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DEMAZURE MODULES OF LEVEL TWO AND PRIME REPRESENTATIONS OF QUANTUM AFFINE $\mathfrak{sl}_{n+1}$

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Abstract We study the classical limit of a family of irreducible representations of the quantum affine algebra associated to $\mathfrak{sl}_{n+1}$. After a suitable twist, the limit is a module for $\mathfrak{sl}_{n+1}[t]$, i.e., for the maximal standard parabolic subalgebra of the affine Lie algebra. Our first result is about the family of prime representations introduced in Hernandez and Leclerc (*Duke Math. J.* **154** (2010), 265–341; *Symmetries, Integrable Systems and Representations, Springer Proceedings in Mathematics & Statistics*, Volume 40, pp. 175–193 (2013)), in the context of a monoidal categorification of cluster algebras. We show that these representations specialize (after twisting) to $\mathfrak{sl}_{n+1}[t]$-stable prime Demazure modules in level-two integrable highest-weight representations of the classical affine Lie algebra. It was proved in Chari *et al.* (arXiv:1408.4090) that a stable Demazure module is isomorphic to the fusion product of stable prime Demazure modules. Our next result proves that such a fusion product is the limit of the tensor product of the corresponding irreducible prime representations of quantum affine $\mathfrak{sl}_{n+1}$.

Keywords: Demazure modules; quantum affine algebras; prime representations; graded limits

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Introduction

The classification of finite-dimensional irreducible representations of quantum affine algebras was given in [10, 13]. Since that time, many different and deep approaches have been developed to study these modules. However, outside the simplest case of the quantum affine algebra associated to $\mathfrak{sl}_2$, even the answer to a basic question such as a dimension formula is not known for an arbitrary irreducible representation. Thus, the focus of the study has been on particular families of modules: amongst the best known are the standard or local Weyl modules, Kirillov–Reshetikhin modules, and their generalizations: the minimal affinizations (see [26, 27, 41, 42] for instance). More recently, motivated by categorification of cluster algebras, Hernandez and Leclerc identified in [28, 30] an interesting class of 'prime' irreducible modules for quantum affine algebras. (Recall that a representation is said to be prime if it is not isomorphic to a tensor product of non-trivial representations.) This class includes some of the known and well-studied prime representations, but also included new families of examples. One of the goals of our paper is to explore the structure of these prime representations by studying their $q \to 1$ specialization and to show that the character of these modules is given by the Demazure character formula.

The systematic study of the classical limit of finite-dimensional representations of a quantum affine algebra was begun in [17], where a necessary and sufficient condition for the existence of the limit was proved. All the interesting representations mentioned earlier admit classical limits. The limit, when it exists, is a finite-dimensional module for the corresponding affine Lie algebra. Hence it is also a module for the current algebra $\mathfrak{g}[t]$ of polynomial maps $\mathbb{C} \to \mathfrak{g}$, which is naturally a subalgebra of the affine algebra. (Here $\mathfrak{g}$ is the underlying simple Lie algebra of the affine Lie algebra.) Equivalently the current algebra is the commutator subalgebra of the standard maximal parabolic subalgebra of the affine Lie algebra. The scaling element d of the affine Lie algebra defines an integer grading on the affine Lie algebra, and the current algebra is a graded subalgebra. The notion of the graded limit of a representation of the quantum affine algebra was developed in [4, 8]. It was shown in these papers that, when $\mathfrak{g}$ is of classical type, the Kirillov–Reshetikhin modules could be regarded (after pulling back by a suitable automorphism) as graded representations of the current algebra. The result is now known in all types, and we refer the reader to [33] for a detailed discussion. In [39], it was shown in some cases and conjectured in general that minimal affinizations could also be regarded as a graded representation of the current algebra. The conjecture was established in [45, 46] when $\mathfrak{g}$ is of classical type.

There is a well-known family of graded modules for the current algebra which arises as follows. Consider a highest-weight irreducible integrable representation $V(\Lambda)$ of level m for the affine Lie algebra. Fix a Borel subalgebra of the affine Lie algebra. A Demazure module of level m is the module for the Borel subalgebra generated by an extremal element of $V(\Lambda)$. Under natural conditions on the extremal vector, the Demazure module admits an action of the standard maximal parabolic subalgebra containing the chosen Borel subalgebra. Since d is an element of the Borel subalgebra, it follows that such Demazure modules, which we call stable, are graded modules for $\mathfrak{g}[t]$.

The relation between graded limits and Demazure modules was made in [7]. In that paper, it was shown, using results in [17], that any stable level one Demazure module is the graded limit of an irreducible local Weyl module (or an irreducible standard module) for the quantum affine algebra. The results of [24] imply that this remains true for simply laced Lie algebras. In the non-simply laced case, this is no longer true; however, it was shown in [44] that the graded limit admits a flag by level-one Demazure modules. The work of [4, 8] shows that the graded limit of Kirillov–Reshetikhin modules, in the case when $\mathfrak{g}$ is classical, is a higher-level stable Demazure module. Finally, the work of [45, 46] shows that Demazure modules and generalized Demazure modules appear as graded limits of minimal affinizations.

We turn now to the new family of prime representations identified by Hernandez and Leclerc. The highest weight of such a representation satisfies a condition, which is best described as minimal affinization by parts (see the next section for details). In this paper we establish, in the case of $\mathfrak{sl}_{n+1}$, that the classical limit of such a prime representation is a stable level-two prime Demazure module. The first hint of this connection comes from the work of [19], which gives a refined presentation of the stable Demazure modules.

We also answer in the affirmative the following question: does any level-two Demazure module associated with affine $\mathfrak{sl}_{n+1}$ appear as a classical limit of a (not necessarily prime) representation of the quantum affine algebra? This is a delicate question since the classical limit of a tensor product is not the tensor product of the classical limits. The notion of a fusion product of $\mathfrak{g}[t]$-modules was introduced in [22], and there are many examples which suggest that it is closely related to this question. In this paper we prove that certain fusion products of prime level-two Demazure modules are the graded limits of the tensor products of the corresponding representations of the quantum affine algebra.

