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BMO-Estimates for Maximal Operators via Approximations of the Identity with Non-Doubling Measures

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Abstract. Let μ be a nonnegative Radon measure on $\mathbb{R}^d$ that satisfies the growth condition that there exist constants $C_0 > 0$ and $n \in (0, d]$ such that for all $x \in \mathbb{R}^d$ and $r > 0$, $\mu(B(x, r)) \le C_0 r^n$, where $B(x, r)$ is the open ball centered at x and having radius r. In this paper, the authors prove that if f belongs to the BMO-type space RBMO(μ) of Tolsa, then the homogeneous maximal function $\dot{\mathcal{M}}_S(f)$ (when $\mathbb{R}^d$ is not an initial cube) and the inhomogeneous maximal function $\mathcal{M}_S(f)$ (when $\mathbb{R}^d$ is an initial cube) associated with a given approximation of the identity S of Tolsa are either infinite everywhere or finite almost everywhere, and in the latter case, $\dot{\mathcal{M}}_S$ and $\mathcal{M}_S$ are bounded from RBMO(μ) to the BLO-type space RBLO(μ). The authors also prove that the inhomogeneous maximal operator $\mathcal{M}_S$ is bounded from the local BMO-type space rbmo(μ) to the local BLO-type space rblo(μ).

1 Introduction

In recent years, it has been shown that many results on the Calderón–Zygmund theory remain valid for non-doubling measures; see, for example, [3–7] and their references. Recall that a non-doubling measure μ on $\mathbb{R}^d$ means that μ is a nonnegative Radon measure that only satisfies the following growth condition: there exist constants $C_0 > 0$ and $n \in (0, d]$ such that for all $x \in \mathbb{R}^d$ and $r > 0$,

$$\mu\,(B(x, r)) \le C_0 r^n,$$

where $B(x, r)$ is the open ball centered at x and having radius r. Such a measure μ is not necessarily doubling, which is a key assumption in the classical theory of harmonic analysis. One of main motivations for extending the classical theory to the non-doubling context was the solution of several questions related to analytic capacity, like Vitushkin's conjecture or Painlevé's problem; see [8, 9, 11] or survey papers [10, 12, 13] for more details.

In particular, Tolsa [6] constructed a class of approximation of the identity, and as applications, Tolsa developed a Littlewood–Paley theory with non-doubling measures for functions in $L^p(\mu)$ when $p \in (1, \infty)$ and established some $T(1)$ theorems. In [15], the authors introduced the homogeneous and inhomogeneous maximal operators $\dot{\mathcal{M}}_S$ and $\mathcal{M}_S$ associated with a given approximation of the identity S of Tolsa

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in [6], and proved that both $\dot{\mathcal{M}}_S$ and $\mathcal{M}_S$ are bounded on $L^p(\mu)$ for $p \in (1,\infty)$, that $\dot{\mathcal{M}}_S$ is bounded from the Hardy space $H^1(\mu)$ of Tolsa [5] to $L^1(\mu)$, and that $\mathcal{M}_S$ is bounded from a local version of the Hardy space $h^{1,\infty}_{\rm atb}(\mu)$, which was introduced in [2], to $L^1(\mu)$. The main purpose of this paper is to consider the BMO-boundedness of $\dot{\mathcal{M}}_S$ and $\mathcal{M}_S$ at another extremal case, namely, when $p = \infty$. To be precise, we first prove that if f belongs to the BMO-type space RBMO(μ) of Tolsa [5], then $\dot{\mathcal{M}}_S(f)$ (when $\mathbb{R}^d$ is not an initial cube) and $\mathcal{M}_S(f)$ (when $\mathbb{R}^d$ is an initial cube) are either infinite everywhere or finite almost everywhere, and in the latter case, $\dot{\mathcal{M}}_S$ and $\mathcal{M}_S$ are bounded from RBMO(μ) to the BLO-type space RBLO(μ) of Jiang [3]. We also prove that $\mathcal{M}_S$ is bounded from the local BMO-type space rbmo(μ) to the local BLO-type space rblo(μ), which were introduced in [2]. It is known that rblo$(\mu) \subset$ rbmo(μ) and RBLO$(\mu) \subset$ RBMO(μ). On the other hand, even in the case that μ is the d-dimensional Lebesgue measure, a BMO$(\mathbb{R}^d)$ function with essential lower bound is not necessary to belong to BLO$(\mathbb{R}^d)$; for example, consider the function $(\log|x|)\chi_{\{x\in\mathbb{R}^d:\,|x|\geq 1\}}(x)$, where and in what follows, for any $D \subset \mathbb{R}^d$, χ_D denotes the characteristic function of D. An interesting open problem is whether $\dot{\mathcal{M}}_S$ (or $\mathcal{M}_S$) can characterize the Hardy space $H^1(\mu)$ (or $h^{1,\infty}_{\rm atb}(\mu)$). Recall that the dual spaces of $H^1(\mu)$ and $h^{1,\infty}_{\rm atb}(\mu)$ are proved, respectively, to be the spaces RBMO(μ) and rbmo(μ) in [5] and [2].

The organization of this paper is as follows. In Section 2, we recall some necessary notions and notation. In Section 3, we prove that if f belongs to RBMO(μ), then $\dot{\mathcal{M}}_S(f)$ (when $\mathbb{R}^d$ is not an initial cube) and $\mathcal{M}_S(f)$ (when $\mathbb{R}^d$ is an initial cube) are either infinite everywhere or finite almost everywhere, and in the latter case, $\dot{\mathcal{M}}_S(f)$ and $\mathcal{M}_S(f)$ are bounded from RBMO(μ) to RBLO(μ). In this section, we also establish the boundedness of $\mathcal{M}_S$ from rbmo(μ) to rblo(μ). Differently from the homogeneous case, for any $f \in$ rbmo(μ), $\mathcal{M}_S(f)(x) < \infty$ for μ-almost everywhere $x \in \mathbb{R}^d$. The results in this paper are also new even when μ is the d-dimensional Lebesgue measure.

Throughout the paper, we always denote by C a positive constant that is independent of the main parameters, but it may vary from line to line. Constants with subscripts such as C_1, do not change in different occurrences. The symbol $Y \lesssim Z$ means that there exists a positive constant C such that $Y \leq CZ$. For any $f \in L^1_{\rm loc}(\mu)$ and cube Q, $m_Q(f)$ denotes the mean of f over Q, namely, $m_Q(f) \equiv \frac{1}{\mu(Q)}\int_Q f(y)\,d\mu(y)$.

2 Preliminaries

In this section, we recall some necessary notions and notation. By a cube $Q \subset \mathbb{R}^d$, we mean a closed cube whose sides are parallel to the axes and centered at some point of supp(μ), and we denote its side length by $l(Q)$ and its center by x_Q. If $\mu(\mathbb{R}^d) < \infty$, we also regard $\mathbb{R}^d$ as a cube. Let $\alpha,\ \beta$ be two positive constants, $\alpha \in (1,\infty)$ and $\beta \in (\alpha^n,\infty)$. A cube Q is said to be an $(\alpha,\ \beta)$-doubling cube if it satisfies $\mu(\alpha Q) \leq \beta\mu(Q)$, where and in what follows, given $\lambda > 0$ and any cube Q, λQ denotes the cube concentric with Q and having side length $\lambda l(Q)$. It was pointed out by Tolsa (see [5, pp. 95-96] or [6, Remark 3.1]) that if $\beta > \alpha^n$, then for any $x \in$ supp(μ) and any $R > 0$, there exists some $(\alpha,\ \beta)$-doubling cube Q centered at x with $l(Q) \geq R$, and that if $\beta > \alpha^d$, then for μ-almost everywhere $x \in \mathbb{R}^d$, there exists a sequence of (α, β)-doubling cubes, $\{Q_k\}_{k\in\mathbb{N}}$, centered at x with $l(Q_k) \to 0$ as $k \to \infty$. Let $\rho \in (1, \infty)$. Throughout this paper, we always take $\beta_\rho \equiv \rho^{d+1}$. For any cube Q, let $\widetilde{Q}^\rho$ be the smallest (ρ, β_ρ)-doubling cube that has the form $\rho^k Q$ with $k \in \mathbb{N} \cup \{0\}$. We denote $\widetilde{Q}^2$ simply by $\widetilde{Q}$. Moreover, by a doubling cube Q, we always mean a $(2, 2^{d+1})$-doubling cube.

