation when computing waiting costs for passengers traveling in services with headways higher than 15 min. Fosgerau (2009) defines the value of headway as the costs of waiting time and schedule delay when passengers randomly arrive at a bus stop. However, in his article, schedule delay costs arise because of the discrete nature of bus arrivals at the bus stop. In our case, schedule delay costs would disappear without excess demand.

7.5 min. We assume the cost of a minute of early arrival equals the cost of a minute of late arrival; this value equals V_{SD} .

In our setting, bus companies cannot afford to purchase or acquire more buses because of financial constraints. Bus companies also choose the capacity of buses without considering social costs. Suppose travel demand exceeds the supply capacity in the inbound direction. In that case, the bus system can only serve demand by allowing overcrowding and *extending the peak hour period* – as in a regular server operation. Overcrowding contributes to diminishing waiting time and schedule delays. However, if demand vastly exceeds capacity, overcrowding reaches a maximum, given by the maximum number of standing passengers that permits a safe operation. Then, the only way to serve demand is to extend the time the whole fleet of buses has to be in operation. The relevant point from the above analysis is that frequencies cannot be increased, nor the fleet of buses; hence, an additional passenger does not create a marginal cost of capital. Following Parry and Small (2009), if frequencies are not increased, the marginal supply capital cost of transporting an additional passenger is zero. In addition, as the level of overcrowding is at its maximum, the marginal cost an additional passenger creates in terms of discomfort for her fellow passengers is also zero.¹⁶ This should not be interpreted as overcrowding not deserving to be addressed. Our model clearly shows that overcrowding negatively affects the welfare of passengers, and reducing overcrowding will also reduce TCs.

Regarding external costs, an additional passenger in the inbound direction increases waiting time and schedule delays. These are the two relevant externalities we need to calculate in our modeling. To calculate external costs per passenger in the inbound direction, we differentiate Equation (3) with respect to y_i and subtract those costs the passenger bears as user costs. Waiting time marginal external cost is given by, $V_{WT} \frac{y_i}{Khf^2} \frac{1}{L}$ and schedule delays marginal external costs by $V_{SD} \frac{1}{2} \frac{1}{Khf} \frac{1}{L} \frac{d \text{ arrival}}{2} y_i$.

Regarding operating costs, we assume that the cost per passenger-km per bus is independent of the number of passengers traveling: if occupancy is higher, operating costs per bus increase proportionally to occupancy. In other words, if a bus travels with Kh passengers, the cost of operating that bus will be h times higher than that of a bus traveling with K passengers. This could be attributed to the higher maintenance costs of the bus itself, more complicated labor scheduling corresponding to peak hours, and/or greater fuel costs. This assumption implies that marginal supply operating costs per passenger kilometer are constant and equal to average costs (see Appendix A). Regarding the marginal capital costs per passenger, we have already explained that these are zero.

We assume the following values to calculate the two marginal external costs described above. Frequencies f are 12 services per hour in the peak (5-min headway), and that demand exceeds the capacity by a factor of 1.5. Buses can accommodate 60 passengers by design.¹⁷ If buses carry 20 % of the excess capacity, 72 passengers will travel. Route length in each direction equals 24 km, and passengers make trips of 12 km on average, which gives a renewal rate of 2. A typical bus line transports 1,728 passengers per hour. We assume that twice this amount of people want to travel over an 80-min period in the inbound direction.

¹⁶ Because of this, the value of γ is irrelevant for the cost–benefit analysis of the subsidy at the margin.

¹⁷ Following Jara-Díaz and Schwender (2009), bus companies choose low frequencies and large-size buses to minimize costs. The 60-passenger capacity bus will thus be the largest bus, enabling smooth operation given the street layout.

As this is not possible, the morning peak for bus operation will span two hours. In the outbound direction, ridership is 1,440 (accounting for demand renewal) for the 2-h peak period. Demand in the inbound direction accounts for 71 % of the total demand per bus per cycle; return trips account for the other 29 % of total demand.

Because of the high level of congestion at peak times and the lack of bus-traffic control systems, we assume a value of r equal to 1; for example, the arrival times between consecutive buses distribute exponentially. The value of travel time is transferred from Chile (MDSF, 2022), adjusted by the difference in per capita GDP, and converted to dollars. The benefit of saving 1 h of travel time equals USD 1.12. The value of waiting times duplicates the value of travel time ($V_{WT}=2 V_{TT}=\text{USD}2.24$), and the value of schedule delay equals $0.4 V_{TT}$.

With our assumptions and the adopted parameters, the marginal external cost of increased waiting time is 3.1 cents¹⁸ per passenger-km, and the marginal external cost of additional schedule delay is 2.5 cents per passenger-km. In the outbound direction, an extra passenger only creates an external cost of travel time, which is given by the expression $a V_{TT} t \frac{\gamma}{f} \frac{1}{L}$ and equals 0.6 cents. Considering the proportion of trips in each direction, marginal external costs from an additional passenger equals 4.1 cents.

