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MIXING PROPERTIES FOR STIT TESSELLATIONS

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Abstract

The so-called STIT tessellations form a class of homogeneous (spatially stationary) tessellations in $\mathbb{R}^d$ which are stable under the nesting/iteration operation. In this paper we establish the mixing property for these tessellations and give the decay rate of $\mathrm{P}(A \cap M = \varnothing,\ T_h B \cap M = \varnothing)/\mathrm{P}(A \cap Y = \varnothing)\,\mathrm{P}(B \cap Y = \varnothing) - 1$, where A and B are both compact connected sets, h is a vector of $\mathbb{R}^d$, T_h is the corresponding translation operator, and M is a STIT tessellation.

Keywords: Stochastic geometry; random tessellation; STIT tessellation; space ergodicity; mixing property

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1. Introduction and notation

Random tessellations, or mosaics, form an important class of objects from stochastic geometry. They have proven to be a useful tool for modelling geometrical structures appearing in biology, geology, and medical sciences. The Poisson hyperplane tessellation, the Poisson–Voronoi tessellation, and its dual, the Poisson–Delaunay tessellation, are the most celebrated and tractable models investigated to date, and all are defined using a Poisson point process on an appropriate space. In the 1980s, Ambartzumian had the idea of applying an operation to the mosaics of $\mathbb{R}^d$, namely the operation of iteration (also called the nesting operation). In $\mathbb{R}^2$, the class of T-noded tessellations is stable under iteration, while the class of X-noded tessellations, such as the Poisson line tessellation, is stable under superposition. Nagel, Mecke, and Weiss (see [6], [7], and [8]) introduced the STIT (stable under iteration) tessellation model, motivated by Cowan's [3] work. The STIT tessellation can be used as a model for crack patterns, such as those seen on old pottery or on drying soil.

The STIT tessellation is homogeneous, i.e. space stationary, and a proper choice of parameters can make it isotropic, which yields a very interesting model. Many geometrical features of STIT tessellations have been investigated, including moments of variables related to typical faces of the tessellation. Cowan [1], [2] emphasised the importance of ergodic properties for random tessellations. In this paper we establish that all STIT tessellations possess the mixing property, which implies ergodicity. Namely, if A and B are two Borel sets, and M is the closed set of boundaries of the cells of a STIT tessellation, then

$$\mathrm{P}(M \cap A = \varnothing,\ M \cap T_h B = \varnothing) - \mathrm{P}(M \cap A = \varnothing)\,\mathrm{P}(M \cap B = \varnothing) \to 0 \tag{1}$$

as $\|h\| \to +\infty$, where $T_h B$ is the set B translated by the vector h. In the general case, the decay is $o(\|h\|^{-1})$.

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The paper is organised as follows. In the rest of this section we introduce the notation we will use in the following sections. In Section 2 we give a brief description of the construction and the properties of STIT tessellations. In Section 3 we establish mixing properties for STIT tessellations. Section 4 contains the proofs of the results obtained in the paper.

1.1. Notation

In the sequel, $\mathrm{int}(A)$ denotes the interior of a set A, $\mathrm{conv}(A)$ denotes its convex hull, ∂A denotes its boundary, and $\mathrm{span}(A)$ denotes the smallest subspace of $\mathbb{R}^d$ containing A.

Since we are interested in hitting and missing probabilities here, we introduce the corresponding families. For $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\mathcal{F}_A = \{C \in \mathcal{F} : C \cap A \neq \varnothing\}, \qquad \mathcal{F}^A = \{C \in \mathcal{F} : C \cap A = \varnothing\}.$$

Denote by $\mathcal{K}$ the class of compact sets of $\mathbb{R}^d$. The *topology of closed convergence* on $\mathcal{F}$, or the *Fell topology*, is the topology generated by $\mathcal{F}^K$, $K \in \mathcal{K}$, and $\mathcal{F}_O$, for open O. For any subclass $\mathcal{C} \subset \mathcal{F}$, we can define the induced Fell topology and the corresponding Borel σ-algebra $\mathcal{B}(\mathcal{C})$. It is known that the σ-algebra $\mathcal{B}(\mathcal{F})$ is generated by the $\mathcal{F}^K$, $K \in \mathcal{K}$; see [9, Lemma 2.1.1]. Let us give definitions for tessellations and random tessellations.