Given two cubes $Q,\ R \subset \mathbb{R}^d$, let x_Q be the center of Q, and Q_R be the smallest cube concentric with Q containing Q and R. The following coefficients were first introduced by Tolsa in [5]; see also [6, 7].

Definition 2.1 Given two cubes $Q,\ R \subset \mathbb{R}^d$, we define

$$\delta(Q, R) \equiv \max\left\{ \int_{Q_R\setminus Q} \frac{1}{|x - x_Q|^n}\, d\mu(x),\ \int_{R_Q\setminus R} \frac{1}{|x - x_R|^n}\, d\mu(x) \right\}.$$

We may treat points $x \in \mathbb{R}^d$ as if they were cubes (with side length $l(x) = 0$). So, for $x,\ y \in \mathbb{R}^d$ and some cube Q, the notations $\delta(x, Q)$ and $\delta(x, y)$ make sense.

The following useful properties of $\delta(\cdot, \cdot)$, which were proved in [7, pp. 320-321] (see also [6, Lemma 3.1]), play important roles throughout the paper.

Lemma 2.2 *There exists a positive constant C, which only depends on C_0, n, d, and ρ, such that the following properties hold:*

(i) *If $l(Q) \sim l(R)$ and $\operatorname{dist}(Q, R) \le Cl(Q)$, then $\delta(Q, R) \le C$. Moreover, for any $\eta \in (1, \infty)$, $\delta(Q, \eta Q) \le C_0 2^n \eta^n$.*

(ii) *Let $\rho \in (1, \infty)$ and $Q \subset R$ be concentric cubes such that there exist no (ρ, β_ρ)-doubling cubes of the form $\rho^k Q$, $k \ge 0$, with $Q \subset \rho^k Q \subset R$. Then $\delta(Q, R) \le C$.*

(iii) *If $Q \subset R$, then $\delta(Q, R) \le C[1 + \log \frac{l(R)}{l(Q)}]$.*

(iv) *There exists a positive ϵ_0 such that if $P \subset Q \subset R$, then*

$$|\delta(P, R) - [\delta(P, Q) + \delta(Q, R)]| \le \epsilon_0.$$

In particular, $\delta(P, Q) \le \delta(P, R) + \epsilon_0$ and $\delta(Q, R) \le \delta(P, R) + \epsilon_0$. Moreover, if P and Q are concentric, then $\epsilon_0 = 0$.

(v) *For any $P,\ Q,\ R \subset \mathbb{R}^d$, $\delta(P, R) \le C + \delta(P, Q) + \delta(Q, R)$.*

We now recall the notion of cubes of generations; see [6, 7] for more details.

Definition 2.3 We say that $x \in \mathbb{R}^d$ is a stopping point (or stopping cube) if $\delta(x, Q) < \infty$ for some cube $Q \ni x$ with $l(Q) \in (0, \infty)$. We say that $\mathbb{R}^d$ is an initial cube if $\delta(Q, \mathbb{R}^d) < \infty$ for some cube Q with $l(Q) \in (0, \infty)$. The cubes Q such that $l(Q) \in (0, \infty)$ are called transit cubes.

Remark 2.4 In [6, p. 67], it was pointed out that if $\delta(x, Q) < \infty$ for some transit cube Q containing x, then $\delta(x, Q') < \infty$ for any other transit cube Q' containing x. Also, if $\delta(Q, \mathbb{R}^d) < \infty$ for some transit cube Q, then $\delta(Q', \mathbb{R}^d) < \infty$ for any transit cube Q'.

Let A be some big positive constant. In particular, we assume that A is much bigger than the constants ϵ_0, ϵ_1, and γ_0, which appear, respectively, in [6, Lemmas 3.1–3.3]. Moreover, the constants A, ϵ_0, ϵ_1, and γ_0 depend only on C_0, n, and d.

Definition 2.5 Assume that $\mathbb{R}^d$ is not an initial cube. We fix some doubling cube $R_0 \subset \mathbb{R}^d$. This will be our "reference" cube. For each $j \in \mathbb{N}$, let R_{-j} be some doubling cube concentric with R_0, containing R_0, and such that $|\delta(R_0, R_{-j}) - jA| \le \epsilon_1$ (which exists because of [6, Lemma 3.3]). If Q is a transit cube, we say that Q is a cube of generation $k \in \mathbb{Z}$ if it is a doubling cube, and for some cube R_{-j} containing Q, we have $|\delta(Q, R_{-j}) - (j + k)A| \le \epsilon_1$. If $Q \equiv \{x\}$ is a stopping cube, we say that Q is a cube of generation $k \in \mathbb{Z}$ if for some cube R_{-j} containing x, we have $\delta(Q, R_{-j}) \le (j + k)A + \epsilon_1$.

We remark that the definition of cubes of generations is proved in [6, p. 68] to be independent of the chosen reference cubes R_{-j} in the sense modulo some small errors.

Definition 2.6 Assume that $\mathbb{R}^d$ is an initial cube. Then we choose $\mathbb{R}^d$ as our "reference" cube: If Q is a transit cube, we say that Q is a cube of generation $k \ge 1$, if Q is doubling and $|\delta(Q, \mathbb{R}^d) - kA| \le \epsilon_1$. If $Q \equiv \{x\}$ is a stopping cube, we say that Q is a cube of generation $k \ge 1$ if $\delta(x, \mathbb{R}^d) \le kA + \epsilon_1$. Moreover, for all $k \le 0$, we say that $\mathbb{R}^d$ is a cube of generation k.

Using [6, Lemma 3.2], it is easy to verify that for any $x \in \operatorname{supp}(\mu)$ and $k \in \mathbb{Z}$, there exists a doubling cube of generation k; see [6, p. 68]. Moreover, from [14, Proposition 2.1] and Definition 2.6, it follows that for any $x \in \operatorname{supp}(\mu)$, $l(Q_{x,k}) \to \infty$ as $k \to -\infty$. Throughout this paper, for any $x \in \operatorname{supp}(\mu)$ and $k \in \mathbb{Z}$, we denote by $Q_{x,k}$ a fixed doubling cube centered at x of generation k.

Remark 2.7 We should point out that when $\mathbb{R}^d$ is an initial cube, cubes of generations in [6] were not assumed to be doubling. However, by using [6, Lemma 3.2], it is easy to check that doubling cubes of generations exist even in this case.