A detailed overview of the results of this paper, a discussion of the natural questions arising from our work, and a description of the overall organization of the paper are given in § 1.

1. The main results

We describe the main results of our paper and discuss the connections with those of [18, 28, 30]. Throughout the paper, we denote by $\mathbb{C}$ the field of complex numbers, by $\mathbb{Z}$ the set of integers, and by $\mathbb{Z}_+$ and $\mathbb{N}$ the set of non-negative and positive integers, respectively.

1.1. Simple, affine, and current algebras

We shall only be interested in the Lie algebra $\mathfrak{s}l_{n+1}$ (denoted from now on as $\mathfrak{g}$) of $(n+1)\times(n+1)$ complex matrices of trace zero. Let $I=\{1,\ldots,n\}$ be the set of vertices of the Dynkin diagram of $\mathfrak{g}$, and let $\{\alpha_i : i\in I\}$ and $\{\omega_i : i\in I\}$ be a set of simple roots and the corresponding set of fundamental weights, respectively. The $\mathbb{Z}$ (respectively, $\mathbb{Z}_+$) span of the simple roots will be denoted by Q (respectively, Q^+), and the $\mathbb{Z}$ (respectively, $\mathbb{Z}_+$) span of the fundamental weights is denoted by P (respectively, P^+). Define a partial order on P by $\lambda\leqslant\mu$ iff $\mu-\lambda\in Q^+$. The positive roots are

$$R^+=\{\alpha_{i,j}=\alpha_i+\alpha_{i+1}+\cdots+\alpha_j : 1\leqslant i\leqslant j\leqslant n\}.$$

Given $\alpha_{i,j}=\alpha_i+\cdots+\alpha_j\in R^+$, let $x^{\pm}_{ij}$ be the corresponding root vector of $\mathfrak{g}$, and set $x^{\pm}_i=x^{\pm}_{i,i}$ and $h_i=[x^+_i,x^-_i]$. The elements $x^{\pm}_i, h_i$, $1\leqslant i\leqslant n$, generate $\mathfrak{g}$ as Lie algebra.

Let $\widehat{\mathfrak{g}}$ be the untwisted affine Lie algebra associated to $\mathfrak{g}$ which can be realized as follows. Let t be an indeterminate, and let $\mathbb{C}[t,t^{-1}]$ be the corresponding algebra of Laurent polynomials. Define a Lie algebra structure on the vector space $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c\oplus\mathbb{C}d$ by requiring c to be central and setting

$$[x\otimes t^r, y\otimes t^s]=[x,y]\otimes t^{r+s}+\mathrm{tr}(xy)c,\quad [d,x\otimes t^r]=r(x\otimes t^r), x,y\in\mathfrak{g}, r,s\in\mathbb{Z}.$$

The commutator subalgebra is $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c$, and we shall denote it by $\widetilde{\mathfrak{g}}$. We shall frequently regard the action of d as defining a $\mathbb{Z}$-grading on $\widetilde{\mathfrak{g}}$.

The current algebra is the $\mathbb{Z}_+$-graded subalgebra $\mathfrak{g}\otimes\mathbb{C}[t]$ of $\widetilde{\mathfrak{g}}$, and it will be denoted as $\mathfrak{g}[t]$. We shall be interested in graded representations of the current algebra: namely,

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$\mathbb{Z}$-graded vector spaces $V = \oplus_{s\in\mathbb{Z}} V[s]$ which admit an action of $\mathfrak{g}[t]$ which is compatible with the grading, $(\mathfrak{g}\otimes t^r)V[s] \subset V[r+s]$, for all $r, s \in \mathbb{Z}$. A map of graded modules is a grade-preserving map of $\mathfrak{g}[t]$-modules.

1.2. Quantized enveloping algebras and their A-forms

Let $\mathbb{C}(q)$ be the field of rational functions in an indeterminate q, and set $\mathbf{A} = \mathbb{Z}[q, q^{-1}]$. Let $\mathbf{U}_q(\mathfrak{g})$ and $\mathbf{U}_q(\widetilde{\mathfrak{g}})$ be the quantized enveloping algebras (defined over $\mathbb{C}(q)$) associated to $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$, respectively. The algebra $\mathbf{U}_q(\mathfrak{g})$ is isomorphic to a subalgebra of $\mathbf{U}_q(\widetilde{\mathfrak{g}})$. Let $\mathbf{U}_\mathbf{A}(\mathfrak{g})$ and $\mathbf{U}_\mathbf{A}(\widetilde{\mathfrak{g}})$ be the $\mathbf{A}$-forms of $\mathbf{U}_q(\mathfrak{g})$ and $\mathbf{U}_q(\widetilde{\mathfrak{g}})$ defined in [37]; these are free $\mathbf{A}$-submodules such that

$$\mathbf{U}_q(\mathfrak{g}) \cong \mathbf{U}_\mathbf{A}(\mathfrak{g}) \otimes_\mathbf{A} \mathbb{C}(q), \quad \mathbf{U}_q(\widetilde{\mathfrak{g}}) \cong \mathbf{U}_\mathbf{A}(\widetilde{\mathfrak{g}}) \otimes_\mathbf{A} \mathbb{C}(q).$$

Regard $\mathbb{C}$ as an $\mathbf{A}$-module by letting q act as 1. Then, $\mathbf{U}_\mathbf{A}(\widetilde{\mathfrak{g}}) \otimes_\mathbf{A} \mathbb{C}$ and $\mathbf{U}_\mathbf{A}(\mathfrak{g}) \otimes_\mathbf{A} \mathbb{C}$ are algebras over $\mathbb{C}$ which have the universal enveloping algebra $\mathbf{U}(\widetilde{\mathfrak{g}})$ and $\mathbf{U}(\mathfrak{g})$ as canonical quotients. Finally, recall that $\mathbf{U}_q(\widetilde{\mathfrak{g}})$ is a Hopf algebra and that $\mathbf{U}_q(\mathfrak{g})$, $\mathbf{U}_\mathbf{A}(\widetilde{\mathfrak{g}})$, and $\mathbf{U}_\mathbf{A}(\mathfrak{g})$ are all Hopf subalgebras.