A tessellation of $\mathbb{R}^d$ is a countable set R of convex polytopes of $\mathbb{R}^d$ that satisfies

(i) for all $C \in R$, $\mathrm{int}(C) \neq \varnothing$;

(ii) $\bigcup_{C\in R} C = \mathbb{R}^d$;

(iii) for all $C, C' \in R$ such that $C \neq C'$, we have $\mathrm{int}(C) \cap \mathrm{int}(C') = \varnothing$;

(iv) for all $K \in \mathcal{K}$, $\mathrm{card}(\mathcal{F}_K \cap R) < \infty$.

It is equivalent and more convenient to work with the closed set $M = \bigcup_{C\in R} \partial C$ formed by the boundaries of the cells of R. Reciprocally, it is possible to retrieve R from M by taking the closures of all connected components of M's complement. Call $\mathcal{M}$ the subclass of $\mathcal{F}$ formed by all tessellations.

A random tessellation is a random element with values in $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$. According to the Choquet theorem (see [5] and [9, Theorem 2.2.1]), the law P of a random closed set M is characterised by its capacity functional, defined by

$$\mathfrak{T}_M(K) = \mathrm{P}(\mathcal{F}_K), \qquad K \in \mathcal{K}.$$

The equality in law, denoted by '$\overset{\mathrm{D}}{=}$', occurs if and only if there is equality of the corresponding capacity functionals.

2. STIT tessellations

A STIT tessellation, as a Poisson hyperplane tessellation, is constructed from a hyperplane Poisson process. Details concerning the construction of STIT tessellations can be found in [6], [7], and [8].

We give here the construction of a STIT tessellation seen through a compact window W. We do not use the elegant construction from a hyperplane Poisson process, because it is less explicit, but the reader is referred to [7] for such a description. Let us begin with some definitions.

Call $\mathcal{H}$ the class of all hyperplanes of $\mathbb{R}^d$. For $A \in \mathcal{B}$, set

$$[A] = \mathcal{H} \cap \mathcal{F}_A,$$

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the family of all hyperplanes hitting A. For another Borel set B, define $[A|B]$ as the family of all hyperplanes γ that strictly separate A and B, i.e. such that A and B are contained in different open half-spaces of $\mathbb{R}^d \setminus \gamma$. We write in short, for $x, y \in \mathbb{R}^d$,

$$[x] = [\{x\}], \qquad [x|y] = [\{x\}|\{y\}].$$

Given $\gamma \in \mathcal{H} \setminus [0]$, denote by γ^+ the closed half-space delimited by γ not containing 0, and denote by γ^- the other closed half-space. Let $\mathcal{S}^{d-1}$ be the unit sphere of $\mathbb{R}^d$. We identify $\mathcal{H}$ with $\mathbb{R}^+ \times \mathcal{S}^{d-1}$, where an element (r, u) of $\mathbb{R}^+ \times \mathcal{S}^{d-1}$ stands for the hyperplane γ at distance r from the origin with normal exterior vector u (i.e. a normal vector directed towards γ^+). If $\gamma \in [0]$, we take the arbitrary convention that γ^+ is the half-space that contains $(1, 0, \dots, 0)$ in its interior, or $(0, \dots, 0, 1)$ if $(1, 0, \dots, 0) \in \gamma$.

Equip $\mathcal{H}$ with the topology inherited from this identification and the corresponding σ-algebra. Let Λ be a translation invariant locally finite measure on $\mathcal{H}$. Note that $\Lambda([0]) = 0$, since Λ is stationary and locally finite. By stationarity we can write, along with the identification,

$$\Lambda = \lambda^+ \otimes \nu,$$

where λ^+ is the Lebesgue measure restricted to $\mathbb{R}^+$ and ν is a finite measure on $\mathcal{S}^{d-1}$. Also, assume that

$$\operatorname{span}(\operatorname{supp}(\nu)) = \mathbb{R}^d. \tag{2}$$

The latter property ensures that tessellations generated by the hyperplane processes with distribution Λ will almost surely have compact cells (whether it is a STIT tessellation or a Poisson hyperplane tessellation).