In [6], Tolsa constructed a class of approximation of the identity $\{S_k\}_{k=-\infty}^{\infty}$ related to $\{Q_{x,k}\}_{x\in\mathbb{R}^d,\, k\in\mathbb{Z}}$, which are integral operators given by kernels $S_k(x, y)$ on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying the following properties:

(A-1) $S_k(x, y) = S_k(y, x)$ for all $x,\ y \in \mathbb{R}^d$;

(A-2) for any $k \in \mathbb{Z}$ and any $x \in \operatorname{supp}(\mu)$, if $Q_{x,k}$ is a transit cube, then

$$\int_{\mathbb{R}^d} S_k(x, y)\, d\mu(y) = 1;$$

(A-3) if $Q_{x,k}$ is a transit cube, then $\operatorname{supp}(S_k(x, \cdot)) \subset Q_{x,k-1}$;

(A-4) if $Q_{x,k}$ and $Q_{y,k}$ are transit cubes, then there exists a constant $C > 0$ such that

$$0 \le S_k(x, y) \le \frac{C}{[l(Q_{x,k}) + l(Q_{y,k}) + |x - y|]^n};$$

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(A-5) if $Q_{x,k}$, $Q_{x',k}$, and $Q_{y,k}$ are transit cubes, and $x, x' \in Q_{x_0,k}$ for some $x_0 \in \operatorname{supp}(\mu)$, then there exists a constant $C > 0$ such that

$$|S_k(x,y) - S_k(x',y)| \le C \frac{|x-x'|}{l(Q_{x_0,k})} \frac{1}{[l(Q_{x,k}) + l(Q_{y,k}) + |x-y|]^n}.$$

Moreover, Tolsa [6] pointed out that properties (A-1)–(A-5) also hold if any of $Q_{x,k}$, $Q_{x',k}$, and $Q_{y,k}$ is a stopping cube, and that (A-1) and (A-3)–(A-5) also hold if any of $Q_{x,k}$, $Q_{x',k}$, and $Q_{y,k}$ coincides with $\mathbb{R}^d$, except that (A-2) is replaced by (A-2)': if $Q_{x,k} = \mathbb{R}^d$ for some $x \in \operatorname{supp}(\mu)$, then $S_k = 0$. In what follows, without loss of generality, for any $x \in \operatorname{supp}(\mu)$, we may always assume that $Q_{x,k}$ is not a stopping cube, since the proofs for stopping cubes are similar.

For any $k \in \mathbb{Z}$, $f \in L^1_{\mathrm{loc}}(\mu)$, and $x \in \operatorname{supp}(\mu)$, define

$$S_k f(x) \equiv \int_{\mathbb{R}^d} S_k(x,y) f(y)\, d\mu(y).$$

Let $D_k \equiv S_k - S_{k-1}$ for $k \in \mathbb{Z}$, and we also use D_k to denote the corresponding integral operator with kernel D_k.

We next recall the notions of the space $\mathrm{RBMO}(\mu)$ in [5] and $\mathrm{RBLO}(\mu)$ in [3].

Definition 2.8 Let $\eta, \rho \in (1,\infty)$ and $\beta_\rho \equiv \rho^{d+1}$. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the space $\mathrm{RBMO}(\mu)$ if there exists some nonnegative constant $\widetilde{C}$ such that for any cube Q centered at some point of $\operatorname{supp}(\mu)$,

$$\frac{1}{\mu(\eta Q)} \int_Q \left| f(y) - m_{\widetilde{Q}^\rho}(f) \right| d\mu(y) \le \widetilde{C},$$

and for any two (ρ, β_ρ)-doubling cubes $Q \subset R$,

$$|m_Q(f) - m_R(f)| \le \widetilde{C}[1 + \delta(Q,R)].$$

Moreover, the minimal constant $\widetilde{C}$ as above is defined to be the norm of f in the space $\mathrm{RBMO}(\mu)$ and denoted by $\|f\|_{\mathrm{RBMO}(\mu)}$.

Remark 2.9 It was proved by Tolsa [5] that the definition of $\mathrm{RBMO}(\mu)$ is independent of the choices of η and ρ. As a result, unless explicitly pointed out, we always assume $\eta = \rho = 2$ in Definition 2.8.

Definition 2.10 We say that $f \in L^1_{\mathrm{loc}}(\mu)$ belongs to the space $\mathrm{RBLO}(\mu)$ if there exists some nonnegative constant $\widetilde{C}$ such that for any doubling cube Q,

$$m_Q(f) - \operatorname{essinf}_{x \in Q} f(x) \le \widetilde{C}, \tag{2.1}$$

and for any two doubling cubes $Q \subset R$,

$$m_Q(f) - m_R(f) \le \widetilde{C}[1 + \delta(Q,R)]. \tag{2.2}$$

Moreover, the minimal constant $\widetilde{C}$ as above is defined to be the norm of f in the space $\mathrm{RBLO}(\mu)$ and denoted by $\|f\|_{\mathrm{RBLO}(\mu)}$.

Remark 2.11 It was proved in [2] that we obtain an equivalent norm of $\mathrm{RBLO}(\mu)$ if (2.1) in Definition 2.10 is replaced by: for fixed $\eta \in (1, \infty)$ and any cube Q centered at some point of $\mathrm{supp}(\mu)$,

$$\frac{1}{\mu(\eta Q)} \int_Q \left[f(x) - \mathrm{essinf}_{\widetilde{Q}} f(y) \right] d\mu(x) \leq \widetilde{C}. \tag{2.3}$$

Moreover, we obtain an equivalent norm of $\mathrm{RBLO}(\mu)$ if the $(2, 2^{d+1})$-doubling cubes in (2.2) and (2.3) are replaced by (ρ, ρ^{d+1})-doubling cubes for any fixed $\rho \in (1, \infty)$.

To recall the notions of the local spaces $\mathrm{rbmo}(\mu)$ and $\mathrm{rblo}(\mu)$, we need first to recall the set $\mathcal{D}$ of cubes with "large size", which was introduced in [2]. If $\mathbb{R}^d$ is not an initial cube, letting $\{R_{-j}\}_{j=0}^{\infty}$ be as in Definition 2.5, we then define the set

$$\mathcal{D} \equiv \Big\{ Q \subset \mathbb{R}^d : \text{ there exists a cube } P \subset Q \text{ and } j \in \mathbb{N} \cup \{0\} \text{ such that } P \subset R_{-j} \text{ with } \delta(P, R_{-j}) \leq (j+1)A + \epsilon_1 \Big\}.$$

If $\mathbb{R}^d$ is an initial cube, we define the set

$$\mathcal{D} \equiv \Big\{ Q \subset \mathbb{R}^d : \text{ there exists a cube } P \subset Q \text{ such that } \delta(P, \mathbb{R}^d) \leq A + \epsilon_1 \Big\}.$$

In [2], it was pointed out that if $Q \in \mathcal{D}$, then any R containing Q is also in $\mathcal{D}$ and the definition of the set $\mathcal{D}$ is independent of the chosen reference cubes $\{R_{-j}\}_{j\in\mathbb{N}\cup\{0\}}$ in the sense modulo some small error (the error is no more than $2\epsilon_1 + \epsilon_0$); see also [6, p. 68]. Moreover, it was also proved in [2] that if μ is the d-dimensional Lebesgue measure on $\mathbb{R}^d$, then for any cube $Q \subset \mathbb{R}^d$, $Q \in \mathcal{D}$ if and only if $l(Q) \gtrsim 1$.

