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On Hessian Limit Directions along Gradient Trajectories

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Abstract. Given a non-oscillating gradient trajectory $|\gamma|$ of a real analytic function f, we show that the limit ν of the secants at the limit point $\mathbf{0}$ of $|\gamma|$ along the trajectory $|\gamma|$ is an eigenvector of the limit of the direction of the Hessian matrix $\operatorname{Hess}(f)$ at $\mathbf{0}$ along $|\gamma|$. The same holds true at infinity if the function is globally sub-analytic. We also deduce some interesting estimates along the trajectory. Away from the ends of the ambient space, this property is of metric nature and still holds in a general Riemannian analytic setting.

1 Introduction

The famous Thom Gradient Conjecture states that *a gradient trajectory of a real analytic function has a tangent at its ω-limit point* $\mathbf{0}$. It was eventually proved by Kurdyka, Mostowski, and Parusiński [9].

When trying to address the asymptotic analytical properties of gradient trajectories of real analytic functions at their limit point, nothing else is known within the whole category of real analytic functions.

Nevertheless, there exist gradient trajectories presenting a rigid asymptotic regularity (yet to be fully explored) in the following sense.

A gradient trajectory γ of a real analytic function germ $f\colon (\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}, 0)$ is *semi-analytically non-oscillating* at its limit point $\mathbf{0}$, a critical point of the function, if, given any semi-analytic set H (germ at $\mathbf{0}$), either the trajectory γ is contained in H, or it does not intersect with H nearby its limit point $\mathbf{0}$.

Since bounded plane gradients curves are definable in the Pfaffian closure of $\mathbb{R}_{an}$, they are non-oscillating at their limit point. In dimension at least 3, there are always trajectories that are analytic half-branches at their limit point [13], thus also non-oscillating.

Whether any gradient trajectory is oscillating or not at its limit point is a very hard problem in general, which is not understood beyond the special cases dealt with in [5, 8, 15].

The aim of this note is to present a property happening along gradient trajectories that are non-oscillating at their limit point. It is also a somewhat unexpected new result, since, to our best knowledge, it was not even known in the plane case, where it applies. Our main result, Theorem 5.5, is of metric nature. Although it is just stated below for the Euclidean metric, it actually holds true in any analytic Riemannian setting (see Section 8). This result provides some useful information along

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non-oscillating gradient trajectories about where and how a gradient trajectory terminates. Our main result is the following theorem.

Theorem 5.5 *Let γ be a gradient trajectory of a real analytic function $f\colon (\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}, 0)$. Assume the trajectory γ is non-oscillating at its ω-limit point $\mathbf{0}$, a critical point of the function f. Let $\nu \in \mathbf{S}^{n-1}$ be the limit at $\mathbf{0}$ of the secant lines along γ.*

(i) *The oriented direction of the matrix* Hess(f) *along γ has a (non-zero) limit $\mathcal{H}$ at $\mathbf{0}$.*

(ii)
$$\lim_{|\gamma| \ni x \to \mathbf{0}} \frac{\mathrm{Hess}(f)(x) \cdot \nabla f(x)}{|\mathrm{Hess}(f)(x) \cdot \nabla f(x)|} = \nu.$$

Thus the unit vector ν is an eigenvector of $\mathcal{H}$.

The paper is organized as follows.

Section 2 introduces the few useful definitions, notions, and notations we will deal with. In Section 3 we present very quickly the framework in which we will operate, and recall a technical result from [9] that is interesting for us. Section 4 suggests that the result presented here may not be so surprising in the lights of some elementary examples emphasizing some links between gradient trajectories and Hessian matrices, which, we believe, are known folklore. The main result is stated and proved in Section 5. It is followed by a few corollaries in Section 6 presenting some estimates along the trajectory. Section 7 deals with the analog result at the limit point at infinity of a non-oscillating (at infinity) unbounded gradient trajectory of a globally sub-analytic real analytic function. We also provide the corresponding estimates expected from those found in the previous section. In Section 8 we state and prove our main result in the general (real analytic) Riemannian case. The last section presents some remarks and also briefly discusses the question in the wider setting of an o-minimal structure over the real numbers.

2 Background–Notations

Let $\gamma\colon I \to \mathbb{R}^n$ be a C^1 parameterized curve over a connected interval I with non-empty interior. We will write $|\gamma|$ for the image of the parameterized curve γ, that is

$$|\gamma| := \left\{ \gamma(t), t \in I \right\}.$$

Definition 2.1 Assume that the origin $\mathbf{0}$ belongs to clos$(|\gamma|) \setminus |\gamma|$. The curve $|\gamma|$ is *non-oscillating* at the origin if, for any semi-analytic subset H, the germ at the origin of the intersection $H \cap |\gamma|$ is either empty or is the germ of the curve $|\gamma|$ at the origin.

This notion was re-explored recently in order to study the asymptotic dynamics of real analytic vector fields on real analytic 3-manifolds at their singular points in certain situations, namely, to discuss the notion of spiraling [2, 5, 15], but also to discuss the o-minimality of structures expanding the real field and extended by means of a non-oscillating trajectory of a real analytic vector field [14].

It is an elementary exercise to check that plane gradient trajectories do not oscillate at their limit point. Moreover, we recall that in higher dimensions non-oscillating gradient trajectories exist as well [5, 8, 13, 15].

The pointed vector space $\mathrm{Sym}(\mathbb{R}^n)^*$ of non-zero real symmetric matrices of $\mathbb{R}^n$ is an open cone on the unit sphere of $\mathrm{Sym}(\mathbb{R}^n)$ with vertex the null matrix and on which the group $(\mathbb{R}_{>0}, \cdot)$ acts naturally, smoothly, and semi-algebraically by the homotheties of positive ratio. The resulting quotient space $\mathbf{Sym}(n)$ is the space of *oriented directions of the real symmetric n-matrices* and is a real algebraic manifold isomorphic to $\mathbf{S}^{n(n+1)/2-1}$. The quotient mapping $\sigma\colon \mathrm{Sym}(\mathbb{R}^n)^* \to \mathbf{Sym}(n)$ is a smooth semi-algebraic submersion.