We also compute the benefits of increasing frequencies by one unit during peak hours. To do so, we differentiate Equation (3) with respect to frequency and calculate the value of this derivative. This will increase the operator's costs¹⁹ and decrease users' costs as the waiting time and schedule delays diminish. Travel times will only decrease for passengers traveling in the outbound direction, as buses will still travel at maximum overcapacity in the inbound direction. To do these calculations, we need a few additional parameters: $t = 7$ s (time to board and alight), $T = 4$ h (total driving time), and $\gamma = 0.4$ – the value of travel time for overcrowding increases 40 %.²⁰ The benefit of one additional hourly bus service in the peak amounts to a daily figure of USD 11,126 – this figure considers the benefits of the morning and the evening peaks. The yearly figure is USD 2.8 million – assuming 250 working days. If frequencies were optimally adjusted, this benefit should be zero. This evidences how strong the government's financial constraint is. This benefit would accrue for a single bus line; the benefit for the system would be more significant.

Another way to assess the underinvestment in the bus fleet is to calculate the costs this creates in terms of additional waiting time, overcrowding, and schedule delay. In other words, if frequencies were such that demand could be accommodated without overcrowding, these costs would not occur. For the whole system, the annual extra waiting, schedule delay, and overcrowding costs equal USD 3.5 million, USD 2.8 million, and USD 7.1 million, respectively. The cost of overcrowding is the largest; it is a cost that bus users are willing to incur to travel in peak periods. To put these values in perspective, a new bus's annual capital cost (depreciation plus opportunity cost of capital) plus its annual operating costs amount to around USD 100,000.

¹⁸ When we write “cents” alone, we refer to USD cents.

¹⁹ The hourly operator costs are around USD18 per bus during the peak.

²⁰ This last value is lower than those reported in the literature (Batarce *et al.*, 2016; Tirachini *et al.*, 2017). Most values reported about the crowding penalty in the value of travel time are from countries with higher per capita income than Paraguay. If travel comfort is a luxury, willingness to pay for it will increase with income more than proportionally.

Table 1. Marginal congestion external costs per passenger-km (USD cent)

	Car	Motorcycle	Bus
Peak	12.2	6.1	0.7
Off-Peak	2.6	1.3	0.2

4.2. External costs among bus users – off-peak period

During the off-peak, companies determine the number of buses to run according to demand. In this case, frequency should be equal to or higher than demand, such that $Y \ll (KL) \leq f$. Once again, the buses in operation in the off-peak are given by multiplying the frequency and cycle time. We assume that off-peak demand is even in both directions (inbound and outbound). Equation 4 determines the TC of operating services in the off-peak.

$$TC = \frac{y_v}{K} \left(T + tK \left(1 + \frac{y_i}{y_v} \right) \right) C + V_{WT} r \frac{y_i + y_o}{\frac{y_v}{K} \frac{l}{L}} + V_{TT} \left(\frac{1}{2} T + tK \right) (y_i + y_v) \frac{l}{L} \quad (4)$$

We have substituted $(y/K)(l/L)$ for frequency. The marginal external cost of an additional passenger in the inbound direction is negative: this is the typical Mohring effect – decreasing waiting times–, whereby an additional passenger decreases the waiting time of all passengers. This marginal external benefit is given by $-V_{WT} r \frac{K}{y_i} \frac{l}{L} \frac{y_i + y_o}{y_i}$. By symmetry, this benefit also applies to passengers traveling in the outbound direction.

We assume an off-peak period of six buses per hour and that buses travel at total capacity in both directions.²¹ During the off-peak, total demand equals 720, taking into account the renewal rate. Because of lower congestion, in the off-peak, r equals 0.8. The external benefit of an additional passenger in diminishing waiting times is 5.0 cents.

4.3. Congestion costs

Rizzi and De la Maza (2017) estimated congestion costs for Santiago, Chile. These authors provide a set of values per kilometer and passenger-kilometer for light cars and buses for the peak and the off-peak periods. We adjust those values to account (i) for differences in occupancy rates per vehicle and (ii) for differences in per capita income and convert these values to USD from October 2022. Regarding the first adjustment, occupancy rates are higher for Asunción: bus-peak, 72 passengers; bus-off peak, 60 passengers; cars, 1.4 passengers and motorcycles, 1.3 passengers. Regarding the second adjustment, we proceeded in the same way we explained above to transfer the value of waiting time and the value of travel time. The marginal external congestion costs of motorcycles are estimated assuming that a motorcycle has a congestion effect equivalent to 0.5 cars. Table 1 shows these marginal external costs.

4.4. Road crashes

Jansson (1994) develops a simple model to determine the costs of road accident externalities. Assume that accidents per unit of time (Acc) are given by a function such as $Acc = \alpha V^\beta$, with

²¹ According to our calculations, the current off-peak frequency is optimal.