The following spaces $\mathrm{rbmo}(\mu)$ and $\mathrm{rblo}(\mu)$ were introduced in [2]. It is not difficult to see that $\mathrm{rblo}(\mu) \subset \mathrm{rbmo}(\mu) \subset \mathrm{RBMO}(\mu)$ and $\mathrm{rblo}(\mu) \subset \mathrm{RBLO}(\mu) \subset \mathrm{RBMO}(\mu)$; see [2].

Definition 2.12 Let $\eta \in (1, \infty)$, $\rho \in [\eta, \infty)$, and $\beta_\rho \equiv \rho^{d+1}$. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the space $\mathrm{rbmo}(\mu)$, if there exists a nonnegative constant $\widetilde{C}$ such that for any cube $Q \notin \mathcal{D}$,

$$\frac{1}{\mu(\eta Q)} \int_Q \left| f(y) - m_{\widetilde{Q}^\rho}(f) \right| d\mu(y) \leq \widetilde{C},$$

that for any two (ρ, β_ρ)-doubling cubes $Q \subset R$ with $Q \notin \mathcal{D}$,

$$|m_Q(f) - m_R(f)| \leq \widetilde{C}[1 + \delta(Q, R)],$$

and that for any cube $Q \in \mathcal{D}$,

$$\frac{1}{\mu(\eta Q)} \int_Q |f(y)| \, d\mu(y) \leq \widetilde{C}.$$

Moreover, the minimal constant $\widetilde{C}$ as above is defined to be the norm of f in the space $\mathrm{rbmo}(\mu)$ and denoted by $\|f\|_{\mathrm{rbmo}(\mu)}$.

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Remark 2.13 It was proved in [2] that the definition of $\mathrm{rbmo}(\mu)$ is independent of the choices of $\eta \in (1, \infty)$ and $\rho \in [\eta, \infty)$. Therefore, in what follows, we always assume $\eta = \rho = 2$ in Definition 2.12.

Definition 2.14 A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to belong to the space $\mathrm{rblo}(\mu)$ if there exists a nonnegative constant $\widetilde{C}$ such that for any cube $Q \notin \mathcal{D}$,

$$\frac{1}{\mu(2Q)} \int_Q \Big[f(x) - \operatorname{essinf}_{\widetilde{Q}} f \Big] \, d\mu(x) \le \widetilde{C},$$

that for any two doubling cubes $Q \subset R$ with $Q \notin \mathcal{D}$,

$$m_Q(f) - m_R(f) \le \widetilde{C}[1 + \delta(Q, R)],$$

that for any cube $Q \in \mathcal{D}$,

$$\frac{1}{\mu(2Q)} \int_Q |f(y)| \, d\mu(y) \le \widetilde{C},$$

and that for any cube $Q \in \mathcal{D}$,

$$\Big| \operatorname*{essinf}_{\widetilde{Q}} f \Big| \le \widetilde{C}.$$

Moreover, the minimal constant $\widetilde{C}$ as above is defined to be the norm of f in the space $\mathrm{rblo}(\mu)$ and denoted by $\|f\|_{\mathrm{rblo}(\mu)}$.

In what follows, for any cube R and $x \in R \cap \mathrm{supp}(\mu)$, let H^x_R be the largest integer k such that $R \subset Q_{x,k}$. The following properties on H^x_R, which were established in [2], are useful in applications.

Lemma 2.15 *The following properties hold:*

(i) *For any cube R and $x \in R \cap \mathrm{supp}(\mu)$, $Q_{x, H^x_R+1} \subset 3R$ and $5R \subset Q_{x, H^x_R-1}$.*
(ii) *For any cube R, $x \in R \cap \mathrm{supp}(\mu)$, and $k \in \mathbb{Z}$ with $k \ge H^x_R + 2$, $Q_{x,k} \subset \frac{7}{5}R$.*
(iii) *For any cube R and $x \in R \cap \mathrm{supp}(\mu)$, $H^x_R \ge 0$ when $R \notin \mathcal{D}$; moreover, $H^x_R \le 1$ when $\mathbb{R}^d$ is not an initial cube and $R \in \mathcal{D}$, and $0 \le H^x_R \le 1$ when $\mathbb{R}^d$ is an initial cube and $R \in \mathcal{D}$.*
(iv) *When $k \ge 2$, for any $x \in \mathrm{supp}(\mu)$, $Q_{x,k} \notin \mathcal{D}$.*
(v) *For any cube R and $x \in R \cap \mathrm{supp}(\mu)$, there exists a positive constant C such that $\delta(R, Q_{x,H^x_R}) \le C$ and $\delta(Q_{x,H^x_R+1}, R) \le C$.*

3 Main Results and their Proofs

Let $S \equiv \{S_k\}_{k\in\mathbb{Z}}$ be an approximation of the identity as in Section 2. We then consider the following maximal operators: for any locally integrable function f and $x \in \mathbb{R}^d$, define

$$\dot{\mathcal{M}}_S(f)(x) \equiv \sup_{k\in\mathbb{Z}} |S_k(f)(x)| \quad \text{and} \quad \mathcal{M}_S(f)(x) \equiv \sup_{k\in\mathbb{N}} |S_k(f)(x)|.$$

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These two operators were introduced in [15]. Moreover, $\dot{\mathcal{M}}_S$ was proved to be bounded on $L^p(\mu)$ for $p \in (1,\infty)$ and from $H^1(\mu)$ to $L^1(\mu)$, and $\mathcal{M}_S$ was proved to be bounded on $L^p(\mu)$ for $p \in (1,\infty)$ and from $h^{1,\infty}_{\mathrm{atb}}(\mu)$ to $L^1(\mu)$; see [15]. In this section, we consider their boundedness in $\mathrm{RBMO}(\mu)$ and $\mathrm{rbmo}(\mu)$, respectively.

Theorem 3.1 *If $\mathbb{R}^d$ is not an initial cube, for any $f \in \mathrm{RBMO}(\mu)$, $\dot{\mathcal{M}}_S(f)$ is either infinite everywhere or finite almost everywhere, and in the latter case, there exists a positive constant C independent of f such that $\|\dot{\mathcal{M}}_S(f)\|_{\mathrm{RBLO}(\mu)} \le C\|f\|_{\mathrm{RBMO}(\mu)}$.*

If $\mathbb{R}^d$ is an initial cube, the same conclusions as above are true if $\dot{\mathcal{M}}_S$ is replaced by $\mathcal{M}_S$.

Proof We use some basic ideas from [1]. By homogeneity, we may assume that

$$\|f\|_{\mathrm{RBMO}(\mu)} = 1.$$

Moreover, when $\mathbb{R}^d$ is an initial cube, by the convention, we have $S_k = 0$ when $k \le 0$. Thus, using this convention, we can also write $\mathcal{M}_S$ into $\dot{\mathcal{M}}_S$.

