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ON STEENROD BUNDLES AND THE VAN KAMPEN THEOREM

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ABSTRACT. We present a general method based on the van Kampen theorem for computing the fundamental group of the total space in certain Steenrod bundles. The method is applied to mapping spaces and Grassmann bundles.

1. **Introduction.** A principal tool for computing the fundamental group of a space is the theorem of van Kampen [10]. Most often the theorem is applied to a finite complex; in particular, it seems never to have been used in the context of mapping spaces. The origin of the present paper is a successful computation of the fundamental group of a mapping space using the van Kampen theorem. The method of computation is most clearly explained by generalizing the problem, and this leads to further results.

In Section 2 we present a general method for computing the fundamental group of the total space in a Steenrod bundle, which can be decomposed into trivial bundles along two subspaces in the base space satisfying the requirements for using the van Kampen theorem.

We derive three useful corollaries from the general theorem. As an application of the first corollary, we present in Section 3 a completely elementary computation of the fundamental group of the space of maps of degree k on the 2-sphere. This group was originally determined by Hu [4] using properties of Whitehead products. The computation presented here is the aforementioned origin of the paper. The second corollary is applied to certain Grassmann bundles in Section 4. The third corollary may be useful in connection with some of the $SU(n)$-bundles occurring in gauge field theory.

The idea of using the van Kampen theorem in the context of mapping spaces was conceived of during a very pleasant visit to the University of Wales in Bangor supported by the British Council. In particular, I am grateful to Professor Ronnie Brown for arranging this visit and for stimulating discussions.

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2. **The method.** Let $p:E \to B$ be a bundle in the sense of Steenrod [9] with group G and fibre F. Let B_1 and B_2 be a pair of subspaces in B which covers B. In other words, B decomposes as the union $B = B_1 \cup B_2$. Throughout the paper we shall make the following

Basic assumptions.

1) The subspaces B_1 and B_2 are deformation retracts of open sets in B.

2) The subspaces B_1, B_2 and the intersection $S = B_1 \cap B_2$ are nonempty and arcwise connected.

3) The fibre F of p is arcwise connected.

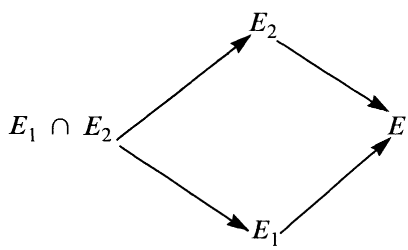
4) The bundle p is trivial over both B_1 and B_2.

5) The change of trivialization, when we pass from B_1 to B_2, is governed by the map $\varphi:S \to G$. (We arrange it so, that if $x_0 \in S$ is a base point in S, then $\varphi(x_0)$ is the identity element in G.)

Consider the inverse images $E_1 = p^{-1}(B_1)$ and $E_2 = p^{-1}(B_2)$. We note that $E_1 \cap E_2 = p^{-1}(S)$. Choose a base point $e_0 \in E$ in the fibre F over the base point $x_0 \in S$.

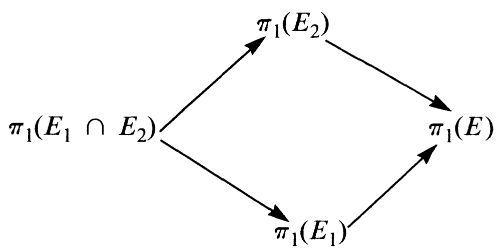
The basic assumptions ensure that the van Kampen theorem can be applied to the push-out

(1)



see e.g. ([6], Chapter 4), and hence we get an induced push-out diagram of fundamental groups

(2)



In other words, $\pi_1(E)$ is the free product of $\pi_1(E_1)$ and $\pi_1(E_2)$ with amalgation induced by the homomorphisms of $\pi_1(E_1 \cap E_2)$ into $\pi_1(E_1)$ and $\pi_1(E_2)$.