Let a be a positive number. On a compact window $W \in \mathcal{K}$ with nonempty interior, we define the tessellation with parameters a and Λ, where a is the time parameter, as a stochastic process taking values in $\mathcal{M}$. Let $\mathbb{B}$ be Young's binary infinite tree, where the leaves are labelled so that the top leaf is attributed the number 1, and the k-labelled leaf has $2k$- and $(2k+1)$-labelled daughter leaves. We now attach to each leaf a pair (ϵ_k, γ_k), where $(\epsilon_k)_{k\ge 1}$ is a family of independent and identically distributed (i.i.d.) random exponential variables with parameter $\Lambda([W])$, and the $(\gamma_k)_{k\ge 1}$ are i.i.d. random hyperplanes with law $\Lambda([W] \cap \cdot)/\Lambda[W]$.

The tessellation is defined as a process of cell division, where each cell is identified with a leaf of the tree, and the first cell is the window $C_1 = W$ itself. We describe the tesselation in terms of a birth-and-death process, where a cell dies when it is divided into two daughter cells. The death times (d_k), birth times (b_k), and daughter cells of $(C_k)_{k\ge 1}$ are defined recursively as follows (here $\lfloor s \rfloor$ is the integer part of s):

$$b_1 = 0, \qquad b_k = d_{\lfloor k/2 \rfloor}, \qquad d_k = b_k + \epsilon_k, \qquad C_{2k} = C_k \cap \gamma_k^-, \qquad C_{2k+1} = C_k \cap \gamma_k^+.$$

We then define

$$M_{a,W} = \bigcup_{b_k \le a} \partial C_k.$$

In the sequel we denote the law of this tessellation by $\mathrm{P}_{a,W}$. Note that Λ is an implicit parameter of the model. We also write for short

$$\mathfrak{T}_{a,W} = \mathfrak{T}_{M_{a,W}} \quad \text{and} \quad \mathfrak{U}_{a,W} = 1 - \mathfrak{T}_{a,W}.$$

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Remarks. 1. The only nonintuitive feature of this construction is that γ_k might not be in $[C_k]$, which would mean that one daughter cell is $\varnothing$ and the other is C_k itself. There is no theoretical objection to this feature, but it can be remedied in the following way. Instead of choosing an i.i.d. family (ϵ_k), independently attach to each cell C_k the death rate $\Lambda([C_k])$ and a hyperplane drawn from $\Lambda([C_k] \cap \cdot)/\Lambda[C_k]$. The resulting law of the tessellation is not modified and each hyperplane indeed hits the cell to which it is attached.

2. If Λ is isotropic, the death rate $\Lambda([C])$ of a cell C is proportional to the perimeter of C.

3. We now mention a construction from a Poisson hyperplane process. Let Θ be a Poisson process on $\mathbb{R}^+ \times \mathcal{H}$ with intensity $\Lambda([W])\lambda^+ \otimes \Lambda([W] \cap \cdot)/\Lambda[W]$. Since, almost surely, for all $t \geq 0$, $\operatorname{card}((\{t\} \times \mathcal{H}) \cap \Theta) \leq 1$, we can define a random sequence $(\tau_k, \gamma_k)_{k \geq 1}$ such that $\tau_k < \tau_{k+1}$ for all $k \geq 1$, $\Theta = \{(\tau_k, \gamma_k) \mid k \geq 1\}$, and $(\tau_{k+1} - \tau_k, \gamma_k)_{k \geq 1}$ satisfy the hypotheses of the previous construction almost surely.

Nagel and Weiss [8] established the consistency property.