We first claim that if there exists a point $x_0 \in \mathbb{R}^d$ such that $\dot{\mathcal{M}}_S(f)(x_0) < \infty$, then for any doubling cube $Q \ni x_0$,

$$\frac{1}{\mu(Q)}\int_Q \left[\dot{\mathcal{M}}_S(f)(x) - \inf_Q \dot{\mathcal{M}}_S(f)\right] d\mu(x) \lesssim 1. \tag{3.1}$$

In fact, for any cube Q and $f \in L^1_{\mathrm{loc}}(\mu)$, define

$$\dot{\mathcal{M}}_{S,Q,1}(f)(x) \equiv \sup_{k \ge H^x_Q+1} |S_k(f)(x)|, \tag{3.2}$$

$$\dot{\mathcal{M}}_{S,Q,2}(f)(x) \equiv \sup_{k \le H^x_Q} |S_k(f)(x)|, \tag{3.3}$$

$Q_1 \equiv \{x \in Q : \dot{\mathcal{M}}_{S,Q,1}(f)(x) \ge \dot{\mathcal{M}}_{S,Q,2}(f)(x)\}$ and $Q_2 \equiv Q \setminus Q_1$. We then have that $\dot{\mathcal{M}}_S(f) = \max(\dot{\mathcal{M}}_{S,Q,1}(f), \dot{\mathcal{M}}_{S,Q,2}(f))$. Write $f_1 \equiv [f - m_Q(f)]\chi_{\frac{4}{3}Q}$ and

$$f_2 \equiv [f - m_Q(f)]\chi_{\mathbb{R}^d \setminus \frac{4}{3}Q}.$$

Since $\dot{\mathcal{M}}_{S,Q,1}$ is sublinear, we see that

$$\begin{aligned}
&\frac{1}{\mu(Q)}\int_Q \left[\dot{\mathcal{M}}_S(f)(x) - \inf_Q \dot{\mathcal{M}}_S(f)\right] d\mu(x)\\
&\quad \le \frac{1}{\mu(Q)}\int_{Q_1} [\dot{\mathcal{M}}_{S,Q,1}(f_1)(x) + \dot{\mathcal{M}}_{S,Q,1}(f_2)(x)]\, d\mu(x) + \left[|m_Q(f)| - \inf_Q \dot{\mathcal{M}}_S(f)\right]\\
&\qquad + \frac{1}{\mu(Q)}\int_{Q_2} \left[\dot{\mathcal{M}}_{S,Q,2}(f)(x) - \inf_Q \dot{\mathcal{M}}_S(f)\right] d\mu(x) \equiv \mathrm{E}_1 + \mathrm{E}_2 + \mathrm{E}_3.
\end{aligned}$$

To estimate E_1, recall that there exists a positive constant C such that for any $f \in \mathrm{RBMO}(\mu)$ with $\|f\|_{\mathrm{RBMO}(\mu)} = 1$ and doubling cubes Q and R,

$$|m_Q(f) - m_R(f)| \le C + 2\delta(Q,R) \tag{3.4}$$

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(see [7, Proposition 2.6]). On the other hand, by [15, Lemma 4.1], $\dot{\mathcal{M}}_S$ is bounded on $L^2(\mu)$.

From this fact, the Hölder inequality, the doubling property of Q, Lemma 2.2, (3.4), and [5, Corollary 3.5], it follows that

$$\begin{aligned}\frac{1}{\mu(Q)}\int_{Q_1}&\dot{\mathcal{M}}_{S,Q,1}(f_1)(x)\,d\mu(x)\\&\lesssim\Big\{\frac{1}{\mu(Q)}\int_{\frac43 Q}|f(x)-m_Q(f)|^2\,d\mu(x)\Big\}^{1/2}\\&\lesssim\Big\{\frac{1}{\mu(Q)}\int_{\frac43 Q}\big|f(x)-m_{\widetilde{\frac43 Q}}(f)\big|^2\,d\mu(x)\Big\}^{1/2}+\big|m_Q(f)-m_{\widetilde{\frac43 Q}}(f)\big|\lesssim 1.\end{aligned}$$

By this inequality, the estimate for E_1 is reduced to showing that

$$\frac{1}{\mu(Q)}\int_{Q_1}\dot{\mathcal{M}}_{S,Q,1}(f_2)(x)\,d\mu(x)\lesssim 1. \tag{3.5}$$

From the construction of $\{Q_{x,k}\}_{k\in\mathbb{Z}}$, it is easy to see that for any $k\in\mathbb{Z}$ and $x\in\operatorname{supp}(\mu)$, $\delta(Q_{x,k},Q_{x,k-1})\lesssim 1$. Then by applying [14, Lemma 3.1], we see that

$$\int_{Q_{x,k-1}}\frac{|f(z)-m_{Q_{x,k}}(f)|}{[|z-x|+l(Q_{x,k})]^n}\,d\mu(z)\lesssim[1+\delta(Q_{x,k},Q_{x,k-1})]^2\lesssim 1. \tag{3.6}$$

Moreover, if $k\geq H_Q^x+4$, then $Q_{x,k-1}\subset\frac43 Q$. This can be seen by applying Lemma 2.15(ii) together with the fact that $l(Q_{x,k-1})\leq\frac{1}{10}l(Q_{x,k-2})$ for any $x\in\operatorname{supp}(\mu)$ and $k\in\mathbb{Z}$ (see [6, p. 69]). Then from this fact, (A-2)– (A-4), (3.6), (3.4), Lemma 2.2(iv), and Lemma 2.15(v), it follows that for any $x\in Q_1$,

$$\begin{aligned}\dot{\mathcal{M}}_{S,Q,1}(f_2)(x)&=\sup_{H_Q^x+1\leq k\leq H_Q^x+3}|S_k(f_2)(x)|\\&\leq\sup_{H_Q^x+1\leq k\leq H_Q^x+3}\Big[\big|S_k\big((f-m_{Q_{x,k}}(f))\chi_{\mathbb{R}^d\setminus\frac43 Q}\big)(x)\big|\\&\quad+\big|m_{Q_{x,k}}(f)-m_{Q_{x,H_Q^x+1}}(f)\big|+\big|m_{Q_{x,H_Q^x+1}}(f)-m_Q(f)\big|\Big]\lesssim 1.\end{aligned}$$

This implies (3.5).

Now we estimate E_2. From (A-2)–(A-4) and (3.6), it follows that for any $k\in\mathbb{Z}$ and $x\in\operatorname{supp}(\mu)$,

$$|S_k(f)(x)-m_{Q_{x,k}}(f)|\lesssim 1. \tag{3.7}$$

Then applying this, together with Lemma 2.15(v) and (3.4), we see that for any $y\in Q$,

$$\begin{aligned}|m_Q(f)|-\dot{\mathcal{M}}_S(f)(y)&\leq\big|m_Q(f)-S_{H_Q^y+1}(f)(y)\big|\\&\leq\big|m_Q(f)-m_{Q_{y,H_Q^y+1}}(f)\big|+\big|m_{Q_{y,H_Q^y+1}}(f)-S_{H_Q^y+1}(f)(y)\big|\\&\lesssim 1.\end{aligned}$$

Then we have that $\mathrm{E}_2 \lesssim 1$.

On the other hand, for any $x, y \in Q$ and $k \le H_Q^x$, [6, Lemma 4.2] implies that $Q_{y,k} \subset Q_{x,k-1} \subset Q_{y,k-2}$. Then Lemma 2.2(iv) and (v) yield that $\delta(Q_{y,k}, Q_{x,k}) \lesssim 1$. Therefore, it follows from (3.4) and (3.7) that

$$\begin{aligned}
&|S_k f(x)| - \dot{\mathcal{M}}_S(f)(y)\\
&\quad \le |S_k f(x) - S_k f(y)|\\
&\quad \le |S_k f(x) - m_{Q_{x,k}}(f)| + |m_{Q_{x,k}}(f) - m_{Q_{y,k}}(f)| + |m_{Q_{y,k}}(f) - S_k f(y)|\\
&\quad \lesssim 1,
\end{aligned}$$

which implies $\mathrm{E}_3 \lesssim 1$. Combining estimates for E_1 through E_3 leads to (3.1).