We now bring into play the trivializations of the bundle p over B_1 and B_2. If we take the trivialization over $S = B_1 \cap B_2$ to be that induced from the

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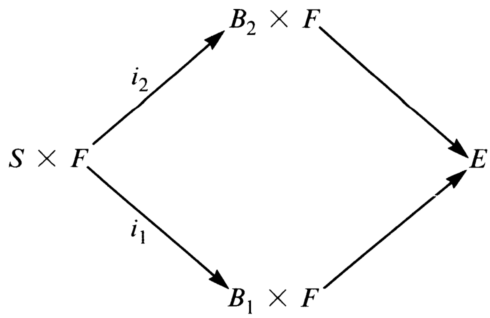
trivialization over B_1, the diagram (1) corresponds to the push-out diagram of spaces

(3)



where

$$i_1(x, e) = (x, e)$$
$$i_2(x, e) = (x, \varphi(x)e)$$

for $x \in S$, $e \in F$.

The change of trivialization map $\varphi: S \to G$ induces a map $\bar{\varphi}: S \to F$ defined by $\bar{\varphi}(x) = \varphi(x)e_0$ for $x \in S$.

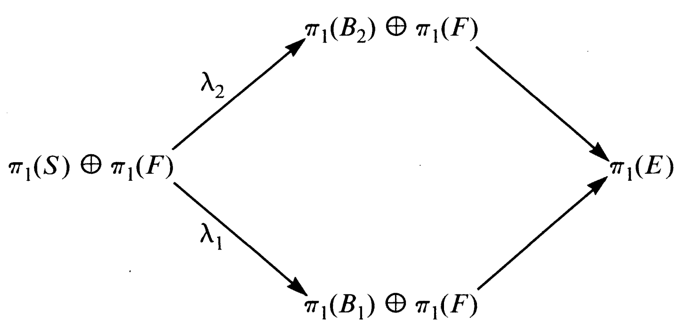
Clearly, $\bar{\varphi}: S \to F$ is just the composition

$$S \xrightarrow{\varphi} G \xrightarrow{r} F,$$

where r is defined by $r(g) = ge_0$ for $g \in G$.

The push-out diagram of spaces (3) induces a push-out diagram of fundamental groups corresponding to the diagram (2). Thereby we get the following

THEOREM. *Let $p: E \to B$ be a bundle in the sense of Steenrod with group G and fibre F, which satisfies the above basic assumptions. Using additive notation in the groups, the fundamental group of the total space, $\pi_1(E)$, is determined by the push-out diagram of fundamental groups*



where

$$\lambda_1(\alpha, \beta) = (i_{1*}(\alpha), \beta)$$
$$\lambda_2(\alpha, \beta) = (i_{2*}(\alpha), \bar{\varphi}_*(\alpha) + \beta)$$

for $\alpha \in \pi_1(S)$, $\beta \in \pi_1(F)$.

The theorem presents $\pi_1(E)$ as a free product with amalgation. We note that the elements $(0, \beta) \in \pi_1(B_1) \oplus \pi_1(F)$ are identified with the corresponding elements in $\pi_1(B_2) \oplus \pi_1(F)$.

From the theorem we can easily derive the following useful corollaries.

COROLLARY 1. *Suppose in addition to the basic assumptions on* $p:E \to B$ *that both* B_1 *and* B_2 *are simply connected. Then we have an isomorphism*

$$\pi_1(E) \cong \pi_1(F)/\text{Image } \bar{\varphi}_*,$$

where Image $\bar{\varphi}_*$ *denotes the normal closure in* $\pi_1(F)$ *for the image of the induced homomorphism* $\bar{\varphi}_*:\pi_1(S) \to \pi_1(F)$.

Corollary 1 is related to ([5], Theorem 10.4, p. 90). An application of Corollary 1 to mapping spaces will be given in Section 3.

COROLLARY 2. *Suppose in addition to the basic assumptions on* $p:E \to B$ *that* $S = B_1 \cap B_2$ *and either* B_1 *or* B_2 *are simply connected. Then we have an isomorphism*

$$\pi_1(E) \cong \pi_1(B) \oplus \pi_1(F).$$

An application of Corollary 2 to Grassmann bundles will be given in Section 4.