Theorem 1. *If $W \subset W'$ are two compact sets with nonempty interiors then*

$$M_{a,W} \cap \operatorname{int}(W) \stackrel{\mathrm{D}}{=} M_{a,W'} \cap \operatorname{int}(W).$$

The statement of Theorem 1 means that, for any compact set $K \subset \operatorname{int}(W)$, $\mathfrak{T}_{a,W}(K) = \mathfrak{T}_{a,W'}(K)$, and on $\mathcal{K}$ we can define

$$\mathfrak{T}_a(K) = \mathfrak{T}_{a,W}(K), \qquad \mathfrak{U}_a(K) = \mathfrak{U}_{a,W}(K), \qquad \text{for any } W \text{ containing } K \text{ in its interior.}$$

Theorem 2.3.1 of [9], which is a consequence of the Choquet theorem, allows us to define a tessellation M_a on the whole space, with capacity functional $\mathfrak{T}_a$, such that

$$M_a \cap \operatorname{int}(W) \stackrel{\mathrm{D}}{=} M_{a,W} \cap \operatorname{int}(W), \qquad W \in \mathcal{K},$$

which is called the STIT tessellation with parameters a and Λ. Denote by P_a its law on $\mathcal{M}$. Mecke *et al.* [6] also provided an explicit global construction on $\mathbb{R}^d$, i.e. without the help of a general extension theorem.

3. Mixing property

Here $\Lambda = \lambda^+ \otimes \nu$ is a measure on $\mathcal{H}$ satisfying assumption (2) and a is a strictly positive number.

Consider the set $\mathcal{T} = \{T_h; h \in \mathbb{R}^d\}$ of all translations, seen as operators on $\mathcal{F}$. Their action is naturally lifted to $\mathcal{B}(\mathcal{F})$, for which we keep the same notation. The stationarity of a random closed set X with law P means that, for all $T \in \mathcal{T}$, P is invariant under T. Now we call every Borel set $\mathcal{C}$ of the σ-algebra $\mathcal{B}(\mathcal{F})$ a $\mathcal{T}$-invariant set of $\mathcal{B}(\mathcal{F})$ if, for all T in $\mathcal{T}$, $T\mathcal{C} = \mathcal{C}$. For instance, $\mathcal{K}$, $\mathcal{H}$, $\mathcal{M}$, or 'the class of all tessellations having a cube as one of their cells' are invariant sets. Given a stationary law P on $\mathcal{F}$, the dynamical system $(\mathcal{F}, \mathcal{B}(\mathcal{F}), \mathrm{P}, \mathcal{T})$ is said to be *ergodic* if every $\mathcal{T}$-invariant set has probability 0 or 1, and *mixing* if

$$\mathrm{P}(\mathcal{C} \cap T_h\mathcal{C}') \to \mathrm{P}(\mathcal{C})\,\mathrm{P}(\mathcal{C}') \qquad \text{as } \|h\| \to +\infty \tag{3}$$

for all $\mathcal{C}$, $\mathcal{C}'$ in $\mathcal{F}$. If X is a random closed set with law P, we simply speak of the stationarity, the ergodicity, or the mixing property for X. Roughly speaking, the mixing property yields the asymptotic independence of relatively distant sets. Since any $\mathcal{T}$-invariant set $\mathcal{C}$ satisfies $\mathrm{P}(\mathcal{C} \cap T_h \mathcal{C}) = \mathrm{P}(\mathcal{C})$, the mixing property implies ergodicity.

According to Lemma 9.3.1 of [9], it suffices to show (3) for sets $\mathcal{C}, \mathcal{C}'$ drawn from a semialgebra $\mathcal{A}$ generating $\mathcal{B}(\mathcal{F})$. Therefore, the mixing property for M_a is a consequence of the following theorem.