Let f be a C^2 function defined on some open set U of $\mathbb{R}^n$ equipped with some orthonomal coordinates. We denote by $\mathrm{Hess}(f) := [h_{i,j}]$ the Hessian matrix of the function f, and define its size function h on U as $h(x) := [\sum_{i,j} h_{i,j}(x)^2]^{1/2}$. We will write $H(f)$ for the matrix $h^{-1}\mathrm{Hess}(f)$ where it is defined. We obviously get $\sigma(\mathrm{Hess}(f)) = \sigma(H(f))$. Note that $H(f)$ is not $\sigma(H(f))$, but has a similar property: let $(\mathrm{Hess}(f)(x_k))_k \in \mathrm{Sym}(\mathbb{R}^n)^*$ be a sequence converging to the null matrix. By definition of the matrix $H(f)$, the accumulation values of the sequence $(H(f)(x_k))_k$ are bounded, and moreover none of them can be the null matrix. In particular if the sequence $(H(f)(x_k))_k$ has a unique accumulation value $\mathcal{H}$, the sequence $(\sigma(H(f)(x_k)))_k$ then converges to $\sigma(\mathcal{H}) = \lambda\mathcal{H}$ for some positive real number λ. In the sequel it is more convenient to work with $H(f)$ than with $\sigma(\mathrm{Hess}(f))$. That is why with an obvious but small abuse of language we will also call such a limit $\mathcal{H}$ an *oriented limit direction* of the sequence $(H(f)(x_k))_k$.

Concerning notation, unless explicitly mentioned, $\mathbb{R}^n$ comes equipped with the Euclidean structure and we will write $\langle \cdot, \cdot \rangle$ for the corresponding scalar product and $|\cdot|$ for the associated norm.

3 Setting

Let $f\colon U \to \mathbb{R}$ be a real analytic function defined on an open subset U of $\mathbb{R}^n$ containing the origin $\mathbf{0}$. We assume the origin is a critical point of the function and that $f(\mathbf{0}) = 0$. Up to shrinking the open subset U, we can also assume that 0 is the only critical value of the function.

Let ν_f be the unitary gradient vector field defined on $\mathbb{R}^n \setminus \mathrm{crit}(f)$, where the subset $\mathrm{crit}(f)$ is the critical locus of the function f. But its orientation, the unitary gradient vector field, depends only on the foliation by the connected components of the levels of the function, not on the chosen function giving this foliation.

The arc-length, parameterized, gradient, differential equation writes:

$$\dot{x}(s) = \nu_f(x(s)) \text{ with } x(0) = x_0 \quad \text{and} \quad x_0 \notin \mathrm{crit}(f). \tag{3.1}$$

Assume that we are given a gradient trajectory $|\gamma|$ that is a solution of the differential equation (3.1). Up to taking $-f$ instead of f, we can also assume that the function $s \to f(\gamma(s))$ increases to 0 and that its ω-limit set $\omega(|\gamma|)$ contains $\mathbf{0}$. Łojasiewicz found that the length $l(|\gamma|)$ of the curve $|\gamma|$ between $\gamma(0)$ and $\mathbf{0}$ is finite [12]. Consequently its ω-limit set reduces to $\{\mathbf{0}\}$.

But Łojasiewicz's result on the length, the next theorem, by Kurdyka, Mostowski, and Parusiński, is up to now the only other result on the behavior of gradient trajectories at their limit point holding true within the whole category of real analytic functions.

Theorem 3.1 ([9]) *The length of the radial projection* $s \to \frac{\gamma(s)}{|\gamma(s)|}$ *of the gradient curve* $|\gamma|$ *onto* $\mathbf{S}^{n-1}$ *is finite.*

The straightforward and looked-for consequence of Theorem 3.1 is that the following well known Thom Gradient Conjecture holds true.

Thom Gradient Conjecture There exist $\nu \in \mathbf{S}^{n-1}$ such that $\lim_{s \to l(|\gamma|)} \frac{\gamma(s)}{|\gamma(s)|} = \nu$.

Let r be the function Euclidean distance to the origin $\mathbf{0}$, and let ∂_r be the unitary radial vector field. The gradient vector field ∇f decomposes as the orthogonal sum of its radial component $\partial_r f \partial_r$ and its spherical part $\nabla' f$:

$$\nabla f = \partial_r f \partial_r + \nabla' f.$$

The demonstration of Kurdyka, Mostowski, and Parusiński used fine estimates along the trajectory. The following ones, key for their proof, will be of interest for our purpose as well:

Proposition 3.2 ([9]) (i) *There exists a rational number* $m := m(|\gamma|)$ *and a positive real number* $\mathbf{a} := \mathbf{a}(|\gamma|)$ *such that along* $|\gamma|$,

$$f = -\mathbf{a}r^m + o(r^m) \quad \textit{and} \quad \partial_r f = -m\mathbf{a}r^{m-1} + o(r^{m-1}).$$

(ii) *There exists a finite subset* $S \subset \mathbb{Q}_{>0} \times \mathbb{R}_{>0}$ *of pairs* (l, b), *such that for any trajectory* α *such that* $\omega(|\alpha|) = \{\mathbf{0}\}$ *and* f *is negative along* $|\alpha|$, *the pair* $(m(|\alpha|), \mathbf{a}(|\alpha|))$ *as in property* (i) *corresponding to* $|\alpha|$ *belongs to* S.

4 Hessian Matrices and Gradients

This section is devoted to highlighting some connection between limits of direction of Hessian matrices and gradient vector fields behavior at singular points. This material should be known and is part of the folklore of this subject. We present it in a fashion serving our point of view.