V flow (vehicles/unit of time). Risk (r) is defined as ACC/V in this simple representation. $\beta-1$ is the elasticity of risk with respect to flow. If $\beta-1$ is greater than zero, the risk rises as flow increases, whereas if $\beta-1$ is less than zero, the risk decreases with flow. If $\beta-1$ equals zero, then the risk is proportional to flow. Hence, depending on the value of β , there could be negative, positive, or no externalities.

If there are heterogeneous traffic flows, say light vehicles (L) and heavy vehicles (H), the modeling of road crashes is somewhat more complex. Following Jansson (1994), accidents can be represented by this equation: $Acc(L, H) = \kappa L^\gamma H^\delta$. In terms of risk, there are two functions: one is the risks for light vehicles and the other is the risk for heavy vehicles, respectively: $r(L) = \kappa L^{\gamma-1} H^\delta$ and $r(H) = \kappa L^\gamma H^{\delta-1}$. There are four relevant elasticities of accident risk: (i) elasticity of risk for light vehicles with respect to light vehicles flow ($\gamma-1$), (ii) elasticity of risk for light vehicles with respect to heavy vehicles flow (δ), (iii) elasticity of risk for heavy vehicles with respect to light vehicles flow (γ) and (iv) elasticity of risk for heavy vehicles with respect to heavy vehicles flow ($\delta-1$). For instance, if both δ and γ equaled 1, there would be negative cross-externalities between the flows and no externalities within flows. Following Lindberg (2001), we assume that, in crashes involving vehicles of different masses, the TCs of the accidents are borne by the vulnerable user, the one with less mass. This means that the relevant risk elasticity values are $\gamma-1$ and δ .

We estimate the marginal external costs of road crashes following the theoretical models of Jansson (1994) and Lindberg (2001), considering four categories of users: cars, motorcycles, buses, and pedestrians. Pedestrians are included because they are victims of the other three categories in many road crashes. Table 2 shows the elasticities adopted for calculating marginal costs for every two-user crash category and every single-user crash category. We assume away crashes between buses and crashes involving three categories or the four category users.

From Lindberg, we adopt the elasticity of crash risk for a lower mass user category with respect to the higher mass user category flow (δ) as 0.5 and the elasticity of crash risk for a lower mass user category with respect to its own flow ($\gamma-1$) as -0.5 . The elasticity of risk of crashes for motorcyclists – both as a single motorcycle crash or a crash with another motorcycle – is 0.2; idem for cars.

The other key parameter is the value of a statistical life (VSL). Ample international evidence (Lindhjem *et al.*, 2011) suggests multiple values. In addition, there is a discussion on the relationship between the VSL regarding traffic-related and health-related risks (Dekker *et al.*, 2011). Lindhjem *et al.* (2011) found no clear evidence in any direction. Chilean evidence on the VSL is scarce (Greenlab UC, 2014), but also suggests a diversity of

Table 2. Elasticities of risk for the lower mass category in two-user category crashes

	Pedestrian	Motorcycle	Car	Bus
Pedestrian		0.5	0.5	0.5
Motorcycle	-0.5		0.5	0.5
Car	-0.5	-0.5		0.5
Bus	-0.5	-0.5	-0.5	

Note: The above-diagonal values represent the elasticity of risk of the lower mass category with respect to the flow of the higher mass category; the below-diagonal values represent the elasticity of risk of the lower mass category with respect to its own flow. The diagonal values are the elasticities assumed for crashes within categories or solo crashes for that category.

values. After weighing different pieces of evidence, we consider the value proposed by OECD (2014) for Chile adjusted to Paraguay using the formula suggested in the OECD report, updating the value to 2022 prices and converting to USD. Including other costs (police, firefighters, medical inputs, etc.), the VSL amounts to USD 605,000. The value of saving a statistical severe injury, a statistical less serious injury, and a statistical mild injury are assumed to be 24 %, 11 %, and 1 % of the VSL, respectively.

To complete the estimation of marginal road crashes external costs, we need annual crashes classified by vehicle category and injury severity. The Agencia Nacional de Tránsito y Seguridad provides these figures. Dividing by the kilometers driven by each vehicle category and by the average number of occupants by type of vehicle, we estimate the marginal external cost per vehicle category expressed as a cost per passenger kilometer. These values are 1.4 cents, 1.17 cents, and 2.6 cents, respectively, for buses, cars, and motorcycles. The motorcycle creates the greatest externality cost.

4.5. Local pollution, noise, and CO₂ emissions

The marginal external costs for local pollutants and noise are adapted from Rizzi and De la Maza (2017). For local pollutants, we consider PM_{2.5}. Metropolitan Asunción is less polluted than Metropolitan Santiago, with a third of the population. Daily average annual PM_{2.5} concentrations in the latter were 29.14 μm^3 (Osse, 2018). According to Recalde et al. (2021), most PM monitoring stations report annual daily average values below 12 μm^3 , values still above those recommended by the World Health Organization. On the other hand, the fuel quality in Paraguay is lower than in Chile. Combining all these factors, we assume the marginal external damage in terms of PM_{2.5} emission of a vehicle kilometer is a third of the values estimated for Santiago. We also assume the noise marginal damage of a vehicle kilometer equals that for Santiago for light vehicles and buses. Finally, PM_{2.5} and noise marginal external costs per vehicle-km are adjusted by per capita income difference and vehicle occupancy rates and converted to USD.