Throughout the paper, we shall, as is usual, only be working with type-one representations of quantized enveloping algebras, and we will make no further mention of this fact.

1.3. Finite-dimensional representations of $\mathfrak{g}$, $\mathbf{U}_q(\mathfrak{g})$, and their A-forms

It is well known that the isomorphism classes of irreducible finite-dimensional representations of $\mathfrak{g}$ and $\mathbf{U}_q(\mathfrak{g})$ are indexed by elements of P^+: given $\lambda \in P^+$, we denote by $V(\lambda)$ and $V_q(\lambda)$ an element of the corresponding isomorphism class.

It is also known that the category of finite-dimensional representations of these algebras is semisimple. Further, the $\mathbf{U}_q(\mathfrak{g})$-module $V_q(\lambda)$ admits an $\mathbf{A}$-form $V_\mathbf{A}(\lambda)$ which is a representation of $\mathbf{U}_\mathbf{A}(\mathfrak{g})$. The space $V_\mathbf{A}(\lambda) \otimes_\mathbf{A} \mathbb{C}$ is an irreducible module for $\mathbf{U}(\mathfrak{g})$, and we have

$$V_\mathbf{A}(\lambda) \otimes_\mathbf{A} \mathbb{C} \cong_\mathfrak{g} V(\lambda), \quad \lambda \in P^+.$$

1.4. The sets $\mathcal{P}^+$, $\mathcal{P}^+_\mathbb{Z}$, $\mathcal{P}^+_\mathbb{Z}(1)$, and the weight function

Let $\mathcal{P}^+$ be the monoid consisting of n-tuples of polynomials with coefficients in $\mathbb{C}(q)[u]$ with constant term one and with coordinate-wise multiplication. For $1 \leqslant i \leqslant n$ and $a \in \mathbb{C}(q)$, we take $\boldsymbol{\omega}_{i,a}$ to be the n-tuple of polynomials where the only non-constant entry is the element $(1 - au)$ in the ith coordinate. Let $\mathcal{P}^+_\mathbb{Z}$ be the submonoid of $\mathcal{P}^+$ generated by the elements $\boldsymbol{\omega}_{i,a}$, $1 \leqslant i \leqslant n$ and $a \in q^\mathbb{Z}$. Define $\mathrm{wt} : \mathcal{P}^+ \to P^+$ by $\mathrm{wt}\,\boldsymbol{\pi} = \sum_{i=1}^n (\deg \pi_i)\omega_i$.

Definition. Let $\mathcal{P}^+_\mathbb{Z}(1)$ be the subset of $\mathcal{P}^+_\mathbb{Z}$ consisting of the constant n-tuple and elements of the form $\boldsymbol{\omega}_{i_1,a_1} \cdots \boldsymbol{\omega}_{i_k,a_k}$, where $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$, $a_j \in \mathbb{C}(q)$, $1 \leqslant k \leqslant n$, such that

$$a_{i_j} a_{i_{j+1}}^{-1} = q^{\pm(i_{j+1} - i_j + 2)}, \quad k \geqslant 2, \tag{1.1}$$

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and, if $j \leqslant k-2$,

$$a_{i_j} a_{i_{j+1}}^{-1} = q^{\pm(i_{j+1}-i_j+2)} \implies a_{i_{j+1}} a_{i_{j+2}}^{-1} = q^{\mp(i_{j+2}-i_{j+1}+2)}. \tag{1.2}$$

□

Note that

$$\operatorname{wt} \mathcal{P}_{\mathbb{Z}}^{+}(1) = P^{+}(1) = \{\lambda \in P^{+} : \lambda(h_i) \leqslant 1, 1 \leqslant i \leqslant n\}. \tag{1.3}$$

1.5. Prime representations and prime factors

It was shown in [10, 13, 15] that the isomorphism classes of irreducible finite-dimensional representations of $\mathbf{U}_q(\widetilde{\mathfrak{g}})$ are indexed by $\mathcal{P}^+$; for $\boldsymbol{\pi} \in \mathcal{P}^+$, let $V(\boldsymbol{\pi})$ be an element of the corresponding isomorphism class. Note that the trivial representation corresponds to the constant n-tuple. Given $\boldsymbol{\pi}, \boldsymbol{\pi}' \in \mathcal{P}^+$, the tensor product $V(\boldsymbol{\pi}) \otimes V(\boldsymbol{\pi}')$ is generically irreducible and isomorphic to $V(\boldsymbol{\pi}\boldsymbol{\pi}')$. However, necessary and sufficient conditions for this to hold are not known outside the case $n = 1$, and this motivated the interest in understanding the prime irreducible representations.

Definition. We say that $V(\boldsymbol{\pi})$ is a prime irreducible representation if it cannot be written in a non-trivial way as a tensor product of irreducible $\mathbf{U}_q(\widetilde{\mathfrak{g}})$ representations. We shall say that $\boldsymbol{\pi}_1, \dots, \boldsymbol{\pi}_s$ form a multiset of prime factors of $\boldsymbol{\pi}$, if $V(\boldsymbol{\pi}_j)$ is prime and non-trivial, for all $1 \leqslant j \leqslant s$, and we have an isomorphism of $\mathbf{U}_q(\widetilde{\mathfrak{g}})$-modules

$$V(\boldsymbol{\pi}) \cong V(\boldsymbol{\pi}_1) \otimes \cdots \otimes V(\boldsymbol{\pi}_s).$$

□

Since $V(\boldsymbol{\pi})$ is finite dimensional, it is clear that it is either prime or can be written as a tensor product of two or more non-trivial prime representations. It is, however, not clear that the set of prime factors of $V(\boldsymbol{\pi})$ is unique: even in the case of simple Lie algebras, a unique factorization theorem for the tensor product of finite-dimensional irreducible modules was proved relatively recently in [48].

