ON THE STRONGLY COUNTABLE-DIMENSIONALITY OF μ-SPACES

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Nagata in [3] defined *strongly countable-dimensional spaces* which are the countable union of closed finite-dimensional subspaces. Walker and Wenner in [7] characterized such metric spaces as follows: a space X is a strongly countable-dimensional metric space if and only if there exists a finite-to-one closed mapping of a zero-dimensional metric space onto X with weak local order.

In this paper, we consider strongly countable-dimensionality for the class of μ-spaces in the sense of Nagami [5] and show that the above characterization is generalized to this class. To begin with, we give the definition of a mapping with weak local order, which is introduced in [7]. A mapping $f: X \to Y$ is called to have *weak local order* if for each point $y \in Y$ there exist a point $x \in f^{-1}(y)$, an open neighborhood U of x and a positive integer n such that $\operatorname{ord} f \mid U \leq n$. All spaces are assumed to be Hausdorff and all mappings to be continuous and onto. N always denotes the set of all positive integers. An open collection of a space X means a collection of open sets of X. For a point p of a space X and for a collection $\mathcal{U}$ of sets of X we denote by $\operatorname{ord}_p \mathcal{U}$ the largest integer n such that there exist n members of $\mathcal{U}$ which contain p, and denote $\sup\{\operatorname{ord}_p \mathcal{U} : p \in X\}$ by $\operatorname{ord} \mathcal{U}$.

LEMMA 1. *Let X be a hereditarily collectionwise normal space. Then X satisfies the following* (α):

(α) *Let F be a closed subspace of X and $\mathcal{H} = \{H_\alpha : \alpha \in A\}$ a collection of pairwise disjoint open sets of F. Then there exists a collection $\mathcal{H}' = \{H'_\alpha : \alpha \in A\}$ of pairwise disjoint open sets of X such that*

$$H'_\alpha \cap F = H_\alpha, \qquad \alpha \in A.$$

Proof. Observe that $\mathcal{H}$ is a discrete collection of closed sets of the subspace

$$X' = \bigcup \{H_\alpha : \alpha \in A\} \cup (X - F).$$

Since X' is collectionwise normal, there exists a collection $\mathcal{H}' = \{H'_\alpha : \alpha \in A\}$ of pairwise disjoint open sets of X', and hence of X, such that $H_\alpha \subset H'_\alpha$, $\alpha \in A$. This implies

$$H'_\alpha \cap F = H_\alpha, \qquad \alpha \in A.$$

LEMMA 2. *Let X be a hereditarily collectionwise normal space. Then X satisfies the following* (β):

(β) *Let F be a closed subspace of X and $\mathcal{H} = \{H_\alpha : \alpha \in A\}$ an open cover of F with* $\operatorname{ord} \mathcal{H} \leq m$, *where $m \in N$. Then there exists an open collection $\mathcal{G} = \{G_\alpha : \alpha \in A\}$ of X covering*

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F such that

$$G_\alpha \cap F = H_\alpha, \qquad \alpha \in A,$$
$$\operatorname{ord} \mathcal{G} \le \frac{m(m+1)}{2}.$$

Proof. We shall prove (β) by induction on m. Consider the case $m=1$. In this case, $\mathcal{H}$ is a collection of pairwise disjoint open sets of F. By Lemma 1, there exists a collection $\mathcal{G}=\{G_\alpha : \alpha \in A\}$ of pairwise disjoint open sets of X with the required property. Thus (β) is true for $m=1$. Assume that the theorem is true for all open covers with ord$\le m$ of a closed subspace of X. Let $\mathcal{H}=\{H_\alpha : \alpha \in A\}$ be an open cover of F with ord $\mathcal{H} \le m+1$. Let

$$A^* = \{B \subset A : |B| = m+1\},$$
$$H_B = \bigcap \{H_\alpha : \alpha \in B\}, \qquad B \in A^*.$$