Regarding motorcycles, the typical model has a two-stroke engine. These engines are dirtier in terms of emissions – though fuel consumption is much lower – and louder. Regarding emissions, we assume a motorcycle-km pollutes a third of the amount polluted by a light vehicle-km; regarding noise, we assume the same level of sound intensity per kilometer as a light vehicle.

CO₂ emission values are taken from a web page²² and the cost of emitting a ton of CO₂ is USD 30, the value used in Chile for transport project appraisal for the year 2023. This value is not adjusted by per capita income. Table 3 provides PM_{2.5}, noise, and CO₂ marginal congestion external costs per passenger-km (USD cent).

4.6. Elasticity values

We need three elasticities to perform our calculations. First, we need the own price elasticity of public transport demand. As subsidies will decrease the price of riding a bus, new trips will be made. Second, bus fare subsidies may promote modal switching from car and motorcycle;

²² <https://ecoscore.be/>.

Table 3. Marginal congestion external costs per passenger-km (USD cent)

	Car	Motorcycle	Bus peak	Bus off-peak
PM2.5	0.25	0.08	0,04	0.05
Noise	0.12	0.13	0,02	0.03
CO ₂	0.51	0.17	0.04	0.05

hence, we also need the cross-price elasticity of the car and motorcycle demand with respect to the price of public transport.

Cross-price elasticity values are key parameters in our calculations, especially about the benefits from mode switching induced by bus fare subsidies. The percentage of users switching from the car (motorcycle) to the bus – the diversion factors – is given by $\frac{\varepsilon_a^{pb}}{\varepsilon_b^{pb}} \frac{q_a}{q_b} \left(\frac{\varepsilon_m^{pb}}{\varepsilon_b^{pb}} \frac{q_m}{q_b} \right)$. The numerator has the car (motorcycle) cross-elasticity and the own-price bus elasticity in the denominator. In addition, we must multiply by the quotient between car (motorcycle) travel demand and bus travel demand. For consistency, each of these two expressions needs to be greater than zero and less than one, and the sum of these needs to be less than one as they account for the proportion of new bus passengers switching from the car and the motorcycle to the bus at the margin. $1 - \frac{\varepsilon_a^{pb}}{\varepsilon_b^{pb}} \frac{q_a}{q_b} - \frac{\varepsilon_m^{pb}}{\varepsilon_b^{pb}} \frac{q_m}{q_b}$ accounts for the proportion of new trips (induced demand) that would not have been made had the bus fare not gone down. In the case of Metropolitan Asunción, total bus trips are lower than total trips by car and motorcycle, in an approximate proportion of 1–2.8 and 1–1.4, respectively. Hence, elasticity values need to be such that microeconomic consistency is preserved.

The literature provides a wide range of values for these elasticities. Dunkerley *et al.* (2018) provides evidence of bus fare elasticities relevant to the United Kingdom. Litman (2022b) provides a detailed and up-to-date review of urban transport elasticity for different modes. Gomez Gélvez and Mojica (2022) report values estimated by other authors for Bogota and Santiago. Parry and Timilsina (2010) report values estimated within the context of the United States. None of these studies report cross-price elasticities for the motorcycle. From the values reported by Dunkerley *et al.* and Litman, we adopt -0.4 as the own-price bus elasticity and, from the values reported in Litman, 0.05 as the cross-price car elasticity. The former value is close to that of Parry and Timilsina for México City and higher than that adopted by Gómez Gélvez and Mojica for Bogota. The latter value corresponds to the lowest values reported by Litman (2022b) and was adopted by Tirachini and Proost (2021). We assume motorcyclists are more sensitive to bus fare variations than car drivers and passengers: the cross-price motorcycle elasticity is 0.15 .²³ With these elasticity values and the travel demand figures by mode, microeconomic consistency is guaranteed: at the margin, 34% ($0.05/(-0.4 \times 2.75)$) and 54% ($0.15/(-0.4 \times 1.44)$) of the additional bus users come from the car and motorcycle, respectively,²⁴ and 12% are new trips.

²³ According to local data, the average personal income of a motorcycle user is lower 50–60% less than that of a car user (INE, 2021).

²⁴ The diversion factors depend on the total number of trips per mode. From Section 2, these figures are 1.73 million daily trips by car, 910,000 daily trips by motorcycle, and 630,000 daily trips by bus. The more people travel by car and motorcycle, the fewer people travel by bus, the greater the diversion factors.

4.7. Fuel taxes

Based on tax rates, fuel prices, fuel efficiency (10 km/l), occupancy rates, and average distance of a trip (12 km), we estimate that cars and motorcycles are charged 1.5 cents and 0.2 cents per passenger kilometers in fuel taxes respectively; for a bus, this value would 0.03 cents in the peak and 0.15 cents in the off-peak. We subtract these values from each mode's marginal external costs per passenger-km.