1.6. Some examples of prime representations

Regarding $V(\boldsymbol{\pi})$ as a finite-dimensional $\mathbf{U}_q(\mathfrak{g})$-module, we can write

$$V(\boldsymbol{\pi}) \cong_{\mathbf{U}_q(\mathfrak{g})} V_q(\operatorname{wt} \boldsymbol{\pi}) \bigoplus_{\mu < \operatorname{wt} \boldsymbol{\pi}} (\dim \operatorname{Hom}_{\mathbf{U}_q(\mathfrak{g})}(V_q(\mu), V(\boldsymbol{\pi}))) V_q(\mu). \tag{1.4}$$

The best-known examples of prime representations are the evaluation representations, namely an irreducible representation $V(\boldsymbol{\pi})$ of $\mathbf{U}_q(\widetilde{\mathfrak{g}})$ which is also irreducible as a $\mathbf{U}_q(\mathfrak{g})$-module, i.e., $V(\boldsymbol{\pi}) \cong_{\mathbf{U}_q(\mathfrak{g})} V_q(\operatorname{wt} \boldsymbol{\pi})$. *It is important to recall here that we are in the case when $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_{n+1}$, since our next assertion is false in the other types.* For every $\lambda \in P^+$, there exist elements $\boldsymbol{\pi} \in \mathcal{P}^+$, $a \in \mathbb{C}(q)$, with $\operatorname{wt} \boldsymbol{\pi} = \lambda$ such that $V(\boldsymbol{\pi}) \cong V_q(\lambda)$ as $\mathbf{U}_q(\mathfrak{g})$-modules. An explicit formula for such elements was given in [12] in complete generality. For the purposes of this paper, we will only need the following special cases.

Lemma. *Let* $\boldsymbol{\pi} \in \mathcal{P}^+$.

(i) *If* $\mathrm{wt}\,\boldsymbol{\pi} = \omega_i$, *for some* $1 \leqslant i \leqslant n$, *then* $\boldsymbol{\pi} = \boldsymbol{\omega}_{i,a}$, *for some* $a \in \mathbb{C}(q)$, *and we have an isomorphism of* $\mathbf{U}_q(\mathfrak{sl}_{n+1})$*-modules,*

$$V(\boldsymbol{\pi}) \cong V(\omega_i).$$

(ii) *If* $\mathrm{wt}\,\boldsymbol{\pi} = \omega_i + \omega_j$, *for* $1 \leqslant i \leqslant j \leqslant n$, *then*

$$V(\boldsymbol{\pi}) \cong V_q(\omega_i + \omega_j) \iff \boldsymbol{\pi} = \boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,aq^{\pm(j-i+2)}}, \quad a \in \mathbb{C}(q).$$

In both cases, the module $V(\boldsymbol{\pi})$ *is prime.* □

Remark. This lemma is what motivates the definition of the set $\mathcal{P}^+_{\mathbb{Z}}(1)$. Moreover, we shall see later that the module corresponding to an element of $V(\boldsymbol{\pi})$, $\boldsymbol{\pi} \in \mathcal{P}^+_{\mathbb{Z}}(1)$, is 'minimal' when restricted to suitable subalgebras of $\mathfrak{g}$.

1.7. Further examples of prime representations

Using Lemma 1.6, it is easy to generate further examples of prime representations as follows. The next result was proved in [28]; another proof can be found in [9].

Lemma. *The module* $V(\boldsymbol{\pi})$ *is prime for all* $\boldsymbol{\pi} \in \mathcal{P}^+_{\mathbb{Z}}(1)$. □

Our main goal in this paper is to understand the $\mathbf{U}_q(\mathfrak{g})$-character of the modules $V(\boldsymbol{\pi})$, $\boldsymbol{\pi} \in \mathcal{P}^+_{\mathbb{Z}}(1)$, and, more generally, to relate it to other well-known modules for affine Lie algebras.

1.8. The A-form $V_{\mathbf{A}}(\boldsymbol{\pi})$ and the representation $L(\boldsymbol{\pi})$ of $\mathfrak{g}[t]$

It is in general not true that an arbitrary irreducible finite-dimensional module for $\mathbf{U}_q(\widetilde{\mathfrak{g}})$ admits an **A**-form. In the special case, when $\boldsymbol{\pi} \in \mathcal{P}^+_{\mathbb{Z}}$, the results of [4, 17] show that $V(\boldsymbol{\pi})$ does admit an **A**-form and that $V_{\mathbf{A}}(\boldsymbol{\pi}) \otimes_{\mathbf{A}} \mathbb{C}$ is an indecomposable and usually reducible module for the enveloping algebra $\mathbf{U}(\tilde{\mathfrak{g}})$ and hence also for the current algebra $\mathfrak{g}[t]$. Consider the automorphism of $\mathfrak{g}[t] \to \mathfrak{g}[t]$ defined by mapping $a \otimes t^r \to a \otimes (t-1)^r$, $a \in \mathfrak{g}$, $r \in \mathbb{Z}_+$. Let $L(\boldsymbol{\pi})$ be the representation of $\mathfrak{g}[t]$ obtained by pulling back the $\mathfrak{g}[t]$-module, $V_{\mathbf{A}}(\boldsymbol{\pi}) \otimes_{\mathbf{A}} \mathbb{C}$, via this automorphism. Then, one can prove (see § 2) that

$$(\mathfrak{g} \otimes t^N \mathbb{C}[t]) L(\boldsymbol{\pi}) = 0, \quad N \gg 0.$$

Let $\delta_{r,s}$ be the usual Kronecker delta symbol. The following is not hard to prove (see [39] for instance).