We decompose the real analytic function f as the sum of its homogeneous parts at $\mathbf{0}$, namely $f = f_p + f_{p+1} + \cdots$. For each $k \geq p$ the following formulae hold:

$$k(k-1) f_k = (k-1)\langle \nabla f_k, r\partial_r \rangle = \langle \mathrm{Hess}(f_k) \cdot r\partial_r, r\partial_r \rangle. \tag{4.1}$$

Let $\beta \colon \mathbf{S}^{n-1} \times \mathbb{R}_+ \to \mathbb{R}^n$ be the spherical blowing-up with center $\mathbf{0}$ defined as $(u, r) \to ru$. Up to a multiplication by $p(p-1)r^{2-p}$ the lifted gradient differential equation by β now reads:

$$\begin{aligned} \dot{r} &= r\langle \mathrm{Hess}(f_p)(u) \cdot u, u \rangle + O(r^2) \\ \dot{u} &= \mathrm{Hess}(f_p)(u) \cdot u - \langle \mathrm{Hess}(f_p)(u) \cdot u, u \rangle u + O(r). \end{aligned} \tag{4.2}$$

The vector field corresponding to this differential equation is known as the *divided gradient vector field* [13]. It is denoted ξ_f and extends analytically on $\mathbf{S}^{n-1} \times \mathbb{R}$.

If a unit vector ν is not an eigenvector of $\text{Hess}(f_p)(\nu)$, then $\dot{u}|_{r=0,u=\nu} \neq 0$. Thus any limit point $(\nu, 0) \in \mathbf{S}^{n-1} \times 0$ of a divided gradient trajectory must be such that ν is an eigenvector of $\text{Hess}(f_p)(\nu)$ (which may well be the null matrix).

Let $|\gamma|$ be a trajectory of the gradient vector field ∇f such that, at its limit point $\mathbf{0}$, its limit of secants ν satisfies $f_p(\nu) = -\mathbf{a} < 0$. Using equation (4.1), we observe that in a neighborhood of $(\nu, 0)$ equation (4.2) is of the form

$$\dot{r} = -\mathbf{a}r + O(r^2) \quad \text{and} \quad \dot{u} = O\big(|(u - \nu, r)|\big).$$

Thus the singular point $(\nu, 0)$ is an elementary singularity of the divided gradient vector field ξ_f. Moreover the following result should be known.

> The limit of secants ν is an eigenvector of the non-zero matrix $\text{Hess}(f_p)(\nu)$ associated with the eigenvalue $-p(p-1)\mathbf{a}$. (Note that this limit is equal to $\lim_{x/|x| \to \nu} |x|^{-p+2}\text{Hess}(f)(x)$).

Along the trajectory $|\gamma|$, we recall that Proposition 3.2 gives the estimate $f = -\mathbf{a}r^p + o(r^p)$ as $r \to 0$, where $r(s) = |\gamma(s)|$. Since along the curve $|\gamma|$ we know that $\text{Hess}(f) = r^{p-2}[\text{Hess}(f_p)(\nu) + O(r)]$, the statement about ν follows from equation (4.1).

As a straightforward consequence of this fact, there exists a positive constant C such that $\lim_{|\gamma| \ni x \to \mathbf{0}} H(f)(x) = C\,\text{Hess}(f_p)(\nu)$. More importantly, the conclusions of our main result Theorem 5.5 are true in this setting. It does not use the non-oscillation of the trajectory but just some rough estimates. If the function f is a homogeneous polynomial, then any gradient trajectory with ω-limit set $\mathbf{0}$ satisfies to the conclusions of Theorem 5.5.

5 Main Result

For this section we suppose that we are given a gradient trajectory $|\gamma|$ of the function f accumulating at $\mathbf{0}$. We require that the following additional property be satisfied.

Non-oscillating hypothesis The trajectory $|\gamma|$ does not oscillate at its end point $\mathbf{0}$.

We begin with an elementary, but key, monotonicity result of functions along a non-oscillating trajectory.

Lemma 5.1 *Let $\psi\colon U \to \mathbb{R}$ be a continuous semi-analytic function, analytic and non zero outside* $\text{crit}(f)$. *The function $\psi|_{|\gamma|}$ has a finite limit at $\mathbf{0}$, and up to shrinking U, the function $\psi|_{|\gamma|}$ is either constant or strictly monotonic on $|\gamma| \cap U$.*

Proof Assume that $\psi|_{|\gamma|}$ is not constant.

Assume there exists a sequence $(s_j)_j$ converging to $l(|\gamma|)$, the length of $|\gamma|$, such that ψ' vanishes and changes sign at each s_j. Since

$$\psi'(s) = \langle \nabla\psi, \nu_f \rangle \circ \gamma(s),$$

the subset $|\gamma| \cap \{\langle \nabla\psi, \nu_f \rangle = 0\}$ has infinitely many connected components. Since $\text{clos}(\{\langle \nabla\psi, \nu_f \rangle = 0\})$ is a closed semi-analytic set and contains $\mathbf{0}$, this fact contradicts the non-oscillation of $|\gamma|$ at $\mathbf{0}$. ■

The function $s \to r = |\gamma(s)|$ is strictly decreasing to 0 as $s \to l(|\gamma|)$, following Lemma 5.1. Moreover, we have

$$\dot{r}(s) = \frac{|\nabla f|}{\partial_r f} \circ \gamma(s) \quad \text{and} \quad \dot{r}(s) \to -1 \text{ as } s \to l(|\gamma|).$$

So we reparameterize the integral curve γ with r:

$$\gamma'(r) = \frac{\mathrm{d}}{\mathrm{d}r}(\gamma(r)) = \Big(\frac{|\nabla f|}{\partial_r f} \nu_f \Big) \circ \gamma(r) = \xi \circ \gamma(r),$$

where ξ is the semi-analytic vector field appearing in the above expression.

Another simple but very useful result is the following

Lemma 5.2 *Let a, b be continuous semi-analytic functions such that the $\{b = 0\}$ is contained in $\{\partial_r f = 0\}$. Assume that the function v defined as $v(r) := (a/b) \circ \gamma(r)$ is bounded. Thus $rv'(r) \to 0$ as $r \to 0$.*

Proof First note that by the nature of the parameterization $r \to \gamma(r)$, we check that $v' = \bar{a}/b^2$ is of the same form as v. From Lemma 5.1 the germ of the function v at 0 is either strictly monotonic or constant. If $v' \to \infty$ as $r \to 0$, then rv' is still strictly monotonic when $r \to 0$. If rv' was not tending to 0, integrating v' would contradict the fact that v is bounded. ■

A first interesting consequence of Lemmas 5.1 and 5.2 is the following corollary.