4.8. Fare subsidies

The technical fare and the fare subsidy are defined by a special Committee convened by the Ministerio de Obras Públicas y Comunicaciones. The technical fare and fare subsidy vary according to whether or not the bus is low or high-standard. Since 2014, most new entrant buses are high-standard; hence, we will do our calculations assuming all current buses will be replaced by new high-standard buses, which are more expensive. The official technical fare is USD 0.78, and the official subsidy is USD 0.31. Curiously, the technical fare does not include capital costs and the opportunity cost of capital; if we add these costs, the technical fare increases to USD 0.85. Hence, the total amount of the fare subsidy equals USD 0.38. This last figure is used in our calculations and amounts to a subsidy of 45 % of the technical fare.

5. Results

In this section, we report the social net benefits of bus fare subsidies. [Table 4](#) provides the essential information to estimate [Equation \(2\)](#), and [Table 5](#) provides our baseline results.

[Table 5](#) shows the efficiency of subsidy in cents/pax-km and its benefit–cost ratio. It provides a breakdown of the four categories of benefits and costs per period of the day. These categories correspond to the different terms of [Equation \(2\)](#).

The subsidies provide positive social value in the peak and the off-peak periods. A cent of fare subsidy offers a return of 0.19 cents (or 19 %) in the peak period and a return of 1.33 cents (133 %) in the off-peak period. In the peak period, the main benefit from fare subsidies is congestion relief and the reduction of other negative externalities. These benefits are countered by the external costs among bus passengers regarding waiting time and schedule delays and, to a lesser extent, by the externalities created by buses.

The efficiency of the subsidy is higher in the off-peak period. Decreasing average users' costs provides the main benefit, as one additional passenger reduces the waiting time for all other passengers. Congestion relief and reduction of other negative externalities benefits are also present, but to a lower extent than in peak periods.

Regarding the origin of the bus trips created by fare subsidies at the margin, for the last 100 new passengers, 34 users ($0.05/-0.4 \times 2.75$) are switching from the car to the bus, and 54 users ($0.15/-0.4 \times 1.44$),²⁵ from the motorcycle to the bus. Hence, 12 of these new bus trips are induced demand. Concerning the plausibility of these numbers, the most disputable could be the first one. According to INE (2021), 35 % of households characterized as poor

²⁵ The two formulas in brackets correspond to the diversion factors. See [footnote 24](#).

Table 4. Relevant parameters to compute the efficiency of fare subsidies (USD cents)

	Peak	Off-peak
Subsidy per trip	38.45	38.45
Subsidy per pax-km	3.20	3.20
External effects among bus passengers (pax-km) ^a	4.14	-4.98
Congestion – bus (pax-km)	0.65	0.17
Other negative externalities – bus (pax-km)	0.13	0.15
Congestion – car (pax-km)	12.21	2.57
Other negative externalities – car (pax-km)	1.72	1.72
Congestion – motorcycle (pax-km)	6.10	1.28
Other negative externalities – motorcycle (pax-km)	2.37	2.37
Bus demand own price elasticity	-0.40	-0.40
Car demand cross-bus price elasticity	0.05	0.05
Motorcycle demand cross-bus price elasticity	0.15	0.15
Car demand/bus demand	2.75	2.75
Motorcycle demand/bus demand	1.44	1.44
Fuel tax – car (pax-km)	1.52	1.52
Fuel tax – motorcycle (pax-km)	0.23	0.23
Fuel tax – bus (pax-km)	0.03	0.04

^aIn the off-peak, this value is negative as the Mohring effect is a positive externality reflecting lower waiting times; for the peak, there are two negative externalities: increasing waiting times and schedule delays.

possess one car. Most users switching from the car would come from these households, which are sensitive to price differences between modes.

If Equation (2) is equalized to zero and assuming that elasticity values remain constant, we obtain the optimal flat fare subsidy. In this case, the bus fare will be subsidized up to 78 % – greater than the current 45 %. If the fare subsidy could be optimally set according to the time of the day, the fare subsidy would marginally increase for the peak hours, and the trips during the off-peak would be free of charge.

5.1. Sensitivity analysis

We turn to analyze the robustness of our results by modifying some key parameters and assumptions.

Table 6 shows how the net benefit/cost ratio changes. The sensitivity analysis considers changes in the (i) value of travel and waiting times, (ii) statistical value of life and injuries and the value of life and limb, (iii) cross elasticities, and (iv) motorcycle car-equivalency.

The most sensitive parameter is the value of the cross-price elasticities. If these values are decreased by 30 %, fare subsidies' net benefit/cost ratio turns negative in the peak period. This would be the case if the travelers switching from cars and motorcycles to the bus were very low. In the off-peak period, the Mohring effect dominates, so subsidies provide welfare gains. Considering that more trips occur during the off-peak (64 % against 36 %), the flat fare subsidy would still yield positive efficiency benefits. This analysis shows how relevant – and contentious – cross-price elasticities are in analyzing bus fare subsidies.