Theorem 2. *For all* $A, B \in \mathcal{B}(\mathbb{R}^d)$,

$$\mathrm{P}_a(\mathcal{F}^A, \mathcal{F}^{T_h B}) \to \mathrm{P}_a(\mathcal{F}^A)\,\mathrm{P}_a(\mathcal{F}^B) \quad \textit{as } \|h\| \to \infty.$$

As $T_h \mathcal{F}^B = \mathcal{F}^{T^{-h} B}$, we state the theorem with h instead of $-h$ for more simplicity, since the role of h is symmetric. Since the $\{\mathcal{F}^K;\, K \in \mathcal{K}\}$ also generate the Borel σ-algebra of the Fell topology, with the help of Theorem 9.3.2 of [9], which is a consequence of [9, Lemma 9.3.1], it suffices to show Theorem 2 for $A, B \in \mathcal{K}$. (It was first proved in [4].) Thus, this theorem is a consequence of Theorem 3 below.

Theorem 3. *For all* $A, B \in \mathcal{K}$,

$$|\mathrm{P}_a(\mathcal{F}^A, \mathcal{F}^{T_h B}) - \mathrm{P}_a(\mathcal{F}^A)\,\mathrm{P}_a(\mathcal{F}^B)| = O(\|h\|^{-1}). \tag{4}$$

To give this bound, we have to estimate the Lipschitz constant for the capacity functional as a function of a. We obtain the following result.

Proposition 1. *For all compact sets K, there exists a $\beta_{K,a} > 0$ such that*

$$0 \le \mathfrak{T}_{a+t}(K) - \mathfrak{T}_a(K) \le t\beta_{K,a}, \qquad t > 0. \tag{5}$$

Furthermore, $\beta_{K,a}$ is invariant under rigid motions of K.

4. Proofs

4.1. Proof of Proposition 1

Lemma 1. *Let $0 < t, a$ be real positive numbers, let K be a compact set, and let*

$$\beta_{K,a} = \Lambda([\mathrm{conv}(K)])(1 + a\Lambda([\mathrm{conv}(K)]))(1 - \mathfrak{T}_a(K)).$$

Then $0 \le \mathfrak{T}_{a+t}(K) - \mathfrak{T}_a(K) \le t\beta_{K,a}$.

Proof. We use the notation of Section 2, with $W = \mathrm{conv}(K)$. Let $N_a = \mathrm{card}\{k;\, \varepsilon_k \le a\}$ denote the number of times $\mathrm{conv}(K)$ has been 'hit' by a hyperplane up to time a (if the hyperplane falls outside $\mathrm{conv}(K)$, it is still counted as a hit). For $n \in \mathbb{N}$, set $\pi_n(a) = \mathrm{P}(N_a = n)$. Almost surely, conditionally on $(N_a = n)$, $M_{a,W}$ has $n+1$ cells $C_1, \dots, C_{n+1}$, not necessarily distinct, with nonempty interiors. Define $K_i = K \cap C_i$. Then

$$\begin{aligned}
&\mathfrak{T}_{a+t}(K) - \mathfrak{T}_a(K) \\
&\quad = \mathfrak{U}_a(K) - \mathfrak{U}_{a+t}(K) \\
&\quad = \mathfrak{U}_a(K)\,\mathrm{P}(K \cap M_{a+t} \ne \varnothing \mid K \cap M_a = \varnothing) \\
&\quad = \mathfrak{U}_a(K) \\
&\qquad \times \sum_{n \ge 0} \pi_n(a)\,\mathrm{P}(\text{there exists } i,\ 1 \le i \le n+1,\ K_i \cap M_{a+t} \ne \varnothing \mid N_a = n,\ K \cap M_a = \varnothing) \\
&\quad \le \mathfrak{U}_a(K) \sum_{n \ge 0} \pi_n(a) \sum_{i=1}^{n+1} \mathrm{P}(K_i \cap M_{a+t} \ne \varnothing \mid N_a = n,\ K \cap M_a = \varnothing).
\end{aligned}$$