Lemma. *Let* $\boldsymbol{\pi} \in \mathcal{P}^+$ *be such that* $V(\boldsymbol{\pi}) \cong V_q(\mathrm{wt}\,\boldsymbol{\pi})$. *Then*

$$L(\boldsymbol{\pi}) \cong_{\mathfrak{g}} V(\mathrm{wt}\,\boldsymbol{\pi}) \quad \textit{and} \quad (\mathfrak{g} \otimes t\mathbb{C}[t]) L(\boldsymbol{\pi}) = 0.$$

In particular, $L(\boldsymbol{\pi})$ *is the graded* $\mathfrak{g}[t]$*-module generated by an element* $v_{\mathrm{wt}\,\boldsymbol{\pi}}$ *of grade zero and the following defining relations:*

$$(x^-_{1,n+1} \otimes t) v_{\mathrm{wt}\,\boldsymbol{\pi}} = 0,$$

$$(x^+_i \otimes \mathbb{C}[t]) v_{\mathrm{wt}\,\boldsymbol{\pi}} = 0, \quad (h_i \otimes t^s) v_{\mathrm{wt}\,\boldsymbol{\pi}} = \delta_{s,0}(\deg \pi_i) v_{\mathrm{wt}\,\boldsymbol{\pi}}, \quad (x^-_i)^{\deg \pi_i + 1} v_{\mathrm{wt}\,\boldsymbol{\pi}} = 0,$$

for all $1 \leqslant i \leqslant n$. □

1.9. The main result: a presentation of $L(\boldsymbol{\pi})$, $\boldsymbol{\pi} \in \mathcal{P}_{\mathbb{Z}}^+(1)$

Given $\nu \in P^+$ and $\lambda = \omega_{i_1} + \cdots + \omega_{i_k} \in P^+(1)$, $1 \leqslant i_1 < \cdots < i_k \leqslant n$, let $M(\nu, \lambda)$ be the graded $\mathfrak{g}[t]$-module generated by an element $v_{\nu,\lambda}$ of grade zero with the following defining relations: for all $1 \leqslant i \leqslant n$, and $s \in \mathbb{Z}_+$,

$$(x_i^+ \otimes \mathbb{C}[t])v_{\nu,\lambda} = 0, \quad (h_i \otimes t^s)v_{\nu,\lambda} = \delta_{s,0}(2\nu+\lambda)(h_i)v_{\nu,\lambda}, \quad (x_i^-)^{(2\nu+\lambda)(h_i)+1}v_{\nu,\lambda} = 0, \tag{1.5}$$

$$(x_i^- \otimes t^{\nu(h_i)+\lambda(h_i)})v_{\nu,\mu} = 0, \quad 1 \leqslant i \leqslant n \tag{1.6}$$

$$(x_{i_j,i_{j+1}}^- \otimes t^{\nu(h_{i_j}+h_{i_j+1}+\cdots+h_{i_{j+1}})+1})v_{\nu,\lambda} = 0, \quad 1 \leqslant j \leqslant k-1. \tag{1.7}$$

In this paper, we shall prove the following.

Theorem 1. *Let $\boldsymbol{\pi} \in \mathcal{P}^+$ be such that $V(\boldsymbol{\pi})$ has at most one prime factor in $\mathcal{P}_{\mathbb{Z}}^+(1)$ and all its other prime factors are of the form $\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{i,aq^2}$, $1 \leqslant i \leqslant n$, $a \in q^{\mathbb{Z}}$. Then $\mathrm{wt}\,\boldsymbol{\pi} = 2\nu+\lambda$, for a unique choice of $\nu \in P^+$ and $\lambda \in P^+(1)$, and we have an isomorphism of $\mathfrak{g}[t]$-modules*

$$L(\boldsymbol{\pi}) \cong M(\nu, \lambda).$$

In particular, $L(\boldsymbol{\pi})$ acquires the structure of a graded $\mathfrak{g}[t]$-module.

1.10. Demazure modules as fusion products

We assume for the moment that the reader is familiar with the notion of $\mathfrak{g}$-stable Demazure modules $D(\ell, \lambda)$ of level ℓ, which are indexed by pairs $(\ell, \lambda) \in \mathbb{N} \times P^+$. They are graded modules for $\mathfrak{g}[t]$ and are generated by a vector $v_{\ell,\lambda}$: a detailed development can be found in § 3, where we shall also see that we have an isomorphism of $\mathfrak{sl}_{n+1}$-modules,

$$D(\ell, \ell\omega_i) \cong V(\ell\omega_i), \quad 1 \leqslant i \leqslant n, \ell \in \mathbb{N}.$$

The tensor product of level-ℓ Demazure modules is not a level-ℓ module. In [22], the authors introduced a new $\mathfrak{g}[t]$-structure on the tensor product of cyclic graded $\mathfrak{g}[t]$-modules; the resulting $\mathfrak{g}[t]$-module (unlike the tensor product) is a *cyclic* $\mathfrak{g}[t]$-module. This structure, called the fusion product, depends on a choice of complex numbers, a distinct one for each factor in the tensor product. The underlying $\mathfrak{g}$-module structure is unchanged, and in many cases it is known that the fusion product is independent of this choice of parameters; the case of interest to us is the following special case of a result proved in [18].

Proposition. *Let $\nu = \sum_{i=1}^n r_i\omega_i \in P^+$ and $\lambda \in P^+(1)$. Then*

$$D(2, 2\nu+\lambda) \cong D(2, 2\omega_1)^{*r_1} * \cdots * D(2, 2\omega_n)^{*r_n} * D(2, \lambda).$$

Moreover, the module $D(2, \lambda)$ is prime in the sense that it is not isomorphic to a tensor product of non-trivial $\mathfrak{g}$-modules. □

1.11. The connection with Demazure modules

The connection between Theorem 1 and Demazure modules is made via the following proposition, proved in § 4 of this paper.