Corollary 5.3 $\lim_{r \to 0} \nu_f \circ \gamma(r) = -\nu$.

Proof First we write $\gamma(r) = r(\nu + \chi(r))$, where $\chi(r) = (\partial_r - \nu) \circ \gamma(r)$. Since the unitary radial vector field ∂_r is semi-analytic, applying 5.1 we deduce that $|\chi(r)| \to 0$ as $r \to 0$. Thus $\gamma'(r) = s'(r)\nu + \beta(r)$ where $\beta(r) = \chi(r) + r\chi'(r)$. From Lemmas 5.1 and 5.2, we get $\beta(r) = o(1)$. And to conclude we recall that $s'(r) \to -1$ as $r \to 1$, ■

We recall that the matrix $H(f)$, introduced in Section 2 is *the normalized Hessian matrix* of f. The mapping $x \to H(f)(x)$ is thus continuous and semi-analytic. As a straightforward application of Lemma 5.1 we deduce the following lemma.

Lemma 5.4 *There exists* $\mathcal{H} \in \text{Sym}(\mathbb{R}^n)^*$ *such that*

$$\lim_{|\gamma| \ni x \to \mathbf{0}} H(f)(x) = \mathcal{H}.$$

Note that this result can also be deduced from the results of [11]. This fact satisfied, the main result of this note is the following theorem.

Theorem 5.5

$$\lim_{|\gamma|\ni x\to\mathbf{0}} \frac{\mathrm{Hess}(f)(x)\cdot\nu_f(x)}{|\mathrm{Hess}(f)(x)\cdot\nu_f(x)|} = \lim_{|\gamma|\ni x\to\mathbf{0}} \frac{H(f)(x)\cdot\nu_f(x)}{|H(f)(x)\cdot\nu_f(x)|} = \nu.$$

Thus the vector ν is an eigenvector of the matrix $\mathcal{H}$.

The proof will consist of several elementary steps.

Proof In the proof of Corollary 5.3 we wrote $\gamma(r) = r(\nu + \chi(r))$, so that $\gamma' = s'(r)\nu + \beta$ with $\beta = \chi + r\chi'$. Since

$$\chi = \partial_r - \nu \quad \text{and} \quad \chi'(r) = \frac{1}{r\partial_r f}\nabla' f \circ \gamma(r),$$

we can write $\beta'(r) = (a/b)\circ\gamma(r)$ for a semi-analytic function $v = a/b$ with $b^{-1}(0) = \partial_r f^{-1}(0)$. Applying Lemma 5.2 again, we deduce

$$\gamma''(r) = o(r^{-1}). \tag{5.1}$$

Note that

$$\gamma''(r) = v'(r)\nu_f(\gamma(r)) + v^2(r)(\mathrm{d}\nu_f\cdot\nu_f)\circ\gamma(r) \text{ with } v(r) = \frac{|\nabla f|}{\partial_r f}\circ\gamma(r).$$

We recall that $v(r) \to -1$ as $r \to 0$, and so from Lemma 5.2, $rv'(r) \to 0$ as $r \to 0$. We deduce from (5.1),

$$(\mathrm{d}\nu_f\cdot\nu_f)\circ\gamma(r) = o(r^{-1}). \tag{5.2}$$

Since

$$\mathrm{d}\nu_f\cdot\nu_f = \frac{1}{|\nabla f|}\left[\mathrm{Hess}(f)\cdot\nu_f - \langle \mathrm{Hess}(f)\cdot\nu_f, \nu_f\rangle\nu_f\right].$$

Equation (5.2) now reads

$$r\left[\mathrm{Hess}(f)\cdot\nu_f - \langle \mathrm{Hess}(f)\cdot\nu_f, \nu_f\rangle\nu_f\right]\circ\gamma(r) = o(|\nabla f|). \tag{5.3}$$

The classical Bochnak–Łojasiewicz inequality applied to $|\nabla f|$ nearby the origin gives

$$|x|\cdot|\mathrm{Hess}(f)\cdot\nu_f| \geq C|\nabla f| \tag{5.4}$$

for a positive constant C.

Since $r|\mathrm{Hess}(f)\cdot\nu_f| \geq C|\nabla f|$, the only possibility for equation (5.3) to hold true is that, along the curve $|\gamma|$, we get

$$\mathrm{Hess}(f)\cdot\nu_f = \langle \mathrm{Hess}(f)\cdot\nu_f, \nu_f\rangle\nu_f + o(|\langle \mathrm{Hess}(f)\cdot\nu_f, \nu_f\rangle|).$$

Dividing both sides by $|\mathrm{Hess}(f)\cdot\nu_f|$ and using Corollary 5.3 provides that the limit is either ν or $-\nu$. Along $|\gamma|$, the estimates $f = -\mathbf{a}r^m\partial_r + o(r^m)$ from Proposition 3.2 will give that it is ν. ∎

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6 A Few Consequences

We will need the following definition.

Definition 6.1 Let $g\colon U \to \mathbb{R}$ be a C^1 function defined on U an open subset of $\mathbb{R}^n$. Assume that the origin $\mathbf{0}$ lies in $\operatorname{clos}(U) \setminus U$. A value $c \in \mathbb{R}$ is *an asymptotic critical value of g at* $\mathbf{0}$ if there exists a sequence $(x_k)_k$ of points of U converging to the origin such that

$$|x_k| \cdot |\nabla g(x_k)| \to 0 \quad \text{and} \quad g(x_k) \to c \text{ as } k \to +\infty.$$

Since we are only interested in the behavior of some functions at $\mathbf{0}$, we will just say asymptotic critical value. The value $-\mathbf{a}$ in Proposition 3.2 is an asymptotic critical value of the function $r^{-m} f$ [9].

For a given non-oscillating trajectory $|\gamma|$ with ω-limit point $\mathbf{0}$, we deduce from Proposition 3.2 and Lemmas 5.1 and 5.2 that the following estimate holds along $|\gamma|$

$$\nabla f = -m\mathbf{a}r^{m-1}\partial_r + o(r^{m-1}). \tag{6.1}$$

The first result of this section is the following proposition.