By (3.1), if there exists a point $x_0 \in \mathbb{R}^d$ such that $\dot{\mathcal{M}}_S(f)(x_0) < \infty$, then $\dot{\mathcal{M}}_S(f)(x)$ is finite almost everywhere and for any doubling cube Q,

$$\frac{1}{\mu(Q)} \int_Q \left[\dot{\mathcal{M}}_S(f)(x) - \operatorname{essinf}_Q \dot{\mathcal{M}}_S(f)\right] d\mu(x) \lesssim 1.$$

To complete the proof of Theorem 3.1, it suffices to verify that for any doubling cube $Q \subset R$,

$$m_Q[\dot{\mathcal{M}}_S(f)] - m_R[\dot{\mathcal{M}}_S(f)] \lesssim 1 + \delta(Q, R).$$

Let $\dot{\mathcal{M}}_{S,R,1}(f)$ and $\dot{\mathcal{M}}_{S,R,2}(f)$ be as in (3.2) and (3.3) with Q replaced by R,

$$Q_1 \equiv \{x \in Q : \dot{\mathcal{M}}_{S,R,1}(f)(x) \ge \dot{\mathcal{M}}_{S,R,2}(f)(x)\}$$

and $Q_2 \equiv Q \setminus Q_1$. Split

$$f = [f - m_R(f)]\chi_{\frac{4}{3}Q} + [f - m_R(f)]\chi_{\mathbb{R}^d \setminus \frac{4}{3}Q} + m_R(f) \equiv f_1 + f_2 + m_R(f).$$

From the fact that $\dot{\mathcal{M}}_{S,R,1}$ is sublinear, it follows that

$$\begin{aligned}
&m_Q[\dot{\mathcal{M}}_S(f)] - m_R[\dot{\mathcal{M}}_S(f)]\\
&\quad \le \frac{1}{\mu(Q)} \int_{Q_1} \left\{\dot{\mathcal{M}}_{S,R,1}(f_1)(x) + \dot{\mathcal{M}}_{S,R,1}(f_2)(x)\right\} d\mu(x) + \left[|m_R(f)| - m_R[\dot{\mathcal{M}}_S(f)]\right]\\
&\qquad + \frac{1}{\mu(Q)} \int_{Q_2} \left\{\dot{\mathcal{M}}_{S,R,2}(f)(x) - m_R[\dot{\mathcal{M}}_S(f)]\right\} d\mu(x) \equiv \mathrm{F}_1 + \mathrm{F}_2 + \mathrm{F}_3.
\end{aligned}$$

By the boundedness of $\dot{\mathcal{M}}_S$ in $L^2(\mu)$, the Hölder inequality, the doubling property of Q, Lemma 2.2, (3.4), and [5, Corollary 3.5],

$$\begin{aligned}
(3.8)\qquad &\frac{1}{\mu(Q)} \int_{Q_1} \dot{\mathcal{M}}_{S,R,1}(f_1)(x)\, d\mu(x)\\
&\quad \lesssim \left\{\frac{1}{\mu(Q)} \int_{\frac{4}{3}Q} \left|f(x) - m_{\widetilde{\frac{4}{3}Q}}(f)\right|^2 d\mu(x)\right\}^{1/2}\\
&\qquad + \left|m_{\widetilde{\frac{4}{3}Q}}(f) - m_Q(f)\right| + \left|m_Q(f) - m_R(f)\right|\\
&\quad \lesssim 1 + \delta(Q, R).
\end{aligned}$$

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From the fact that $Q_{x,k-1} \subset \frac{4}{3}Q$ for $k \geq H_Q^x + 4$, (A-2)–(A-4), (3.6), Lemma 2.2, (3.4), and Lemma 2.15(i) and (v), we deduce that for any $x \in Q$,

$$\begin{aligned}\mathcal{M}_{S,R,1}(f_2)(x) &\leq \sup_{H_R^x+1\leq k\leq H_Q^x+3} \Big\{ S_k\big[|f - m_{Q_{x,k}}(f)|\big](x) \\ &\quad + |m_{Q_{x,k}}(f) - m_{\widetilde{3R}}(f)| + |m_{\widetilde{3R}}(f) - m_R(f)|\Big\} \\ &\lesssim 1 + \delta(Q_{x,H_Q^x+3}, \widetilde{3R}) \lesssim 1 + \delta(Q,R).\end{aligned}$$

From this and (3.8), it follows that $\mathrm{F}_1 \lesssim 1 + \delta(Q,R)$.

On the other hand, Lemma 2.15(v), (3.4), and (3.7) imply that for any $y \in R$,

$$\begin{aligned}|m_R(f)| - \dot{\mathcal{M}}_S(f)(y) &\leq \big|m_R(f) - S_{H_R^y+1}(f)(y)\big| \\ &\leq \big|m_R(f) - m_{Q_{y,H_R^y+1}}(f)\big| + \big|m_{Q_{y,H_R^y+1}}(f) - S_{H_R^y+1}(f)(y)\big| \lesssim 1.\end{aligned}$$

Taking the average over $y \in R$ yields $\mathrm{F}_2 \lesssim 1$.

Observe that for any x, $y \in R$ and $k \leq H_R^x$, $\delta(Q_{y,k}, Q_{x,k}) \lesssim 1$. Then it follows from (3.4) and (3.7) that

$$\begin{aligned}&|S_k f(x)| - \dot{\mathcal{M}}_S(f)(y) \\ &\quad\leq |S_k f(x) - m_{Q_{x,k}}(f)| + |m_{Q_{x,k}}(f) - m_{Q_{y,k}}(f)| + |m_{Q_{y,k}}(f) - S_k f(y)| \lesssim 1,\end{aligned}$$

which implies $\mathrm{F}_3 \lesssim 1$, and hence completes the proof of Theorem 3.1. ■

Theorem 3.2 *There exists a positive constant C such that for all $f \in \mathrm{rbmo}(\mu)$,*

$$\big\|\mathcal{M}_S(f)\big\|_{\mathrm{rblo}(\mu)} \leq C\|f\|_{\mathrm{rbmo}(\mu)}.$$

Proof By homogeneity, we may assume that $\|f\|_{\mathrm{rbmo}(\mu)} = 1$. We first consider the case that $\mathbb{R}^d$ is an initial cube. In this case, we claim that for any cube $Q \in \mathcal{D}$,

$$\frac{1}{\mu(2Q)}\int_Q \mathcal{M}_S(f)(x)\,d\mu(x) \lesssim 1. \tag{3.9}$$

By [15, Lemma 4.1], $\mathcal{M}_S$ is bounded on $L^2(\mu)$. This fact together with the Hölder inequality and [2, Corollary 3.1] yield that

$$\frac{1}{\mu(2Q)}\int_Q \mathcal{M}_S\big[f\chi_{\frac{4}{3}Q}\big](x)\,d\mu(x) \lesssim \Big\{\frac{1}{\mu(2Q)}\int_{\frac{4}{3}Q} |f(x)|^2\,d\mu(x)\Big\}^{1/2} \lesssim 1.$$