COROLLARY 3. *Suppose in addition to the basic assumptions on* $p:E \to B$ *that the group G is simply connected. Then we have an isomorphism*

$$\pi_1(E) \cong \pi_1(B) \oplus \pi_1(F).$$

For the proof of Corollary 3 note that $\bar{\varphi}_* = r_* \circ \varphi_*$ is the trivial homomorphism. Corollary 3 may be useful in connection with some of the $SU(n)$-bundles for $n \geqq 2$, which are important in gauge field theory, see e.g. [1].

3. **An application to mapping spaces.** Let $M_k(S^2, S^2)$ denote the space of (continuous) maps of degree k on the 2-sphere S^2. We shall present a completely elementary computation of the fundamental group of $M_k(S^2, S^2)$ based on Corollary 1.

All spaces are equipped with a base point when necessary. As base point in S^2 we take the point x_0 corresponding to 1 on the equatorial circle S^1 in S^2.

The spaces $M_k(S^2, S^2)$ for all degrees k are the set of (path)-components in the space of maps $M(S^2, S^2)$ on the 2-sphere. Let $F(S^2, S^2)$ denote the corresponding space of based maps, and $F_k(S^2, S^2)$ its component of based maps of degree k.

It is well known [2] that all the spaces $F_k(S^2, S^2)$ have the same homotopy type, and that evaluation at the base point $x_0 \in S^2$ defines a Hurewicz fibration

$$p_k : M_k(S^2, S^2) \to S^2 \text{ with fibre } F_k(S^2, S^2).$$

Let $SO(3)$ denote the group of orientation preserving isometries on S^2, and denote by $SO(2)$ the subgroup of those isometries, which fix the base point $x_0 \in S^2$. Then $SO(2)$ can be identified with the circle S^1, and evaluation at the base point $x_0 \in S^2$ defines a locally trivial fibration

$$p : SO(3) \to S^2 \text{ with fibre } SO(2) = S^1.$$

The fibration p_k is fibre homotopy equivalent to an associated bundle of the $SO(2)$-bundle p. We can define an explicit fibre homotopy equivalence

$$\begin{array}{ccc} SO(3) \times_{SO(2)} F_k(S^2, S^2) & \xrightarrow{\quad \Phi \quad} & M_k(S^2, S^2) \\ & {\scriptstyle \bar{p}_k} \searrow \quad \swarrow {\scriptstyle p_k} & \\ & S^2 & \end{array}$$

by associating to the isometry $A \in SO(3)$ and the based map $f \in F_k(S^2, S^2)$ the free map $A \circ f \in M_k(S^2, S^2)$. The map Φ, which was studied in [3], is clearly a fibre homotopy equivalence by a fundamental theorem of Dold.

Using the homotopy identification Φ we can consider p_k as a bundle in the sense of Steenrod with group $SO(2)$ and fibre $F_k(S^2, S^2)$. In particular, local trivializations of p_k over subspaces of S^2 are then induced from corresponding local trivializations of p.

We decompose S^2 as the union $S^2 = S^2_- \cup S^2_+$ of the southern hemisphere S^2_- and the northern hemisphere S^2_+ with intersection $S^1 = S^2_- \cap S^2_+$. The subspaces S^2_- and S^2_+ of S^2 are both contractible, and hence the fibrations p_k and p are trivial when restricted to S^2_- or S^2_+.

Let $\varphi : S^1 \to SO(2)$ be the map which changes the trivializations of p and p_k when we pass from S^2_- to S^2_+; so chosen, that $\varphi(x_0)$ is the identity isometry on S^2. Then the basic assumptions in Section 2 are satisfied for the bundles p and p_k with respect to the decomposition $S^2 = S^2_- \cup S^2_+$.

For each degree k choose a based map $g_k : S^2 \to S^2$ of degree k, such that g_1 is the identity map and $g_{-k} = g_{-1} \circ g_k$. We take the map g_k as base point in $F_k(S^2, S^2)$.