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Proposition. *For $\nu \in P^+$ and $\lambda \in P^+(1)$, we have an isomorphism*

$$M(\nu, \lambda) \cong_{\mathfrak{g}[t]} D(2, 2\nu + \lambda).$$

1.12. The connection with the category $\mathcal{C}_\kappa$

We discuss the relationship of our work with that of [28, 30]. Let $\kappa : \{1, 2, \ldots, n\} \to \mathbb{Z}$ be a function satisfying $|\kappa(i+1) - \kappa(i)| = 1$, for $1 \leqslant i \leqslant n-1$. Let $\mathcal{C}_\kappa$ be the full subcategory of finite-dimensional representations of $\mathbf{U}_q(\widetilde{\mathfrak{g}})$ defined as follows: an object of $\mathcal{C}_\kappa$ has all its Jordan–Holder components of the form $V(\boldsymbol{\pi})$, where $\boldsymbol{\pi} \in \mathcal{P}^+$ is a product of terms of the form $\boldsymbol{\omega}_{i,a}$, $a \in \{q^{\kappa(i)}, q^{\kappa(i)+2}\}$, $1 \leqslant i \leqslant n$. The following is a slight reformulation of results proved in [28–30].

Theorem 2. *The category $\mathcal{C}_\kappa$ is closed under taking tensor products. An irreducible object of $\mathcal{C}_\kappa$ is a tensor product of prime irreducible objects of $\mathcal{C}_\kappa$. An irreducible object $V(\boldsymbol{\pi})$ of $\mathcal{C}_\kappa$ is prime only if $\boldsymbol{\pi} \in \mathcal{P}^+_{\mathbb{Z}}(1)$ or if $\boldsymbol{\pi} = \boldsymbol{\omega}_{i,q^{\kappa(i)}}\boldsymbol{\omega}_{i,q^{\kappa(i)+2}}$, for $1 \leqslant i \leqslant n$. Moreover, given $\boldsymbol{\pi} \in \mathcal{P}^+_{\mathbb{Z}}(1)$, there exists a height function κ such that $V(\boldsymbol{\pi})$ is a prime object of $\mathcal{C}_\kappa$.* □

We make some comments about our reformulation. In [30], the authors define a quiver $\mathbf{Q}_\kappa$ whose vertices are the elements of the set $\{1, \ldots, n\}$, and whose edges are the set $i \to i+1$, if $\kappa(i) < \kappa(i+1)$, and $i+1 \to i$, otherwise. Given any subset $J = (1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n)$ of I, consider the connected subquiver determined by this subset, and let $J_<$ and $J_>$ be the set of sinks and sources, respectively, of this subquiver. According to [30], the representation $V(\boldsymbol{\pi})$ associated to

$$\boldsymbol{\pi} = \prod_{j \in J_<} \boldsymbol{\omega}_{j,q^{\kappa(j)+2}} \prod_{j \in J_>} \boldsymbol{\omega}_{j,q^{\kappa(j)}},$$

is prime and, moreover, any prime object in $\mathcal{C}_\kappa$ is either of this form or is isomorphic to $V(\boldsymbol{\omega}_{i,q^{\kappa(i)}}\boldsymbol{\omega}_{i,q^{\kappa(i)+2}})$, for $1 \leqslant i \leqslant n$. It is straightforward to check that the element $\boldsymbol{\pi} \in \mathcal{P}^+_{\mathbb{Z}}(1)$. Conversely, given $\boldsymbol{\pi} = \boldsymbol{\omega}_{i_1,a_1} \cdots \boldsymbol{\omega}_{i_k,a_k} \in \mathcal{P}^+_{\mathbb{Z}}(1)$, consider the height function κ given by requiring the elements $i_1, i_3, \ldots$ to be the sinks of $\mathbf{Q}_\kappa$, and $i_2, i_4, \ldots$ to be the sources of the quiver. Then $V(\boldsymbol{\pi})$ is a prime object of $\mathcal{C}_\kappa$.

Very little has been known so far about prime objects $V(\boldsymbol{\pi})$ in $\mathcal{C}_\kappa$, except in the case when $\boldsymbol{\pi} = \boldsymbol{\omega}_{i,\kappa(i)}\boldsymbol{\omega}_{j,\kappa(j)}$ or $\boldsymbol{\pi} = \boldsymbol{\omega}_{i,q^{\kappa(i)}}\boldsymbol{\omega}_{i,q^{\kappa(i)+2}}$, for $1 \leqslant i \leqslant n$, where we can use Lemma 1.6. As a consequence of Lemma 1.8 and Theorem 1, our results show that if $V(\boldsymbol{\pi})$ is a prime object of $\mathcal{C}_\kappa$ then one knows a presentation of $L(\boldsymbol{\pi})$ and that it is a graded $\mathfrak{g}[t]$-module. Moreover, since the $\mathbf{U}_q(\mathfrak{g})$-character of $V(\boldsymbol{\pi})$ is the same as the $\mathfrak{g}$-character of $L(\boldsymbol{\pi})$, Proposition 1.11 shows that their character is given by the Demazure character formula. Taking this together with Proposition 1.10, we see that $V(\boldsymbol{\pi})$, $\boldsymbol{\pi} \in \mathcal{P}^+_{\mathbb{Z}}(1)$, is *strongly prime*, in the sense that $V(\boldsymbol{\pi})$ is not isomorphic to a tensor product of non-trivial $\mathbf{U}_q(\mathfrak{g})$-modules. To the best of our knowledge all known examples of prime representations are strongly prime. But, it is far from clear that the two notions are equivalent.

1.13. An outline of the proof

We outline the main steps of the proof of Theorem 1. It generalizes ideas in [39] (see also [45]), where a similar question was studied for a different family of irreducible representations.