On the other hand, by Lemma 2.15(iii), $0 \leq H_Q^x \leq 1$ for any $x \in Q$, which in turn implies that $k \geq H_Q^x + 4$ for $k \geq 5$. Then we have $Q_{x,k-1} \subset \frac{4}{3}Q$. Moreover, [2, Lemma 3.10] implies that (3.6) holds for any $f \in \mathrm{rbmo}(\mu)$. From these facts together with (A-2), it follows that for any $x \in Q$,

$$\begin{aligned}\mathcal{M}_S\big[f\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}\big](x) &= \sup_{1\le k\le 4}\left|\int_{Q_{x,\,k-1}} S_k(x,y)f(y)\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}(y)\,d\mu(y)\right|\\ &\le \sup_{1\le k\le 4}\left\{\int_{Q_{x,\,k-1}}\frac{|f(y)-m_{Q_{x,\,k}}(f)|}{[|x-y|+l(Q_{x,\,k})]^n}\,d\mu(y)+\left|m_{Q_{x,\,k}}(f)\right|\right\}\lesssim 1,\end{aligned}$$

where in the last inequality, by Definition 2.12, $\left|m_{Q_{x,\,k}}(f)\right| \le 1$ if $k=1$; and

$$\left|m_{Q_{x,\,k}}(f)\right| \le \left|m_{Q_{x,\,k}}(f)-m_{Q_{x,\,1}}(f)\right| + \left|m_{Q_{x,\,1}}(f)\right| \lesssim 1$$

if $2 \le k \le 4$. Therefore (3.9) follows and $\mathcal{M}_S(f)$ is finite almost everywhere.

We now prove that for any doubling cube $Q \notin \mathcal{D}$,

$$\frac{1}{\mu(Q)}\int_Q\big[\mathcal{M}_S(f)(x)-\operatorname{essinf}_Q\mathcal{M}_S(f)\big]\,d\mu(x)\lesssim 1. \tag{3.10}$$

Let

$$\mathcal{M}_{S,\,Q,\,1}(f)(x)\equiv \sup_{k\ge H_Q^x+1}\left|S_k(f)(x)\right|,\quad \mathcal{M}_{S,\,Q,\,2}(f)(x)\equiv \sup_{1\le k\le H_Q^x}\left|S_k(f)(x)\right|$$

(if $H_Q^x=0$, then $\mathcal{M}_{S,\,Q,\,2}(f)$ disappears),

$$Q_1 \equiv \{x\in Q : \mathcal{M}_{S,\,Q,\,1}(f)(x) \ge \mathcal{M}_{S,\,Q,\,2}(f)(x)\},$$

and $Q_2 \equiv Q\setminus Q_1$. Moreover, write

$$f=[f-m_Q(f)]\chi_{\frac{4}{3}Q}+[f-m_Q(f)]\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}+m_Q(f)\equiv f_1+f_2+m_Q(f).$$

Then we have

$$\begin{aligned}&\frac{1}{\mu(Q)}\int_Q\big[\mathcal{M}_S(f)(x)-\operatorname{essinf}_Q\mathcal{M}_S(f)\big]\,d\mu(x)\\ &\quad\le \frac{1}{\mu(Q)}\int_{Q_1}\big[\mathcal{M}_{S,\,Q,\,1}(f_1)(x)\\ &\qquad+\mathcal{M}_{S,\,Q,\,1}(f_2)(x)\big]\,d\mu(x)+\big[|m_Q(f)|-\operatorname{essinf}_Q\mathcal{M}_S(f)\big]\\ &\qquad+\frac{1}{\mu(Q)}\int_{Q_2}\big[\mathcal{M}_{S,\,Q,\,2}(f)(x)-\operatorname{essinf}_Q\mathcal{M}_S(f)\big]\,d\mu(x)\\ &\quad\equiv \mathrm{G}_1+\mathrm{G}_2+\mathrm{G}_3.\end{aligned}$$

We claim that

$$\frac{1}{\mu(Q)}\int_{Q_1}\mathcal{M}_{S,\,Q,\,1}(f_1)(x)\,d\mu(x)\lesssim 1. \tag{3.11}$$

In fact, an easy computation shows that (3.4) holds for any $f \in \mathrm{rbmo}(\mu)$ with $\|f\|_{\mathrm{rbmo}(\mu)} = 1$. Then by the Hölder inequality, the boundedness of $\mathcal{M}_S$ in $L^2(\mu)$, the doubling property of Q, [2, Corollary 3.1], and Lemma 2.2, if $\frac{4}{3}Q \in \mathcal{D}$,

$$\begin{aligned}\frac{1}{\mu(Q)}\int_{Q_1}\mathcal{M}_{S,Q,1}(f_1)(x)\,d\mu(x)&\\ &\lesssim \left\{\frac{1}{\mu(Q)}\int_{\frac{4}{3}Q}|f(x)-m_Q(f)|^2\,d\mu(x)\right\}^{1/2}\\ &\le \left\{\frac{1}{\mu(Q)}\int_{\frac{4}{3}Q}|f(x)|^2\,d\mu(x)\right\}^{1/2}+\left|m_Q(f)-m_{\widetilde{\frac{4}{3}Q}}(f)\right|+\left|m_{\widetilde{\frac{4}{3}Q}}(f)\right|\\ &\lesssim 1;\end{aligned}$$

and if $\frac{4}{3}Q \notin \mathcal{D}$,

$$\begin{aligned}\frac{1}{\mu(Q)}\int_{Q_1}\mathcal{M}_{S,Q,1}(f_1)(x)\,d\mu(x)&\\ &\le \left\{\frac{1}{\mu(Q)}\int_{\frac{4}{3}Q}\left|f(x)-m_{\widetilde{\frac{4}{3}Q}}(f)\right|^2 d\mu(x)\right\}^{1/2}+\left|m_{\widetilde{\frac{4}{3}Q}}(f)-m_Q(f)\right|\lesssim 1.\end{aligned}$$

Therefore, (3.11) follows.

From (A-2)–(A-4), (3.4), (3.6), Lemma 2.15(v), and the fact that $Q_{x,k-1} \subset \frac{4}{3}Q$ for $k \ge H_Q^x + 4$, it follows that for any $x \in Q_1$,

$$\begin{aligned}\mathcal{M}_{S,Q,1}(f_2)(x)&\\ &\le \sup_{H_Q^x+1\le k\le H_Q^x+3}\Big\{S_k\big[|f-m_{Q_{x,k}}(f)|\big](x)+\left|m_{Q_{x,k}}(f)-m_{Q_{x,H_Q^x+1}}(f)\right|\\ &\qquad\qquad +\left|m_{Q_{x,H_Q^x+1}}(f)-m_Q(f)\right|\Big\}\\ &\lesssim 1.\end{aligned}$$

This and (3.11) lead to $\mathrm{G}_1 \lesssim 1$.

Observe that [2, Lemma 3.10] implies that (3.7) also holds for any $f \in \mathrm{rbmo}(\mu)$. Similarly to the estimate for E_2 in the proof of Theorem 3.1, by Lemma 2.15(v) and (3.7), for any $y \in Q$,

$$|m_Q(f)| - \mathcal{M}_S(f)(y) \le \left|m_Q(f) - S_{H_Q^y+1}(f)(y)\right| \lesssim 1,$$

which implies that $\mathrm{G}_2 \lesssim 1$.

On the other hand, it follows from (3.4) and (3.7) that for any $x, y \in Q$ and $1 \le k \le H_Q^x$,

$$\begin{aligned}|S_kf(x)| - \mathcal{M}_S(f)(y)&\\ &\le |S_kf(x)-m_{Q_{x,k}}(f)|+|m_{Q_{x,k}}(f)-m_{Q_{y,k}}(f)|+|m_{Q_{y,k}}(f)-S_kf(y)|\lesssim 1.\end{aligned}$$

Then we have $\mathrm{G}_3 \lesssim 1$. Combining the estimates for G_1 through G_3 yields (3.10).