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Let us now recall some essential facts concerning the construction of a STIT tessellation. When a cell is divided, the two daughter cells behave independently. Moreover, the division process of a cell has no memory, which means that the probability that it is divided at time $a+t$ knowing it has not been divided at time a equals the probability that it is divided within a period of length t. Since, for each i, K_i is contained in the convex cell C_i, the probability that K_i is divided within a certain period equals the probability that the STIT tessellation defined on C_i touches K_i within a period with the same length. Finally, owing to the consistency property, conditionally on $C_i = C_i^0$, for some C_i^0 in $\mathcal{K}$, the latter probability does not change if the tessellation is considered a random object of $\mathbb{R}^d$ (meaning that the value of C_i^0 does not matter). Thus, we have, for every compact K,

$$\begin{aligned} \mathrm{P}(\mathrm{conv}(K_i) \cap M_{a+t} \neq \varnothing \mid N_a = n,\ \mathrm{conv}(K_i) \cap M_a = \varnothing) &= 1 - \mathfrak{U}_t(\mathrm{conv}(K_i)) \\ &\leq 1 - \mathfrak{U}_t(\mathrm{conv}(K)) \\ &\leq 1 - \mathrm{e}^{-t\Lambda([\mathrm{conv}(K)])}. \end{aligned}$$

This yields

$$\begin{aligned} \mathfrak{U}_a(K) - \mathfrak{U}_{a+t}(K) &\leq \mathfrak{U}_a(K) \sum_{n\geq 0} \pi_n(a) \sum_{i=1}^{n+1} (1 - \mathrm{e}^{-t\Lambda([\mathrm{conv}(K)])}) \\ &\leq \mathfrak{U}_a(K) \sum_{n\geq 0} \pi_n(a) t \sum_{i=1}^{n+1} \Lambda([\mathrm{conv}(K)]) \\ &\leq \mathfrak{U}_a(K) \sum_{n\geq 0} \pi_n(a)(n+1) t \Lambda([\mathrm{conv}(K)]) \\ &\leq \mathfrak{U}_a(K) t \Lambda([\mathrm{conv}(K)])(1 + \mathrm{E}(N_a)). \end{aligned}$$

The number of hyperplanes involved in the cell division process with initial cell $\mathrm{conv}(K)$ up to time a is a Poisson variable with parameter $a\Lambda([\mathrm{conv}(K)])$. Therefore,

$$\mathrm{E}(N_a) = a\Lambda([\mathrm{conv}(K)]),$$

which concludes the proof.

4.2. Proof of Theorem 3

We first establish some inequalities for compact sets A and B, and then add a drift h to give an upper bound when the expressions become too complicated.

Let $W = \mathrm{conv}(A \cup B)$. The key is to consider the tessellation-valued time process $(M_{t,W})_{t\geq 0}$ defined in Section 2. We have the almost-sure identity

$$\epsilon_1 = \inf\{t : M_{0,W} \neq M_{t,W}\}.$$

Let γ_1 be the first hyperplane dividing $C_1 = W$. We introduce the event

$$\Gamma_{A,B} = \{\gamma_1 \in [A|B],\ \epsilon_1 \leq a\}.$$

If A and B are far away from each other, $\Gamma_{A,B}$ is likely to happen, and after $\Gamma_{A,B}$ occurs, the tessellations inside A and B behave independently because they are in disjoint cells. Then

$$\begin{aligned} |\mathrm{P}_a(\mathcal{F}^A, \mathcal{F}^B, \Gamma_{A,B}) - \mathrm{P}_a(\mathcal{F}^A, \mathcal{F}^B)| &\leq \mathrm{P}(\gamma_1 \notin [A|B]) + \mathrm{P}_a(\varepsilon_1 > a) \\ &\leq \frac{\Lambda([A]) + \Lambda([B])}{\Lambda([W])} + \mathrm{e}^{-a\Lambda([W])}. \end{aligned} \tag{6}$$