Table 5. Efficiency of fare subsidies (all values in USD cents/pax-km)

	Peak	Off-peak
Subsidy (A)	-3.2	-3.2
Externalities among fellow bus passengers (B)	-4.14 (increased waiting times and increased schedule delays)	4.98 (decreasing waiting times)
Negative externalities of buses (congestion and others) net of fuel taxes (C)	$-0.65 - 0.13 + 0.03 = -0.75$	$-0.17 - 0.15 + 0.04 = -0.29$
Positive externality for passengers switching from cars to buses net of fuel taxes (D)	$-(12.2 + 1.7 - 1.52) \times 0.05 / (-0.4) \times 2.75 = 4.27$	$-(2.6 + 1.7 - 1.52) \times 0.05 / (-0.4) \times 2.75 = 0.95$
Positive externality for passengers switching from motorcycles to buses net of fuel taxes (E)	$(6.1 + 2.4 - 0.2) \times 0.15 / (-0.4) \times 1.44 = 4.45$	$(1.3 + 2.4 - 0.2) \times 0.15 / (-0.4) \times 1.44 = 1.85$
Subsidy efficiency (A + B + C + D + E) ^a	0.62	4.29
Benefit/cost ratio (subsidy efficiency (pax-km)/subsidy pax-km)	19.3 %	133.7 %

^aThis value corresponds to the net social benefit of transporting one additional bus passenger-km as given by Equation (2) in the main text.

Table 6. Changes to baseline benefit/cost ratios

	Peak	Off-peak
Baseline results	19. %	133. %
Value of travel time, waiting time, and schedule delay		
50 % decrease	-22. %	34. %
50 % increase	61. %	233. %
Cross-price demand elasticity		
30 % decrease	-62. %	107. %
10 % increase	46. %	142. %
Value of life and limb		
50 % decrease	-1. %	112. %
50 % increase	40. %	154. %
Schedule delay as a percentage of the value of travel time		
25 % decrease	33. %	133. %
25 % increase	5. %	133. %

Suppose the travel time, waiting time, and schedule delay values decrease by 50 %. In that case, once again, the net benefit/cost ratio of fare subsidies becomes negative in the peak, and marginally so if the value of life and limb is diminished by 50 %. But once again, a flat fare subsidy throughout the day is efficient in these two cases, as most trips are taken during the off-peak. In the off-peak, fare subsidies are always efficient.

This sensitivity analysis shows that the efficiency of the subsidy is never disputed for the off-peak period. The efficiency of the subsidy is more sensitive to the assumed values for the peak period. However, if the efficiency of the subsidy is computed across the day, it always provides positive welfare.

6. Discussion

We compare our point estimate results to those from Parry and Small²⁶ and Basso and Silva. Parry and Small (2009) obtained net benefit/cost ratios of 20–60 % for public transport subsidies for London, Los Angeles, and Washington metropolitan areas. In these three cases, subsidies cover at least 50 % of operating costs.²⁷ The net benefit/cost ratios reported in Parry and Small are similar for peak hours but lower for the off-peak than ours. In Parry and Small, decreasing average users' costs in the off-peak period is the more significant benefit, followed by reducing negative externalities (including congestion). Also, they conclude that net benefit/cost ratios are higher in the off-peak than in the peak. These results are in agreement with ours. When they searched for the optimal values of subsidies, they found that subsidies would cover up to 90 % of total operating costs, once again a result in accordance with ours.

Basso and Silva (2014) find that for Metropolitan Santiago, in the absence of road pricing and exclusive bus lanes, subsidies to buses should be increased up to 100 % of TCs. In our study, optimal subsidies would cover 78 % of TCs with a flat fare subsidy. We attribute this difference to the fact that we do not model distributional impacts, a point to which we now turn.

We did not address the equity dimension in our modeling results. Equity issues are closely related to general taxation. In a very *ad hoc* manner, we could have introduced a factor of less than one in Equation (2), multiplying the subsidy amount. This would have increased the net benefit/cost ratio of subsidies. On the other hand, we did not consider either the marginal costs of public funds, which would have had the opposite effect, as this would have required multiplying the value of the subsidy by a factor higher than one.²⁸ The demand model in Basso and Silva (2014) considers inequality and the marginal costs of public funds. As mentioned above, they find that a 100 % bus fare subsidy would be optimal without road pricing and dedicated bus lanes. Pavón and Rizzi (2019) and Tirachini and Proost (2017) find similar results. Equity issues were one of the factors contributing to this finding. The main reason to subsidize the bus fare in Metropolitan Asunción was to make the bus fare

²⁶ Parry and Small (2009) compared their results to previous values reported in the literature. They made clear that those comparisons could be tricky, as no previous study is as complete as theirs.

²⁷ They consider rolling stock part of operating costs for rail and underground.