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The hemispheres S^2_- and S^2_+ are in particular simply connected, and hence by Corollary 1 we get the isomorphisms

$$\pi_1(SO(3)) \cong \pi_1(SO(2))/\text{Image } \varphi_*$$

$$\pi_1(M_k(S^2, S^2)) \cong \pi_1(F_k(S^2, S^2))/\text{Image } \bar{\varphi}_*,$$

where $\bar{\varphi}: S^1 \to F_k(S^2, S^2)$ is the composition

$$S^1 \xrightarrow{\varphi} SO(2) \xrightarrow{r_k} F_k(S^2, S^2),$$

when r_k is defined by composition with g_k on the right, i.e.

$$r_k(A) = A \circ g_k \quad \text{for } A \in SO(2).$$

We note that the groups $\pi_1(SO(2))$ and $\pi_1(F_k(S^2, S^2)) \cong \pi_3(S^2)$ are both isomorphic to the integers.

Since $\pi_1(SO(3))$ is the cyclic group of order 2, we get immediately

LEMMA 1. *The induced homomorphism*

$$\varphi_*: \pi_1(S^1) \to \pi_1(SO(2))$$

is multiplication by 2.

Lemma 1 is also well known by other considerations, ([9], Section 23.4, p. 120) or ([5], Theorem 10.1, p. 89).

LEMMA 2. *The induced homomorphism*

$$r_{k*}: \pi_1(SO(2)) \to \pi_1(F_k(S^2, S^2))$$

is multiplication by k.

PROOF. Let $\bar{g}_k: F(S^2, S^2) \to F(S^2, S^2)$ denote composition with g_k on the right, i.e.

$$\bar{g}_k(f) = f \circ g_k \quad \text{for } f \in F(S^2, S^2).$$

The map r_k can be written as the composition

$$SO(2) \xrightarrow{r_1} F_1(S^2, S^2) \xrightarrow{\bar{g}_k} F_k(S^2, S^2).$$

Define for all degrees k a homotopy equivalence, [2] or [8],

$$\theta_k: F_k(S^2, S^2) \to F_0(S^2, S^2)$$

by

$$\theta_k(f) = \nabla \circ (f \vee g_{-k}) \circ \nu,$$

where $\nu: S^2 \to S^2 \vee S^2$ is the coproduct and $\nabla: S^2 \vee S^2 \to S^2$ is the folding map.

Since $g_{-k} = g_{-1} \circ g_k$, the following diagram is commutative

$$\begin{array}{ccc} F_1(S^2, S^2) & \xrightarrow{\bar{g}_k} & F_k(S^2, S^2) \\ \Big\downarrow{\scriptstyle \theta_1} & & \Big\downarrow{\scriptstyle \theta_k} \\ F_0(S^2, S^2) & \xrightarrow[\bar{g}_k]{} & F_0(S^2, S^2) \end{array}$$

The homomorphism $r_{k*} = \bar{g}_{k*} \circ r_{1*}$ can now be analysed using the commutative diagram

$$\begin{array}{ccccc} \pi_1(SO(2)) & \xrightarrow{r_{1*}} & \pi_1(F_1(S^2, S^2)) & \xrightarrow{\bar{g}_{k*}} & \pi_1(F_k(S^2, S^2)) \\ & \searrow{\scriptstyle J} & \Big\downarrow{\scriptstyle \theta_{1*}} & & \Big\downarrow{\scriptstyle \theta_{k*}} \\ & & \pi_1(F_0(S^2, S^2)) & \xrightarrow[\bar{g}_{k*}]{} & \pi_1(F_0(S^2, S^2)) \\ & & \Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle \cong} \\ & & \pi_3(S^2) & \xrightarrow[(\Sigma g_k)^*]{} & \pi_3(S^2) \end{array}$$

In this diagram J is the classical J-homomorphism, $(\Sigma g_k)^*$ is the homomorphism induced by the suspension of g_k, and the unnamed isomorphisms are adjoint isomorphisms.