Then $\{H_B : B \in A^*\}$ is a collection of pairwise disjoint open sets of F. Therefore by Lemma 1 there exists a collection $\{G_B : B \in A^*\}$ of pairwise disjoint open sets of X such that

$$G_B \cap F = H_B, \qquad B \in A^*.$$

Let F_0 be the set of all points x of F such that

$$|\{\alpha \in A : x \in H_\alpha\}| \le m.$$

Then F_0 is closed in F, and hence in X. Since $\{F_0 \cap H_\alpha : \alpha \in A\}$ is an open cover of F_0 with ord$\le m$, by the induction assumption there exists an open collection $\mathcal{M}=\{M_\alpha : \alpha \in A\}$ of X such that

$$M_\alpha \cap F_0 = F_0 \cap H_\alpha, \qquad \alpha \in A,$$
$$\operatorname{ord} \mathcal{M} \le \frac{m(m+1)}{2}.$$

Set

$$G_\alpha = M_\alpha \cup \left(\bigcup \{G_B : \alpha \in B\}\right), \qquad \alpha \in A,$$
$$\mathcal{G} = \{G_\alpha : \alpha \in A\}.$$

Then each G_α is an open set of X such that $G_\alpha \cap F = H_\alpha$. We shall show that ord $\mathcal{G} \le (m+1)(m+2)/2$. Let p be an arbitrary point of X. Since $\{G_B : B \in A^*\}$ is pairwise disjoint, there exists at most one $B \in A^*$ such that $p \in G_B$. Set

$$K = \{\alpha \in A : p \in M_\alpha\}, \qquad K' = K \cup B.$$

Then we have

$$|K'| \le \frac{m(m+1)}{2} + m + 1 = \frac{(m+1)(m+2)}{2}.$$

Since $p \in G_\alpha$, $\alpha \in A$ implies $\alpha \in K'$, we have ord $\mathcal{G} \le (m+1)(m+2)/2$. This completes the proof.

LEMMA 3. *Let X be a hereditarily collectionwise normal space. If X is a strongly countable-dimensional space such that for a closed cover* $\{F_k : k \in N\}$,

$$X = \bigcup_{k=1}^{\infty} F_k, \qquad \dim F_k \leq n_k < \infty, \qquad k \in N,$$

then for every locally finite open cover $\mathcal{U}$ of X there exists an open cover $\mathcal{V}$ of X such that $\mathcal{V}$ is a refinement of $\mathcal{U}$ and

$$\operatorname{ord}_p \mathcal{V} \leq N_k \quad \text{if} \quad p \in F_k, \qquad k \in N,$$

where

$$N_k = m_1 + \ldots + m_k, \qquad m_i = \frac{(n_i+1)(n_i+2)}{2}, \qquad k, i \in N.$$

Proof. Let k be an arbitrary fixed number. By [**6**, Th. 4.3, p. 132] there exists an open cover $\mathcal{U}_k = \{U_\alpha : \alpha \in A\}$ of the subspace F_k satisfying

$$\mathcal{U}_k < \mathcal{U}, \qquad \operatorname{ord} \mathcal{U}_k \leq n_k + 1.$$

By Lemma 2 and its proof, there exists an open collection $\mathcal{U}'_k = \{U'_\alpha : \alpha \in A\}$ of open sets of X such that

$$U'_\alpha \cap F_k = U_\alpha, \qquad \alpha \in A, \qquad \mathcal{U}'_k < \mathcal{U},$$

$$\operatorname{ord} \mathcal{U}'_k \leq \frac{(n_k+1)(n_k+2)}{2} = m_k.$$

Set

$$\mathcal{V}_k = \left\{ U'_\alpha \cap \left(X - \bigcup_{j=1}^{k-1} F_j \right) : \alpha \in A \right\}.$$

Then $\mathcal{V}_k$ is an open cover of $F_k - \bigcup_{j=1}^{k-1} F_j$ in X satisfying

$$\mathcal{V}_k < \mathcal{U},$$

$$\operatorname{ord}_p \mathcal{V}_k \leq m_k \quad \text{if} \quad p \in \bigcup_{j=k}^{\infty} F_j,$$

$$\operatorname{ord}_p \mathcal{V}_k = 0 \quad \text{if} \quad p \in \bigcup_{j=1}^{k-1} F_j.$$