²⁸ As stated by Mayeres and Proost (2001), the income distribution dimension is at the heart of the existing distortionary tax structure in every economy. The marginal cost of public funds measures the efficiency losses that distortionary taxation creates in the economy. Indeed, in models with identical individuals, the optimal tax structure consists of a poll tax.

affordable for the wider population. Hence, if an equity factor adjusted our efficiency ratios, these would likely increase.

We now move to a topic discussed in the literature on how well-targeted fare bus subsidies are. Serebrisky *et al.* (2007) state that “(s)upply side subsidies—provided to the operator—are, for the most part, neutral or regressive; while demand side subsidies—provided to the user—perform better, although many of them do not improve income distribution.” Regarding supply-side subsidies, as in Asunción, these authors made the point that this type of subsidy is regressive as most of the benefits are reaped by the well-off, as the evidence shows.²⁹ However, this analysis misses one point; it is also relevant to account for how taxes are collected regarding their incidence on the population (Kaplow, 2008). If a tax is paid only by the well-off and the collected money is spent, say, 80 % on the well-off and 20 % on the worse-off, although the disbursement of the subsidy would be regressive, as measured by a Lorenz curve, it will still be welfare improving in terms of equity. Borjesson *et al.* (2020) also call into doubt the distributional effects of public transport subsidies in Stockholm. After a detailed analysis, they conclude that subsidies are mildly progressive. However, they do not address how taxes are collected.

There is no comprehensive data to analyze how well-targeted bus fare subsidies are in Metropolitan Asunción. Information collected by the Statistical National Authority (INE, 2021) shows that those who travel by car as drivers or passengers have higher personal income than those who travel by motorcycle as drivers or passengers and those who travel by bus. Bus passengers also earn a higher personal income than motorcycle drivers and passengers. In addition, car possession increases with the level of socioeconomic status. High-income people also consume more goods and services, therefore contributing more value-added taxes and, at the same time, driving more. From this, we conjecture that bus fare subsidies most likely contribute to addressing income inequality. Bus fare subsidies should also benefit motorcycle users, providing them an alternative to travel.

We also make a warning in interpreting our results from Section 5. Efficiency-wise, our results suggest that fare subsidies should increase more in the off-peak. In a context where distributional impacts are relevant, policy decisions should not be only based on efficiency considerations. As Atkinson and Stiglitz (1976) demonstrated, the formula for designing optimal taxes – in our context, subsidies – differs when inequality aversion exists. It could even be that efficiency-wise, fare subsidies should be lower in the peak, but equity-wise, they must be higher.^{30 31} This could be the case if higher bus fares disincentivize participation in the labor market for those who are at the low or near the low end of the salary scale.

Our analysis shows that increasing frequencies in the peak period raises welfare significantly. Buses in Asunción share the streets with other vehicles without any priority in using road space. Dedicated bus lanes and traffic light priorities – low-cost projects – could

²⁹ We agree with Serebrisky *et al.* (2007) in that “it is imperative to move away from supply side subsidies toward demand side subsidies and integrate transport social concerns into wider poverty alleviation efforts.” However, if political conditions are not given for this to happen, bus fare subsidies become a simple alternative to address inequality partially.

³⁰ This result corresponds to Section 4 in Atkinson and Stiglitz (1976), where the government can only charge excise taxes. This setup is relevant in our case study, as the income tax on individuals collects very little money for the Paraguayan Treasury, as mentioned in Section 2.

³¹ This result shows up in Tirachini and Proost (2021).

improve the commercial speed of buses.³² This policy has been applied in other South American cities with success, such as Bogota, Buenos Aires, and Santiago, to name a few. According to Russo et al. (2022), dedicated lanes for buses increase commercial speed by approximately 18 % in a city as congested as Rome. Dedicated bus lanes would improve reliability, reduce cycle time, increase frequency,³³ and reduce users' costs. With reduced cycle times, negative external effects among bus users could turn even positive during peak times, and more users will be attracted to the bus, especially if overcrowding goes down. The efficiency case for bus fare subsidies would be even stronger. At the end of Section 4.1, we calculated the benefits that a marginal increase in frequencies can produce, and they are substantial.

Our analysis also shows the benefits of removing motorcyclists from the roads. This user is more cost-sensitive than car drivers and car passengers; hence, a subsidized bus transport system with a good level of service could be the only alternative to stem the increase in the use of this transport mode.

7. Conclusions

We developed a methodology to conduct a cost–benefit analysis of bus fare subsidies under financial constraints. Our methodology shows a peculiar result: under financial constraints that make it impossible to increase the bus fleet, average users' costs increase at peak periods as one more passenger increases the cost of traveling for all the other passengers, creating a negative externality. The analysis also included the motorcycle as a second private transport mode because of its widespread use in Asunción. Despite the lack of an explicit consideration of inequality, our methodology provides valuable input for decision-makers when considering fare subsidies in settings similar to Asunción.