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Consequently, we can show (4) with $\mathrm{P}_a(\mathcal{F}^A, \mathcal{F}^{T_h B}, \Gamma_{A,T_h B})$ instead of $\mathrm{P}_a(\mathcal{F}^A, \mathcal{F}^{T_h B})$, because their difference has the same magnitude than the expected decay rate. Rigorously, this gives

$$\begin{aligned}\mathrm{P}_a(\mathcal{F}^A, \mathcal{F}^B, \Gamma_{A,B}) &= \int_0^a \mathrm{P}_a(\epsilon_1 \in \mathrm{d}t,\ \gamma_1 \in [A|B], \mathcal{F}^A, \mathcal{F}^B) \\ &= \int_0^a \mathrm{P}_a(\epsilon_1 \in \mathrm{d}t,\ \gamma_1 \in [A|B])\, \mathrm{P}_a(\mathcal{F}^A, \mathcal{F}^B \mid \epsilon_1 = t,\ \gamma_1 \in [A|B]).\end{aligned}$$

If $\epsilon_1 = t$ and $\gamma_1 \in [A|B]$, then A and B are not hit up to time t, and are both still contained in a cell (but the cell encapsulating A is different from that encapsulating B). Owing to the consistency property, and the independent behaviour of distinct cells after their birth, we have

$$\mathrm{P}_a(\mathcal{F}^A, \mathcal{F}^B \mid \epsilon_1 = t,\ \gamma_1 \in [A|B]) = \mathfrak{U}_{a-t}(A)\mathfrak{U}_{a-t}(B).$$

Since the sequence $(\gamma_k)_{k\geq 1}$ is independent of $(\epsilon_k)_{k\geq 1}$, we have

$$\begin{aligned}\mathrm{P}_a(\epsilon_1 \in \mathrm{d}t, \gamma_1 \in [A|B]) &= \mathrm{P}_a(\epsilon_1 \in \mathrm{d}t)\, \mathrm{P}_a(\gamma_1 \in [A|B]) \\ &= \Lambda([W])\mathrm{e}^{-t\Lambda([W])}\, \mathrm{d}t \frac{\Lambda([A|B])}{\Lambda([W])},\end{aligned}$$

and, finally, we have

$$\mathrm{P}_a(\mathcal{F}^A, \mathcal{F}^B, \Gamma_{A,B}) = \Lambda([A|B]) \int_0^a \mathrm{e}^{-t\Lambda([W])}\mathfrak{U}_{a-t}(A)\mathfrak{U}_{a-t}(B)\mathrm{d}t. \tag{7}$$

We give the following relation, which will be useful in subsequent estimations:

$$\Lambda([W]) = \Lambda([A|B]) + \Lambda([A]) + \Lambda([B]) - \Lambda([A] \cap [B]). \tag{8}$$

In order to make a proper upper bound estimate, consider a translation of B. In what follows, we use $T_h B$ instead of B and let $W_h = \mathrm{conv}(\cdot)$. For $u \in \mathcal{S}^d$, define $\zeta(u) = \Lambda([0, u]) = \Lambda([0|u])$ and let

$$\xi(h) = \|h\|\zeta\left(\frac{h}{\|h\|}\right), \qquad h \in \mathbb{R}^d \setminus \{0\}.$$

It is a standard fact from integral geometry that ζ is continuous, and it does not vanish because of assumption (2).

Lemma 2. *We have*

$$\Lambda([W_h]) = \xi(h) + o(\|h\|) \quad \textit{and} \quad \Lambda([A|T_h B]) = \xi(h) + o(\|h\|).$$

Proof. Let $\alpha \in A$ and $\beta \in B$. A hyperplane hitting W_h that does not separate α and $\beta + h$ either hits $\mathrm{conv}(A)$ or $\mathrm{conv}(T_h B)$. Hence,

$$\begin{aligned}|\Lambda([W_h]) - \Lambda([\alpha, \beta + h])| &\leq \Lambda([\mathrm{conv}(A)]) + \Lambda([\mathrm{conv}(T_h B)]) \\ &\leq \Lambda([\mathrm{conv}(A)]) + \Lambda([\mathrm{conv}(B)]) \\ &= o(\|h\|).\end{aligned}$$

Similarly,

$$|\Lambda([A|T_h B]) - \Lambda([\alpha, \beta + h])| = o(\|h\|).$$

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We only need to prove that

$$\Lambda([\alpha, \beta + h]) = \zeta\left(\frac{h}{\|h\|}\right)\|h\|(1 + o(1)).$$