Proposition 6.2 *Let $|\gamma|$, m, and $\mathbf{a}$ be as above. The real number $m \cdot \mathbf{a}$ is an asymptotic critical value of the function $r^{1-m}|\nabla f|$.*

Proof Let us denote by R the function $r \to r^{1-m}|\nabla f(\gamma(r))|$. Since $R \to m\mathbf{a}$ as $r \to 0$, Lemma 5.2 implies $rR' \to 0$ as $r \to 0$.

Since

$$R'(r) = \frac{1}{r^m}\frac{\mathrm{d}s}{\mathrm{d}r}\Big\langle r\mathrm{Hess}(f)\cdot \nu_f - (m-1)|\nabla f|\partial_r, \nu_f\Big\rangle$$

and $\frac{\mathrm{d}s}{\mathrm{d}r} \to -1$, we get

$$\Big\langle r\mathrm{Hess}(f)\cdot \nu_f - (m-1)|\nabla f|\partial_r, \nu_f\Big\rangle = o(r^{m-1}).$$

Along $|\gamma|$ we know that $\nabla f = -m\mathbf{a}r^{m-1}\partial_r + o(r^{m-1})$. Using Bochnak–Łojasiewicz Inequality in the form (5.4), we find: $r|\mathrm{Hess}(f)\cdot\nu_f| \geq Cr^{m-1}$. From Theorem 5.5 we also have that $|\langle \mathrm{Hess}(f)\cdot \nu_f, \nu_f\rangle| = |\mathrm{Hess}(f)\cdot\nu_f| + o(|\mathrm{Hess}(f)\cdot\nu_f|)$, and so we necessarily deduce that

$$r\mathrm{Hess}(f)\cdot\nu_f = (m-1)|\nabla f|\partial_r + o(r^{m-1}).$$

Thus the result is proved. ∎

There are two consequences summarized in the following statements. The first one is already in the proof of Proposition 6.2:

Corollary 6.3 *Along $|\gamma|$, we find*

$$\begin{aligned}\mathrm{Hess}(f)\cdot\nu_f &= -m(m-1)\mathbf{a}r^{m-2}\nu_f + o(r^{m-2}) = m(m-1)\mathbf{a}r^{m-2}\partial_r + o(r^{m-2})\\ &= m(m-1)\mathbf{a}r^{m-2}\nu + o(r^{m-2}).\end{aligned}$$

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From Corollary 6.3 and Lemmas 5.1 and 5.2 we see that along $|\gamma|$, we obtain an estimate similar to equation (4.1):

$$m(m-1)f = \langle \mathrm{Hess}(f)\cdot r\partial_r, r\partial_r\rangle + o(r^m).$$

The second consequence is that the asymptotic critical value $-\mathbf{a}$ may be interpreted as an eigenvalue under some additional hypotheses.

Corollary 6.4 *Assume that along the trajectory $|\gamma|$, the size function h of the Hessian matrix* $\mathrm{Hess}(f)$ *has the estimate $h = Cr^{m-2} + o(r^{m-2})$ for a positive constant C. Then along $|\gamma|$ the matrix $r^{-(m-2)}\mathrm{Hess}(f)$ has a limit at $\mathbf{0}$ in $\mathrm{Sym}(\mathbb{R}^n)^*$, and the eigenvalue in the oriented direction $\mathbb{R}_+\nu$ is exactly $-m(m-1)\mathbf{a}$.*

The case of having 0 as an eigenvalue does happen: consider $f = y^2 - x^3$ and let$|\gamma|$ be any trajectory tangent to the x-axis. Along such a trajectory $m = 3$, while along $|\gamma|$ one coefficient of the matrix $r^{-1}\mathrm{Hess}(f)$ goes to ∞.

7 At Infinity

Even though the results of [6] were obtained in the semi-algebraic context, they also hold true for C^2 globally sub-analytic function.

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a globally sub-analytic real analytic function. We consider again the gradient differential equation $\dot{x}(s) = \nu_f(x(s))$ with $x(0) = x_0$. But we are now interested in gradient trajectories that leave any compact subset of $\mathbb{R}^n$. Namely, for $s \to \gamma(s)$ an arc-length parameterized solution, given any compact subset K of $\mathbb{R}^n$, there exists $s_K > 0$, such that $\gamma(s) \notin K$ for $s > s_K$, or equivalently, $|\gamma(s)| \to +\infty$ as $s \to \infty$. Note that gradient trajectories are either bounded or leave any compact subset, there is no other alternative global behavior.

Assume that $|\gamma|$ is an unbounded gradient trajectory, then $|\gamma|$ leaves any compact subset of $\mathbb{R}^n$. Without loss of generality we suppose that f is increasing along $|\gamma|$ and $\lim_{x\in|\gamma|,|x|\to+\infty} f(x) = 0$.

The main result of [6] is that the gradient conjecture is also true at infinity.

Theorem 7.1 ([6]) *There exists $\nu \in \mathbf{S}^{n-1}$ such that $\lim_{x\in|\gamma|,|x|\to+\infty} \frac{x}{|x|} = \nu$.*

To prove this result, some estimates along the trajectory were necessary. The next one is relevant for this section.

Proposition 7.2 ([6]) *There exists a positive rational number m and a positive real number $\mathbf{a}$ such that along $|\gamma|$,*

$$f = -\mathbf{a}r^{-m} + o(r^{-m}) \quad \textit{and} \quad \partial_r f = m\mathbf{a}r^{-1-m} + o(r^{-1-m}).$$

Now we need to introduce the correct notion of non-oscillation at infinity.

Definition 7.3 The trajectory $|\gamma|$ is non-oscillating at infinity if for any globally sub-analytic unbounded analytic hypersurface H, there exists a radius R such that $|\gamma| \cap H \cap (\mathbb{R}^n \setminus \mathbf{B}(\mathbf{0}, R))$ has finitely many connected components.

The main result in this context, using the notations introduced in the previous sections, is the following theorem.