Notice that by (3.9), for any cube $Q \in \mathcal{D}$,

$$\operatorname{essinf}_{\widetilde{Q}} \mathcal{M}_S(f) \lesssim \frac{1}{\mu(2\widetilde{Q})} \int_{\widetilde{Q}} \mathcal{M}_S(f)(x)\, d\mu(x) \lesssim 1.$$

Then by (3.9) and (3.10), to complete the proof of Theorem 3.2 in the case that $\mathbb{R}^d$ is an initial cube, it suffices to prove that for any doubling cubes $Q \subset R$ with $Q \notin \mathcal{D}$,

$$m_Q[\mathcal{M}(f)] - m_R[\mathcal{M}(f)] \lesssim 1 + \delta(Q, R).$$

Set $\mathcal{M}_{S,R,1}(f)(x) \equiv \sup_{k \ge H_R^x+1} |S_k(f)(x)|$, $\mathcal{M}_{S,R,2}(f)(x) \equiv \sup_{1\le k\le H_R^x} |S_k(f)(x)|$ (if $H_R^x = 0$, then $\mathcal{M}_{S,R,2}(f)$ disappears),

$$Q_1 \equiv \{x \in Q : \mathcal{M}_{S,R,1}(f)(x) \ge \mathcal{M}_{S,R,2}(f)(x)\},$$

and $Q_2 \equiv Q \setminus Q_1$. Since $\mathcal{M}_S$ is sublinear,

$$\begin{aligned}
&m_Q[\mathcal{M}_S(f)] - m_R[\mathcal{M}_S(f)]\\
&\quad\le \frac{1}{\mu(Q)} \int_{Q_1} \left\{\mathcal{M}_{S,R,1}(f_1)(x) + \mathcal{M}_{S,R,1}(f_2)(x)\right\} d\mu(x) + \left[|m_R(f)| - m_R[\mathcal{M}_S(f)]\right]\\
&\qquad + \frac{1}{\mu(Q)} \int_{Q_2} \left\{\mathcal{M}_{S,R,2}(f)(x) - m_R[\mathcal{M}_S(f)]\right\} d\mu(x) \equiv \mathrm{H}_1 + \mathrm{H}_2 + \mathrm{H}_3,
\end{aligned}$$

where $f_1 \equiv [f - m_R(f)]\chi_{\frac{4}{3}Q}$ and $f_2 \equiv [f - m_R(f)]\chi_{\mathbb{R}^d \setminus \frac{4}{3}Q}$.

By the boundedness of $\mathcal{M}_S$ in $L^2(\mu)$, the Hölder inequality, the doubling property of Q, Corollary 3.1 in [2], (3.4), and Lemma 2.2, if $\frac{4}{3}Q \in \mathcal{D}$,

$$\begin{aligned}
&\frac{1}{\mu(Q)} \int_{Q_1} \mathcal{M}_{S,Q,1}(f_1)(x)\, d\mu(x)\\
&\quad\lesssim \left\{\frac{1}{\mu(Q)} \int_{\frac{4}{3}Q} |f(x) - m_R(f)|^2\, d\mu(x)\right\}^{1/2}\\
&\quad\lesssim \left\{\frac{1}{\mu(Q)} \int_{\frac{4}{3}Q} |f(x)|^2\, d\mu(x)\right\}^{1/2} + |m_R(f) - m_Q(f)|\\
&\qquad\qquad + \left|m_Q(f) - m_{\widetilde{\frac{4}{3}Q}}(f)\right| + \left|m_{\widetilde{\frac{4}{3}Q}}(f)\right|\\
&\quad\lesssim 1 + \delta(Q, R);
\end{aligned}$$

and if $\frac{4}{3}Q \notin \mathcal{D}$,

$$\begin{aligned}
\frac{1}{\mu(Q)} \int_{Q_1} \mathcal{M}_{S,Q,1}(f_1)(x)\, d\mu(x) &\lesssim \left\{\frac{1}{\mu(Q)} \int_{\frac{4}{3}Q} \left|f(x) - m_{\widetilde{\frac{4}{3}Q}}(f)\right|^2 d\mu(x)\right\}^{1/2}\\
&\quad + \left|m_{\widetilde{\frac{4}{3}Q}}(f) - m_Q(f)\right| + |m_Q(f) - m_R(f)|\\
&\lesssim 1 + \delta(Q, R).
\end{aligned}$$

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On the other hand, from the fact that $Q_{x,k-1} \subset \frac{4}{3}Q$ for $k \geq H_Q^x + 4$, (A-2)–(A-4), (3.6), Lemma 2.2, and Lemma 2.15, we deduce that for any $x \in Q$,

$$\begin{aligned}
\mathcal{M}_{S,R,1}(f_2)(x) &= \sup_{H_R^x+1 \leq k \leq H_Q^x+3} |S_k(f_2)(x)| \\
&\leq \sup_{H_R^x+1 \leq k \leq H_Q^x+3} \int_{Q_{x,k-1}} S_k(x,z)|f(z) - m_R(f)|\, d\mu(z) \\
&\leq \sup_{H_R^x+1 \leq k \leq H_Q^x+3} \Big[\int_{Q_{x,k-1}} S_k(x,z)|f(z) - m_{Q_{x,k}}(f)|\, d\mu(z) \\
&\quad + \left| m_{Q_{x,k}}(f) - m_{\widetilde{3R}}(f) \right| + \left| m_{\widetilde{3R}}(f) - m_R(f) \right| \Big] \\
&\lesssim 1 + \delta(Q_{x,H_Q^x+3}, \widetilde{3R}) \lesssim 1 + \delta(Q,R).
\end{aligned}$$

Then we have $\frac{1}{\mu(Q)} \int_{Q_1} \mathcal{M}_{S,Q,1}(f_2)(x)\, d\mu(x) \lesssim 1 + \delta(Q,R)$, which together with the estimate for $\mathcal{M}_{S,Q,1}(f_1)(x)$ yields $\mathrm{H}_1 \lesssim 1 + \delta(Q,R)$.

By Lemma 2.15(v), (3.4), and (3.7), for any $y \in R$,

$$\begin{aligned}
|m_R(f)| - \mathcal{M}_S(f)(y) &\leq \left| m_R(f) - S_{H_R^y+1}(f)(y) \right| \\
&\leq \left| m_R(f) - m_{Q_{y,H_R^y+1}}(f) \right| + \left| m_{Q_{y,H_R^y+1}}(f) - S_{H_R^y+1}(f)(y) \right| \lesssim 1.
\end{aligned}$$

This yields that $\mathrm{H}_2 \lesssim 1$.

Moreover, for any $x,\ y \in R$ and $1 \leq k \leq H_R^x$, from the fact that $\delta(Q_{y,k}, Q_{x,k}) \lesssim 1$, (3.4), and (3.7), it follows that

$$\begin{aligned}
&|S_k f(x)| - \mathcal{M}_S(f)(y) \\
&\quad \leq |S_k f(x) - m_{Q_{x,k}}(f)| + |m_{Q_{x,k}}(f) - m_{Q_{y,k}}(f)| + |m_{Q_{y,k}}(f) - S_k f(y)| \lesssim 1.
\end{aligned}$$

Therefore, the proof of Theorem 3.2 in the case that $\mathbb{R}^d$ is an initial cube is completed.

When $\mathbb{R}^d$ is not an initial cube, the proof is similar and we omit the details, which completes the proof of Theorem 3.2. ■

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