It is well known that J is an isomorphism, ([11], Theorem 4). To be specific this follows easily by observing that the standard generator for $\pi_1(SO(2))$ is mapped onto the generator for $\pi_3(S^2)$ represented by the map arising from the Hopf construction applied to the complex multiplication $S^1 \times S^1 \to S^1$. Compare with ([12], Examples 1 and 2, pp. 503-504).

The Hopf fibration induces an isomorphism $\pi_3(S^3) \cong \pi_3(S^2)$. Then it follows immediately that the homomorphism $(\Sigma\, g_k)^*$, and hence the homomorphism $\bar{g}_{k*}$, is multiplication by k. See also ([8], Lemma 2.1).

Altogether it follows that $r_{k*} = \bar{g}_{k*} \circ r_{1*}$ is multiplication by k, and Lemma 2 is proved.

The induced homomorphism

$$\bar{\varphi}_*: \pi_1(S^1) \to \pi_1(F_k(S^2, S^2))$$

can be written as the composition $\bar{\varphi}_* = r_{k*} \circ \varphi_*$. Therefore by Lemma 1 and Lemma 2, $\bar{\varphi}_*$ must be multiplication by $2k$. But then

$$\begin{aligned} \pi_1(M_k(S^2, S^2)) &\cong \pi_1(F_k(S^2, S^2))/\text{Image } \bar{\varphi}_* \\ &\cong Z/2|k| \end{aligned}$$

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is the cyclic group of order $2|k|$ for $k \neq 0$, the integers for $k = 0$. The result is recorded in the following

THEOREM (S. T. Hu). *The fundamental group of $M_k(S^2, S^2)$ is cyclic of order $2|k|$ for $k \neq 0$, and infinite cyclic for $k = 0$.*

The fundamental group $\pi_1(M_k(S^2, S^2))$ was first computed by Hu [4] using properties of Whitehead products.

4. **An application to Grassmann bundles.** Let X be a space which admits the structure of an adjunction space

$$X = A \cup_f D^n,$$

where D^n is an n-cell attached to a CW-complex A of dimension $\leqq n - 1$ along the map $f:S^{n-1} \to A$. As an example, we note that any connected smooth n-dimensional manifold admits such a structure.

Let $p:V \to X$ be an n-dimensional vector bundle over X. For $1 \leqq k < n$ consider the induced bundle $\bar{p}:M_k(V) \to X$, whose fibre over $x \in X$ is the Grassmann manifold of k-planes in the corresponding fibre V_x of p. Then $\bar{p}$ is a bundle in the sense of Steenrod with the general linear group $Gl(n)$ in real n-space as group and the Grassmann manifold $M_{n,k}$ of k-planes in real n-space as fibre.

Choose an n-cell D_1^n embedded in $\operatorname{int} D^n$, where int denotes interior of a space. Then $X \setminus \operatorname{int} D_1^n$ has the homotopy type of A, and the intersection $S = (X \setminus \operatorname{int} D_1^n) \cap D_1^n$ is an $(n - 1)$-sphere.

Suppose now that $p:V \to X$ is stably trivial.

A stably trivial n-dimensional vector bundle is trivial over a complex of dimension $<n$, ([7], Lemma 4). Therefore the vector bundle p, and hence also the Grassmann bundle $\bar{p}$, is trivial over $X \setminus \operatorname{int} D_1^n$ as well as over the contractible space D_1^n.

For $n \geqq 3$, the intersection $S = (X \setminus \operatorname{int} D_1^n) \cap D_1^n$ is simply connected. Using a well-known fact about the fundamental group of Grassmann manifolds ([9], Section 25.8, p. 134), we get then by Corollary 2 the following isomorphisms for $n \geqq 3$,

$$\begin{aligned} \pi_1(M_k(V)) &\cong \pi_1(X) \oplus \pi_1(M_{n,k}) \\ &\cong \pi_1(X) \oplus \mathbf{Z}/2\mathbf{Z}. \end{aligned}$$

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