Theorem 7.4 *Assume that $|\gamma|$ is non-oscillating at infinity. Then*

(i) $$\lim_{x\in|\gamma|,|x|\to+\infty} H(f)(x) = \mathcal{H} \in \mathrm{Sym}(\mathbb{R}^n)^*$$

(ii) $$\lim_{x\in|\gamma|,|x|\to+\infty} \frac{\mathrm{Hess}(f)(x)\cdot\nu_f(x)}{|\mathrm{Hess}(f)(x)\cdot\nu_f(x)|} = \lim_{x\in|\gamma|,|x|\to+\infty} \frac{H(f)(x)\cdot\nu_f(x)}{|H(f)(x)\cdot\nu_f(x)|} = -\nu.$$

Thus ν is an eigenvector of the matrix $\mathcal{H}$.

The proof will work along the same lines as that of Theorem 5.5 with a few modifications that we will describe. A key ingredient in Section 5 is the use of Bochnak–Łojasiewicz Inequality for $|\nabla f|$. Fortunately as a corollary of the main result of the author's joint works [3, 4], we deduced such a Bochnak–Łojasiewicz inequality at infinity.

Proposition 7.5 ([3, 4]) *If $|x| \gg 1$ and $|f(x)| \ll 1$, then $|x|\cdot|\nabla f| \geq C|f|$, for a positive constant C.*

Then using this inequality for the function $|\nabla f|$ we find again that for $|x| \gg 1$ and $|\nabla f| \ll 1$, there is a positive constant C:

$$|x|\cdot|\mathrm{Hess}(f)\cdot\nu_f| \geq C|\nabla f|.$$

Next we need the analogues of Lemmas 5.1 and 5.2.

Lemma 7.6 *Let g be a globally continuous sub-analytic function outside of a compact subset of $\mathbb{R}^n$. Assume that the trajectory $|\gamma|$ is non-oscillating at infinity. Let $v := g\circ\gamma$. As a germ at infinity, if v is not constant, then it is strictly monotonic.*

Proof If Lemma 7.6 did not hold, it would contradict the non-oscillation at infinity of $|\gamma|$. ∎

Since the function $1/r$ is semi-algebraic, we can then reparameterize the solution γ by $r = |\gamma(s)|$.

Lemma 7.7 *Let a, b be globally continuous sub-analytic functions outside of a compact subset of $\mathbb{R}^n$, with $\{b = 0\}$ contained in $\{\partial_r f = 0\}$. Assume that $|\gamma|$ is non-oscillating at infinity. Let $v(r) := (a/b)\circ\gamma(r)$ and assume that $v(r) \to c \in \mathbb{R}$ as $r \to +\infty$. Then the function $r \mapsto r^{-1}v'(r)$ tends to 0 as $r \to +\infty$.*

Proof The proof is similar to that of Lemmas 5.2. ∎

From these two Lemmas, and adapting the proof of Corollary 5.3, it is easy to prove the following corollary.

Corollary 7.8 *Assuming that $|\gamma|$ is non-oscillating at infinity, we get*

$$\lim_{r\to 0} \nu_f\circ\gamma(r) = \nu.$$

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The proof of Theorem 7.4 proceeds along the same lines as the proof of Theorem 5.5, using Lemmas 7.6 and 7.7 and Corollary 7.8 in a similar way.

Along a non-oscillating trajectory $|\gamma|$ going to infinity, we find estimates similar to equation (6.1), Proposition 6.2, Corollary 6.3, and Corollary 6.4, namely

$$\nabla f = m\mathbf{a}r^{-m-1}\partial_r + o(r^{-m-1}).$$

We also find the following proposition.

Proposition 7.9 *Let $|\gamma|$ be a gradient trajectory of the function f non-oscillating at infinity. Let m and $\mathbf{a}$ be as above. The real number $m \cdot \mathbf{a}$ is an asymptotic critical value of the function $r^{1-m}|\nabla f|$.*

Proof The proof works as in the proof of Proposition 6.2 ■

We eventually deduce the following estimates

$$\begin{aligned} -\mathrm{Hess}(f)\cdot\nu_f &= m(m-1)\mathbf{a}r^{-m-2}\nu_f + o(r^{-m-2}) \\ &= m(m-1)\mathbf{a}r^{-m-2}\partial_r + o(r^{-m-2}) \\ &= m(m-1)\mathbf{a}r^{-m-2}\nu + o(r^{-m-2}) \\ m(m-1)f &= \langle \mathrm{Hess}(f)\cdot r\partial_r, r\partial_r\rangle + o(r^{-m}). \end{aligned}$$

8 Riemannian Case

Let $(M, \mathbf{g})$ be a smooth Riemannian manifold of dimension n. Let TM be its tangent bundle and $\mathscr{X}(M)$ be the $C^\infty(M)$-module of the smooth vector fields on M. Let $\langle\cdot,\cdot\rangle_{\mathbf{g}}$ be the scalar product coming with $\mathbf{g}$ and $|\cdot|_{\mathbf{g}}$ be the associated norm. Let D be the canonical connection associated with $\mathbf{g}$.

The gradient vector field $\nabla_{\mathbf{g}}u$ of a smooth function u is defined as the dual of Du for the pairing $\langle\cdot,\cdot\rangle_{\mathbf{g}}$, that is for any smooth vector field X on M,

$$(Du)(X) = D_X u = X\cdot u = \langle \nabla_{\mathbf{g}}u, X\rangle_{\mathbf{g}}.$$

The notion of Hessian of the function u as d^2u, the second differential, is no longer accurate in this Riemannian setting. Since the connection D extends to tensors, the corresponding notion of Hessian in this context comes from the tensor

$$D^2u = D(Du) = (D\mathrm{d}u)\colon \mathscr{X}(M)\times\mathscr{X}(M)\to C^\infty(M)$$

defined as $(D\mathrm{d}u)(X,Y) = (D_X\mathrm{d}u)(Y)$. It is smooth, C^∞-bilinear, and symmetric as well. We also check that

$$(D\mathrm{d}u)(X,Y) = \langle D_X\nabla_{\mathbf{g}}u, Y\rangle_{\mathbf{g}} = \langle D_Y\nabla_{\mathbf{g}}u, X\rangle_{\mathbf{g}},$$

and deduce that $\nabla_{\mathbf{g}}|\nabla_{\mathbf{g}}u|^2_{\mathbf{g}} = 2D_{\nabla_{\mathbf{g}}u}\nabla_{\mathbf{g}}u$. We can also look at $D\mathrm{d}u$ as a $C^\infty(M)$-linear map $\mathscr{X}(M)\to\mathscr{X}(M)$, defined, with an obvious abuse of notation, as $(D\mathrm{d}u)(X) =$