Set

$$\mathcal{V} = \bigcup_{k=1}^{\infty} \mathcal{V}_k.$$

Then $\mathcal{V}$ is an open cover of X such that

$$\mathcal{V} < \mathcal{U},$$

$$\operatorname{ord}_p \mathcal{V} \leq m_1 + \ldots + m_k = N_k \quad \text{if} \quad p \in F_k, \qquad k \in N.$$

This completes the proof.

Note that from the above proof, we can take $\mathcal{V}$ to be locally finite in X.

LEMMA 4. *Let f be a mapping of a hereditarily collectionwise normal, totally normal and strongly countable-dimensional space X onto a metric space Y. Then there exist mappings g, h such that g is that of X onto a strongly countable-dimensional metric space Z and h is that of Z onto Y such that $f = hg$.*

Proof. Let

$$X = \bigcup_{k=1}^{\infty} F_k, \qquad \dim F_k \leq n_k < \infty, \qquad i \in N,$$

where each F_k is closed in X. Since Y is a metric space, there exists a sequence $\{\mathcal{U}_i : i \in N\}$ of locally finite open covers of Y such that mesh$\mathcal{U}_i < 1/i$ for each $i \in N$. By induction and by virtue of Lemma 3 we can define a sequence $\{\mathcal{V}_i : i \in N\}$ of locally finite open covers of X satisfying the following:

(1) $\mathcal{V}_{i+1}$ is a star-refinement of $\mathcal{V}_i \wedge f^{-1}(\mathcal{U}_i)$, $u \in N$,

(2) $\operatorname{ord}_p \mathcal{V}_i \leq N_k$ if $p \in F_k$, $k \in N$, $i \in N$,

where N_k is the number defined in Lemma 3. For this $\{\mathcal{V}_i\}$, construct X', $i : X \to X'$, $f' : X' \to Y$, $\mathcal{V}_i^*$, $\mathcal{W}_i$, $[x]$, $g : X \to Z$ and $h : Z \to Y$ by the same method as in the proof of [**1**, Theorem 4.2.5]. Then the mappings $g : X \to Z$ and $h : Z \to Y$ have the required properties. Since $\{\mathcal{W}_i : i \in N\}$ is easily seen to be a sequence of open covers of Z satisfying the following:

(3) $\{\mathcal{W}_i\}$ is a development of Z,

(4) $\operatorname{ord}_{[p]} \mathcal{W}_i \leq N_k$ for $i \in N$ if $p \in g(F_k)$, $k \in N$.

(5) $\mathcal{W}_{i+1}$ is a star-refinement of $\mathcal{W}_i$ for each $i \in N$.

By [**3**, Theorem 5.3] Z is a strongly countable-dimensional metric space. This completes the proof.

Of course it follows easily that if f is one-to-one, then so are both of g and h.

Let $\rho : X \to \hat{X}$ be a one-to-one mapping of a space X onto a metric space $\hat{X}$ and $g : Y \to \hat{X}$ a mapping of a metric space Y onto X. Construct the subspace Z of $X \times Y$ as follows:

$$Z = \{(x, y) \in X \times Y : \rho(x) = g(y)\}.$$

Let $f : Z \to X$ and $\sigma : Z \to Y$ be the restrictions to Z of the projections.

LEMMA 5. *Let f, g be the same mappings stated above.* (1) *If g is a perfect mapping, then so is f.* (2) *If g is finite-to-one, then so is f.* (3) *If g has weak local order, then so does f.*

Proof. (1) and (2) follow from the argument of [**6**, Lemma 5.13, p. 293]. To see (3), let x be an arbitrary point of X. Since g has weak local order, there exist a point $a \in g^{-1}(\rho(x))$, its open neighbourhood U and $n \in N$ such that $\operatorname{ord} g \mid U \leq n$. Let

$$V = \sigma^{-1}(U), \qquad z = \sigma^{-1}(a) \in Z,$$

where σ is a mapping stated above. Then V is an open neighborhood of z in Z such that $\operatorname{ord} f \mid V \leq n$. Hence f has weak local order.