Suppose that $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2 \in \mathcal{P}_{\mathbb{Z}}^+$ are such that we have an injective map of $\mathbf{U}_q(\widetilde{\mathfrak{g}})$-modules $V(\boldsymbol{\pi}_1\boldsymbol{\pi}_2) \to V(\boldsymbol{\pi}_1)\otimes V(\boldsymbol{\pi}_2)$. Since $\mathbf{U}_\mathbf{A}(\widetilde{\mathfrak{g}})$ is a Hopf subalgebra of $\mathbf{U}_q(\widetilde{\mathfrak{g}})$, we get an injective map $V_\mathbf{A}(\boldsymbol{\pi}_1\boldsymbol{\pi}_2) \to V_\mathbf{A}(\boldsymbol{\pi}_1)\otimes V_\mathbf{A}(\boldsymbol{\pi}_2)$. It was shown in [39, Lemma 2.20, Proposition 3.21] that tensoring with $\otimes_\mathbf{A}\mathbb{C}$, and pulling back by the automorphism of $\mathfrak{g}[t]$ induced by $t \to t-1$, gives rise to a map of $\mathfrak{g}[t]$-modules $L(\boldsymbol{\pi}) \to L(\boldsymbol{\pi}_1)\otimes L(\boldsymbol{\pi}_2)$. This map is neither injective nor surjective even when $V(\boldsymbol{\pi}) \cong V(\boldsymbol{\pi}_1)\otimes V(\boldsymbol{\pi}_2)$, but it plays a big role in this paper. The main steps in the proof of Theorem 1 are the following. Let $\boldsymbol{\pi} \in \mathcal{P}_{\mathbb{Z}}^+(1)$.

(i) There exists a surjective map of $\mathfrak{g}[t]$-modules $\varphi_1 : M(0, \mathrm{wt}\,\boldsymbol{\pi}) \to L(\boldsymbol{\pi}) \to 0$.

(ii) There exist $\boldsymbol{\pi}^o, \boldsymbol{\pi}^e \in \mathcal{P}_{\mathbb{Z}}^+(1)$ with $\boldsymbol{\pi} = \boldsymbol{\pi}^o\boldsymbol{\pi}^e$ such that we have an injective map of $\mathbf{U}_q(\widetilde{\mathfrak{g}})$-modules $V(\boldsymbol{\pi}) \to V(\boldsymbol{\pi}^o)\otimes V(\boldsymbol{\pi}^e)$. Moreover,

$$L(\boldsymbol{\pi}^o) \cong_{\mathfrak{g}[t]} D(1, \mathrm{wt}\,\boldsymbol{\pi}^o), \quad L(\boldsymbol{\pi}^e) \cong_{\mathfrak{g}[t]} D(1, \mathrm{wt}\,\boldsymbol{\pi}^e).$$

The induced map

$$\varphi_2 : L(\boldsymbol{\pi}) \to L(\boldsymbol{\pi}^0)\otimes L(\boldsymbol{\pi}^e) \cong D(1, \mathrm{wt}\,\boldsymbol{\pi}^o)\otimes D(1, \mathrm{wt}\,\boldsymbol{\pi}^e)$$

satisfies

$$\varphi_2(\varphi_1(v_{\mathrm{wt}\,\boldsymbol{\pi}})) = v_{1,\mathrm{wt}\,\boldsymbol{\pi}^o}\otimes v_{1,\mathrm{wt}\,\boldsymbol{\pi}^e}.$$

(iii) There exists an injective map of $\mathfrak{g}[t]$-modules

$$D(2, \mathrm{wt}\,\boldsymbol{\pi}) \to D(1, \mathrm{wt}\,\boldsymbol{\pi}^o)\otimes D(1, \mathrm{wt}\,\boldsymbol{\pi}^e), \quad \text{with } v_{2,\mathrm{wt}\,\boldsymbol{\pi}} \to v_{1,\mathrm{wt}\,\boldsymbol{\pi}^o}\otimes v_{1,\mathrm{wt}\,\boldsymbol{\pi}^e}.$$

Consider the composite map

$$\varphi_2\varphi_1 : M(0, \mathrm{wt}\,\boldsymbol{\pi}) \to D(1, \mathrm{wt}\,\boldsymbol{\pi}^o)\otimes D(1, \mathrm{wt}\,\boldsymbol{\pi}^e).$$

The preceding steps show that the image of this map is $D(2, \mathrm{wt}\,\boldsymbol{\pi})$. Proposition 1.11 proves that $\varphi_2\varphi_1$ must be injective, and hence it follows that φ_1 is an isomorphism, proving Theorem 1 when $\boldsymbol{\pi} \in \mathcal{P}_{\mathbb{Z}}^+(1)$.

(iv) The next step is to prove Proposition 1.11.

(v) The final step is to prove the following. Suppose that V is a cyclic $\mathfrak{g}[t]$-module generated by a vector v satisfying (1.5). Assume that $V \cong D(2, 2\nu+\lambda)$ as $\mathfrak{g}$-modules. Then, $V \cong D(2, 2\nu+\lambda)$ as $\mathfrak{g}[t]$-modules.

We now deduce Theorem 1 in full generality. We prove in Lemma 2.1 that $L(\boldsymbol{\pi})$ is a cyclic $\mathfrak{g}[t]$-module. The assumptions on $\boldsymbol{\pi}$ in Theorem 1 imply that $V(\boldsymbol{\pi})$ is isomorphic, as $\mathbf{U}_q(\widetilde{\mathfrak{g}})$-modules, to a tensor product of modules of the form $V(\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{i,aq^2})$, $1 \leqslant i \leqslant n$, along with a single module $V(\boldsymbol{\pi}_1)$, with $\boldsymbol{\pi}_1 \in \mathcal{P}_{\mathbb{Z}}^+(1)$. Hence, we can write $\mathrm{wt}\,\boldsymbol{\pi} = 2\nu + \mathrm{wt}\,\boldsymbol{\pi}_1$, with $\nu = \sum_{i=1}^n r_i\omega_i \in P^+$ and $\mathrm{wt}\,\boldsymbol{\pi}_1 \in P^+(1)$. Since this is an isomorphism also of $\mathbf{U}_q(\mathfrak{g})$-modules, it follows that we have an isomorphism of $\mathfrak{g}$ (but not of $\mathfrak{g}[t]$)) modules,

$$L(\boldsymbol{\pi}) \cong_{\mathfrak{g}} L(2\omega_1)^{\otimes r_1}\otimes\cdots\otimes L(2\omega_n)^{\otimes r_n}\otimes L(\boldsymbol{\pi}_1).$$