$D_X\nabla_{\mathbf{g}}u$. At each point x of M the tensor Ddu induces a bilinear symmetric form $\mathrm{Hess}^{\mathbf{g}}(u)(x) := (Ddu)_x = T_xM \times T_xM \to \mathbb{R}$. We will call it the Hessian of u at x and the mapping $x \to \mathrm{Hess}^{\mathbf{g}}(u)(x)$ is smooth. We will also look at it as a linear endomorphism of T_xM.

Now we come to our topic.

Assume that $M = U$ is an open subset of $\mathbb{R}^n$ containing the origin $\mathbf{0}$. Assume that the Riemannian metric $\mathbf{g}$ is analytic. Let $f\colon U \to \mathbb{R}$ be a real analytic function such that the origin is a critical point of the function and $f(\mathbf{0}) = 0$. Up to shrinking the open subset U, we can still assume that the function f has single critical value, namely 0. Let $\nu_f^{\mathbf{g}}$ be the unitary gradient vector field defined on $U \setminus \mathrm{crit}(f)$. Let us consider the following gradient differential equation:

$$\dot{x}(s) = \nu_f^{\mathbf{g}}(x(s)) \text{ with } x(0) = x_0 \text{ and } f(x_0) < 0.$$

We take normal coordinates x at $\mathbf{0}$ for $\mathbf{g}$. In a neighborhood of $\mathbf{0}$, the metric writes as $\mathbf{g} = \mathbf{Eucl} + O(|x|^2)$.

If r is the Euclidean distance function to the origin $\mathbf{0}$ and $r^{\mathbf{g}}$ the distance to the origin for the metric $\mathbf{g}$, then $r^{\mathbf{g}}/r \to 1$ as $x \to \mathbf{0}$.

Remark 8.1 In a neighborhood of $\mathbf{0}$, Bochnak-Łojasiewicz inequalities holds, namely

$$r^{\mathbf{g}}|\nabla_{\mathbf{g}} f|_{\mathbf{g}} \geq C|f| \text{ and so } r|\nabla_{\mathbf{g}} f|_{\mathbf{g}} \geq C'|f|,$$

for some positive constant C, C'.

Moreover the results of [9] quoted in Section 3 still hold true, sticking with r in the estimates.

Bochnak–Łojasiewicz inequality for the function $|\nabla_{\mathbf{g}} f|_{\mathbf{g}}$ then reads, in a neighborhood of $\mathbf{0}$,

$$r|D_{\nu_f^{\mathbf{g}}}\nabla_{\mathbf{g}} f|_{\mathbf{g}} \geq C|\nabla_{\mathbf{g}} f|_{\mathbf{g}} \tag{8.1}$$

for some positive constant C. The operator $\mathrm{Hess}^{\mathbf{g}}(f)$ is now analytic, and in the present coordinates, when considered as a linear mapping, it has (symmetric) entries $h_{i,j}^{\mathbf{g}}$. We recall that the (semi-analytic) size function $h^{\mathbf{g}}$ is $h^{\mathbf{g}} = (\sum_{i,j}(h_{i,j}^{\mathbf{g}})^2)^{1/2}$ and we define $H^{\mathbf{g}}(f) = (h^{\mathbf{g}})^{-1}\mathrm{Hess}^{\mathbf{g}}(f)$, so that at each point $x \notin \mathrm{crit}(f)$ nearby $\mathbf{0}$. We deduce $H^{\mathbf{g}}(f)(x)$ is non zero.

Now we can state the main result of this section.

Theorem 8.2 *Assume that $|\gamma|$ is a non-oscillating trajectory of the gradient field associated with f respectively to the analytic Riemannian metric $\mathbf{g}$, with $\omega(|\gamma|) = \{\mathbf{0}\}$. Then*

(i) *Along* $|\gamma|$, $\displaystyle\lim_{x\in|\gamma|,|x|\to+\infty} H^{\mathbf{g}}(f)(x) = \mathcal{H}^{\mathbf{g}} \in \mathrm{Sym}(\mathbb{R}^n)^*$

(ii) $$\lim_{|\gamma|\ni x\to\mathbf{0}} \frac{\mathrm{Hess}^{\mathbf{g}}(f)(x)\cdot\nu_f(x)}{|\mathrm{Hess}^{\mathbf{g}}(f)(x)\cdot\nu_f(x)|} = \lim_{|\gamma|\ni x\to\mathbf{0}} \frac{H^{\mathbf{g}}(f)(x)\cdot\nu_f(x)}{|H^{\mathbf{g}}(f)(x)\cdot\nu_f(x)|} = \nu.$$

Thus the vector ν is an eigenvector of the matrix $\mathcal{H}^{\mathbf{g}}$.

Proof The first point is elementary. The proof of the second point follows the same steps as in the Euclidean case dealt with in Section 5. We nevertheless have to check the form of $\ddot{\gamma}$ in order to recycle the previous proof. We know that

$$\ddot{\gamma}(s) = \mathrm{d}\nu_f^{\mathbf{g}} \cdot \nu_f^{\mathbf{g}}.$$

Thus we find

$$\mathrm{d}\nu_f^{\mathbf{g}} \cdot \nu_f^{\mathbf{g}} = \frac{1}{|\nabla_{\mathbf{g}} f|_{\mathbf{g}}} \left[D_{\nu_f^{\mathbf{g}}} \nabla_{\mathbf{g}} f - \Gamma(\nu_f^{\mathbf{g}}, \nabla_{\mathbf{g}} f) - \langle D_{\nu_f^{\mathbf{g}}} \nabla_{\mathbf{g}} f, \nu_f^{\mathbf{g}} \rangle_{\mathbf{g}} \nu_f^{\mathbf{g}} \right].$$

and we observe that $\Gamma(\nu_f^{\mathbf{g}}, \nabla_{\mathbf{g}} f) = O(|\nabla_{\mathbf{g}} f|)$. Thus applying Bochnak–Łojasiewicz for $D_{\nu_f^{\mathbf{g}}} \nabla_{\mathbf{g}} f$ will complete the proof. ∎

We recall that if $\mathbf{0}$ is a critical point of the function f, then $\mathrm{Hess}(f)(\mathbf{0}) = \mathrm{Hess}^{\mathbf{g}}(f)(\mathbf{0})$. We end this section with the following expected result.