THEOREM. *Let $\mathscr{C}$ be the class of all μ-spaces and $X \in \mathscr{C}$. Then X is a strongly countable-dimensional space if and only if there exists a closed, finite-to-one mapping from $Z \in \mathscr{C}$ with* $\dim Z \leq 0$ *onto X with weak local order.*

Proof. Only if part is proved by the argument parallel to that of the proof of [**2**, Th.1], and therefore we describe the outline. Let X be a strongly countable-dimensional μ-space. Then it is seen that X is the inverse limit of a sequence $\{X_i, g_j^i\}$, where each X_i, $i \geq 2$, is a paracompact σ-metric space with (X_1, g_1^i) as its replica in the sense of [**4**]. Each projection $g_i : X \to X_i$ is a one-to-one mapping. By virtue of Lemma 4, there exist one-to-one mappings $h_1 : X \to Y_1$ and $k_1 : Y_1 \to X_1$ such that Y_1 is a strongly countable-dimensional metric space and $g_1 = k_1 h_1$. For each $i \geq 2$, construct the subspace Y_i of $X_i \times Y_1$ by

$$Y_i = \{(x, y) \in X_i \times Y_1 : g_1^i(x) = k_1(y)\}.$$

Let $k_i : Y_i \to X_i$ and $h_1^i : Y_i \to Y_1$ be the restrictions of the projections. Each Y_i is strongly countable-dimensional because Y_1 is so and (Y_1, h_1^i) is the replica of Y_i, [**4**, Th. 6]. For each pair i, j with $i > j$, h_j^i is defined by $h_j^i = (h_1^j)^{-1} h_1^i$. Since Y_j has the base

$$\{k_j^{-1}(U) \cap (h_1^j)^{-1}(V) : U, V \text{ are open in } X_i, Y_1, \text{ respectively}\},$$

h_j^i is continuous. It follows from this that X is homeomorphic to $\varprojlim\{Y_i, h_j^i\}$, where each Y_i, $i \geq 2$, is a strongly countable-dimensional paracompact σ-metric space. Thus we can write $X = \varprojlim\{Y_i, h_j^i\}$, By the theorem of Walker and Wenner, there exists a closed, finite-to-one mapping f_1 of a zero-dimensional metric space Z_1 onto Y_1 with weak local order. By the similar way to the construction of Y_i and $h_j^i : Y_i \to Y_j$ from X_i, Y_1, g_1^i and k_1 we define a zero-dimensional paracompact σ-metric space Z_i and mappings $m_j^i : Z_1 \to Z_j$, $i > j$. Let $Z = \varprojlim\{Z_i, m_j^i\}$. Then Z is a zero-dimensional μ-space. Let $f : Z \to X$ be defined by

$$f(x) = (f_i(x_i)), \qquad x = (x_i) \in Z.$$

Then f is similarly shown to be a closed, finite-to-one mapping of Z onto X, [**2**, Th. 1]. Since f_i has weak local order by Lemma 5, (3) and m_j^i is one-to-one, it is easily seen that f has weak local order. This completes the proof of the only if part. Since the if part is proved by the same argument as in the proof of [**7**, Th. 2]. Thus we complete the proof.

Nagata called a space X *countable-dimensional* if X is the countable union of zero-dimensional subspaces, and proved the following, [**3**]:

THEOREM. *Let $\mathscr{C}$ be the class of all metric spaces and $X \in \mathscr{C}$. Then X is countable-dimensional if and only if there exists a closed, finite-to-one mapping of a zero-dimensional space $Z \in \mathscr{C}$ onto X.*

The author does not know whether this holds or not for the class of all μ-spaces.

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