It suffices to show that if $\beta - \alpha = \|\beta - \alpha\|u$, with $u \in \mathcal{S}^{d-1}$,

$$\Lambda([\alpha|\beta]) = \|\beta - \alpha\|\zeta(u).$$

This would follow from

$$\Lambda([0|(n+1)\epsilon u]) = \Lambda([0|n\epsilon u]) + \Lambda([0|\epsilon u])$$

for all $\epsilon > 0$, $u \in \mathcal{S}^{d-1}$, and $n \in \mathbb{N}$. Indeed, since Λ is stationary and locally finite, for all $x \in \mathbb{R}^d$, $\Lambda([x]) = 0$, and then we will be able to obtain the result by induction and approximation. Since $[0|(n+1)\epsilon u]$ is the disjoint union of $[0|n\epsilon u]$, $[n\epsilon u|(n+1)\epsilon u]$, and $[n\epsilon u]$, we have

$$\Lambda([0|(n+1)\epsilon u]) = \Lambda([0|n\epsilon u]) + \Lambda([n\epsilon u|(n+1)\epsilon u]),$$

and the result follows by stationarity of Λ.

The continuity of ζ yields

$$\Lambda([W_h]) = \|\alpha - \beta + h\|\zeta\left(\frac{\alpha - \beta + h}{\|\alpha - \beta + h\|}\right)(1 + o(1)) = \|h\|\zeta\left(\frac{h}{\|h\|}\right)(1 + o(1)).$$

Thus, $\Lambda([W_h]) = O(\|h\|^{-1})$, and (7) gives

$$\begin{aligned}
&|\mathrm{P}_a(\mathcal{F}^A, \mathcal{F}^{T_h B}, \Gamma_{A,T_h B}) - \mathrm{P}_a(\mathcal{F}^A)\,\mathrm{P}_a(\mathcal{F}^{T_h B})| \\
&\quad\le \Lambda([A|T_h B]) \int_0^\infty \mathrm{e}^{-t\Lambda([W_h])} |\mathfrak{U}_{a-t}(A)\mathfrak{U}_{a-t}(T_h B) - \mathfrak{U}_a(A)\mathfrak{U}_a(T_h B)|\mathrm{d}t \\
&\qquad + \left|\frac{\Lambda([A|T_h B])}{\Lambda([W_h])} - 1\right| \int_0^\infty \mathrm{e}^{-t\Lambda([W_h])}\mathfrak{U}_a(A)\mathfrak{U}_a(T_h B)\Lambda([W_h])\mathrm{d}t \\
&\qquad + O(\Lambda([W_h])^{-1}\mathrm{e}^{-a\Lambda([W_h])}).
\end{aligned} \tag{9}$$

Using (5) and (8), (9), and Lemma 2, we obtain

$$\begin{aligned}
&|\mathrm{P}_a(\mathcal{F}^A, \mathcal{F}^{T_h B}, \Gamma_{A,T_h B}) - \mathrm{P}_a(\mathcal{F}^A)\,\mathrm{P}_a(\mathcal{F}^B)| \\
&\quad\le \Lambda([A|T_h B])(\lambda_{A,a} + \lambda_{B,a}) \int_0^\infty \mathrm{e}^{-t\Lambda([W_h])} t\,\mathrm{d}t + \left(1 - \frac{\Lambda([A|T_h B])}{\Lambda([W_h])}\right) + o(\|h\|^{-1}) \\
&\quad\le \frac{(\lambda_{A,a} + \lambda_{B,a})\Lambda([A|T_h B])}{\Lambda([W_h])^2} + \frac{\Lambda([\mathrm{conv}(A)]) + \Lambda([\mathrm{conv}(T_h B)])}{\Lambda([W_h])} + o(\|h\|^{-1}) \\
&\quad= O(\|h\|^{-1}).
\end{aligned}$$

Thus, (4) is proved.

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