Proposition 8.3 *Let $|\gamma|$ be a gradient trajectory of $\nabla_{\mathbf{g}} f$ non-oscillating at its limit point $\mathbf{0}$. Let $\mathcal{H}$ be the limit at $\mathbf{0}$ of the normalized "Euclidean" Hessian matrix $H(f)$ along $|\gamma|$. Then $\mathcal{H} = \mathcal{H}^{\mathbf{g}}$.*

Proof Since we have taken normal coordinates x at $\mathbf{0}$ for $\mathbf{g}$, in a neighborhood of $\mathbf{0}$, the matrix $G(x)$ representing the quadratic form $\mathbf{g}(x)$ can be written as $G(x) = Id + O(|x|^2)$ in the (Euclidean) orthonormal basis $\partial_{x_1}(\mathbf{0}), \dots, \partial_{x_n}(\mathbf{0})$. Since $\nabla f = G \nabla_{\mathbf{g}} f$, we deduce that along any arc-length parameterized curve $s \to \alpha(s)$ tending to 0 as $s \to L_\alpha \in \mathbb{R}_{>0}$,

$$\mathrm{Hess}(f) \cdot \alpha'(s) = \mathrm{Hess}^{\mathbf{g}}(f) \cdot \alpha'(s) + O(|\alpha|) \nabla_{\mathbf{g}} f.$$

Thus we deduce that in a neighborhood of $\mathbf{0}$, there exists a positive constant C such that

$$|\mathrm{Hess}(f)(x) - \mathrm{Hess}^{\mathbf{g}}(f)(x)| \leq C |x| |\nabla_{\mathbf{g}} f|. \tag{8.2}$$

Dividing by $|\mathrm{Hess}^{\mathbf{g}}(f)(x)|$ in equation (8.2) and using equation (8.1) along $|\gamma|$ provides the desired conclusion. ∎

Remark 8.4 (i) There are estimates as in Section 6, but it is not clear what information some of them will really carry.

(ii) The papers [6] and [3] only dealt with the Euclidean metric, thus we will not discuss what happens at infinity in a Riemannian context. For such a purpose, we would first need to know some facts about the behavior of the metric $\mathbf{g}$ at infinity. Second, two necessary steps would be to have a Bochnak–Łojasiewicz inequality at infinity and the gradient conjecture at infinity to be true. Third, it is easy to make an example of a smooth semi-algebraic metric on $\mathbb{R}^2 \setminus \mathrm{clos}(\mathbf{B}(\mathbf{0}, 1))$ for which we find a smooth semi-algebraic function whose corresponding gradient trajectories leave any compact subset of the plane in spiraling [7].

9 Remarks and Comments

(1) Gradient trajectories are not attached to a function but to a foliation: the foliation by the connected components of the level of the given function. We have proved, in particular, that the existence of the limit matrix along the non-oscillating trajectory is attached to the considered function. This can be easily seen on the plane functions $f = -x^2 - y^2$ and $g = -f^2$. We can check that along the trajectory $|\gamma| = \{y = 0, x > 0\}$:

$$H(f) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ while } H(g) = 10^{-\frac{1}{2}} \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}$$

What this limit matrix encodes is still mysterious. When the function f is of order at least 3 at $\mathbf{0}$, this matrix may have some connections with the linear part a reduction of the singularities of the gradient vector fields (when we produce one), but in general it is not the linear part of the reduced singular point. This can be checked on any homogeneous polynomial (it is very easy for $f(x, y) = x^3 - y^3$).

(2) We have kept the framework of this note within the category of real analytic functions. But with a few modifications the main results we have proved, namely Theorems 5.5 and 7.4 hold true for C^2 functions (globally) definable in an o-minimal structure $\mathcal{M}$ expanding the real field (see [16]). The most important modification is that in the definition of the non-oscillation we substitute "semi-analytic" by "definable in $\mathcal{M}$".

Regardless of the polynomial boundedness of the o-minimal structure $\mathcal{M}$, everything will work mostly fine in this setting, since there are straightforward analogs of Lemmas 5.1, 5.2, 7.6, and 7.7, with very similar proofs. Thus the non-oscillation property ensures that there is always a limit of secants at the limit point. (Note that nevertheless the Gradient Conjecture is true in polynomially bounded structures [10]). We conclude that with the notion of definable non-oscillation we have chosen, there is an analog version of Theorem 5.5.

At infinity, there is also a limit of secants at infinity along a given non-oscillating gradient curve. Moreover, we can find estimates (monomial, as when the structure $\mathcal{M}$ is polynomially bounded) for the function along the trajectory. To conclude we then use the author's Bochnak–Łojasiewicz inequality at infinity in such an o-minimal setting [4], and thus, with the same arguments, we prove Theorem 7.4 in this case.

Assume that the o-minimal structure $\mathcal{M}$ contains $\mathbb{R}_{an}$; that is, the sub-analytic subsets are definable in $\mathcal{M}$. If we want to detect oscillating behaviors at a given point only with the semi/sub-analytic subsets, which is demanding much less than filtering with definable subsets, it is not absolutely clear to us whether, a gradient curve of a C^2 definable function that is semi/subanalytically non-oscillating at its limit point is also definably non-oscillating at its limit point. Since there are examples of semi-algebraically non-oscillating trajectories of real analytic vector fields that are analytically oscillating, the answer to the previous question, even for a gradient trajectory, may not be so straightforward, and we should also consider producing counter-examples.

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