As a policy result, we conclude that bus fare subsidies provide social benefits in Metropolitan Asunción. As in many other studies, from an efficiency standpoint, our analysis shows greater benefits from bus fare subsidies in the off-peak. Whether or not to provide higher subsidies in the off-peak than in the peak needs careful consideration, as transport authorities should also weigh in on equity issues.

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³² As shown in Basso and Silva (2014), Borjesson *et al.* (2017), and Pavón and Rizzi (2019), dedicated bus lanes would be the second-best urban transport policy. The introduction of road pricing would achieve greater social welfare.

³³ Reducing travel time by 18% would increase frequencies from approximately 12 to 17 services per hour.

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A. APPENDIX A

We differentiate Equation (1) in the main text with respect to q_b , to obtain

$$\begin{aligned} \frac{dSW}{dq_b} = & \int_0^{q_a} \frac{\partial p_a}{\partial q_b} d\rho_a + \int_0^{q_m} \frac{\partial p_m}{\partial q_b} d\rho_m + p_b(0, 0, q_b) + \left(p_a - q_a \frac{\partial c_a}{\partial q_a} - c_a - \frac{\partial Ext}{\partial q_a} \right) \frac{\partial q_a}{\partial q_b} \\ & + \left(p_m - q_m \frac{\partial c_m}{\partial q_m} - c_m - \frac{\partial Ext}{\partial q_m} \right) \frac{\partial q_m}{\partial q_b} - \left(q_b \frac{\partial c_b}{\partial q_b} + c_b + \frac{\partial Ext}{\partial q_b} \right) \end{aligned} \tag{A.1}$$

Without road tolls, $p_a = c_a$ and $p_m = c_m$. Assume modal demand functions are independent of income; hence, $\frac{\partial p_i}{\partial q_b}(q_a, q_m, q_b) = \frac{\partial p_b}{\partial q_i}(q_a, q_m, q_b)$, $i = a, m$ and the first, second, and third terms of the above equations become

$$\begin{aligned} \int_0^{q_a} \frac{\partial p_a}{\partial q_b} d\rho_a + \int_0^{q_m} \frac{\partial p_m}{\partial q_b} d\rho_m + p_b(0, 0, q_b) &= \int_0^{q_a} \frac{\partial p_b}{\partial q_a} d\rho_a + \int_0^{q_m} \frac{\partial p_b}{\partial q_m} d\rho_m + p_b(0, 0, q_b) = \\ &= p_b(q_a, q_m, q_b) - p_b(0, q_m, q_b) + p_b(0, q_m, q_b) - p_b(0, 0, q_b) + p_b(0, 0, q_b) \\ &= p_b(q_a, q_m, q_b) \end{aligned}$$

The average passenger cost is the sum of the operator’s average cost per passenger and the user’s average cost $c_b = c_{op_b} + c_{u_b}$. The user average cost comprises waiting time, travel time, and schedule delay cost. Equation (A.1) can now be written as

$$\begin{aligned} \frac{dSW}{dq_b} = p_b - \left(q_a \frac{\partial c_a}{\partial q_a} + \frac{\partial Ext}{\partial q_a} \right) \frac{\partial q_a}{\partial q_b} - \left(q_m \frac{\partial c_m}{\partial q_m} + \frac{\partial Ext}{\partial q_m} \right) \frac{\partial q_m}{\partial q_b} \\ - \left(q_b \frac{\partial c_b}{\partial q_b} + c_{op_b} + c_{u_b} + \frac{\partial Ext}{\partial q_b} \right) \end{aligned} \tag{A.2}$$

The subsidy fare is the difference (S_b) between the operator’s average cost and the fare. Also, $p_b = c_{u_b} + \text{fare}$; hence, $p_b - c_{op_b} - c_{u_b} = \text{fare} - c_{op_b} = -S_b$. Multiplying and dividing appropriately to complete the elasticity values, we obtain the diversion factors – the percentage of new bus users switching from the i mode – $\frac{\partial q_i}{\partial q_b} = \frac{\varepsilon_i^{p_b}}{\varepsilon_b^{p_b}} \frac{q_i}{q_b}$, $i = a, m$. Our last assumption $q_b \frac{\partial c_b}{\partial q_a} = q_b \frac{\partial c_{u_b}}{\partial q_b}$ implies that operator average costs per passenger are constant. Replacing these values in Equation (A.2), we arrive at Equation (2) in the text.

$$\frac{dWS}{dq_b} = -S_b - q_b \frac{\partial c_b}{\partial q_a} - \frac{\partial Ext}{\partial q_b} - \left(q_a \frac{\partial c_a}{\partial q_a} + \frac{\partial Ext}{\partial q_a} \right) \frac{\varepsilon_a^{p_b}}{\varepsilon_b^{p_b}} \frac{q_a}{q_b} - \left(q_m \frac{\partial c_m}{\partial q_m} + \frac{\partial Ext}{\partial q_m} \right) \frac{\varepsilon_m^{p_b}}{\varepsilon_b^{p_b}} \frac{q_m}{q_b}$$