Using Lemma 1.6, the fact that $D(2, 2\omega_i) \cong_{\mathfrak{g}} V(2\omega_i)$, and Theorem 1 for $\boldsymbol{\pi}_1$, we get

$$L(\boldsymbol{\pi}) \cong_{\mathfrak{g}} D(2, 2\omega_1)^{\otimes r_1} \otimes \cdots \otimes D(2, 2\omega_n)^{\otimes r_n} \otimes D(2, \mathrm{wt}\,\boldsymbol{\pi}_1).$$

Using Proposition 1.10 gives

$$L(\boldsymbol{\pi}) \cong_{\mathfrak{g}} D(2, \mathrm{wt}\,\boldsymbol{\pi}).$$

Using step (v) now proves that $L(\boldsymbol{\pi})$ is isomorphic to $D(2, \mathrm{wt}\,\boldsymbol{\pi})$ as $\mathfrak{g}[t]$-modules, and the proof is complete.

The proof of steps (i) and (ii) are in § 2, the proof of step (iii) is in § 3, the proof of step (iv) is in § 4, and the proof of step (v) is in § 5.

1.14. Further questions

As we have remarked, the isomorphism

$$V(\boldsymbol{\pi}_1\boldsymbol{\pi}_2) \cong V(\boldsymbol{\pi}_1) \otimes V(\boldsymbol{\pi}_2)$$

does not imply that $L(\boldsymbol{\pi}_1\boldsymbol{\pi}_2)$ is isomorphic to $L(\boldsymbol{\pi}_1) \otimes L(\boldsymbol{\pi}_2)$. However, in the case when the $L(\boldsymbol{\pi}_j)$ are graded $\mathfrak{g}[t]$-modules, it is true in many known examples that

$$L(\boldsymbol{\pi}_1\boldsymbol{\pi}_2) \cong L(\boldsymbol{\pi}_1) * L(\boldsymbol{\pi}_2).$$

Theorem 1, and Proposition 1.10 and Proposition 1.11 add to these growing number of examples which support the conjecture that such a statement might be true in general.

We now discuss some natural questions arising from our work. The Demazure character formula is not easy to compute, and an interesting question would be to determine the $\mathfrak{g}$-module decomposition of the prime level-two Demazure modules. Preliminary calculations show that there could be some interesting combinatorics associated with it, along the lines of the formulae given for the well-known Kirillov–Reshetikhin modules.

Another direction is the following: there exist irreducible (but not prime) objects of $\mathcal{C}_\kappa$ which are not level-two Demazure modules. In fact any tensor product of modules which has more than one prime factor of the form $V(\boldsymbol{\pi})$, $\boldsymbol{\pi} \in \mathcal{P}^+_{\mathbb{Z}}(1)$, does not usually specialize to a level-two Demazure module. It is natural to speculate that they too have graded limits which are also related in some way to other known representations for the current algebra. In simple cases, it appears that these are quotients of a family modules for $\mathfrak{sl}_{n+1}[t]$ defined and studied in [19]; they also appear to be related to the generalized Demazure modules studied in [36, 45].

There are two more very obvious questions: can one formulate and prove analogous results for Demazure modules of level at least three for $\mathfrak{sl}_{n+1}[t]$, and are there analogous results for the other simple Lie algebras? It is most convenient to address these two questions together. Assume therefore that $\mathfrak{g}$ is an arbitrary simple Lie algebra. If $\mathfrak{g}$ is simply laced, then one uses results in [7, 17, 24] to prove that any level-one Demazure module is isomorphic to the graded limit of a module for the quantum affine algebra. However, in the non-simply laced case this is false; the level-one Demazure modules are too small. It was shown in [44] that the graded limit usually admits only a flag by Demazure modules.

This phenomenon persists as we move to higher levels. Thus for types D and E, the level-two Demazure modules are again too small; one sees this already in the case of most of the Kirillov–Reshetikhin modules. Similarly for type A beyond level two, some of the Demazure modules appear to be too small. In all these cases there is evidence to suggest that the generalized Demazure modules and the modules defined in [19] might be the correct objects. In particular, in work in progress (see also [2]) we show that the minimal affinizations, which are known through the work of [44] to specialize to generalized Demazure module, are isomorphic to the modules defined in [19].

2. Prime representations and graded limits

In this section we recall the necessary results from the theory of finite-dimensional representation of quantum affine $\mathfrak{sl}_{n+1}$. The results can be stated without introducing, in too much detail, the extensive notation of the quantum affine algebras. We refer the reader to [11] for the basic definitions, and to [45, § 3] for an excellent exposition with detailed references, for the results discussed here, on graded limits and minimal affinizations.

2.1. The modules $L(\boldsymbol{\pi})$ for $\boldsymbol{\pi} \in \mathcal{P}_{\mathbb{Z}}^{+}$

We begin the section by elaborating on the definition of the $\mathfrak{g}[t]$-modules $L(\boldsymbol{\pi})$, $\boldsymbol{\pi} \in \mathcal{P}_{\mathbb{Z}}^{+}$. Special cases of what we are going to say are in the literature [4, 39, 44] but not in the generality we need.

For $\boldsymbol{\pi} \in \mathcal{P}^{+}$, let $\bar{\boldsymbol{\pi}} = (\pi_1(u), \ldots, \pi_n(u)) \in \mathbb{C}[u]$ be the n-tuple of polynomials with complex coefficients obtained from $\boldsymbol{\pi}$ by setting $q = 1$. If $\boldsymbol{\pi} \in \mathcal{P}_{\mathbb{Z}}^{+}$, then it follows that $\bar{\pi}_i(u) = (1-u)^{\deg \pi_i}$, for all $1 \leqslant i \leqslant n$. It was shown in [17, § 4] that the module $V(\boldsymbol{\pi})$, $\boldsymbol{\pi} \in \mathcal{P}_{\mathbb{Z}}^{+}$, has an $\mathbf{A}$-form $V_{\mathbf{A}}(\boldsymbol{\pi})$ which gives an action of $\widetilde{\mathfrak{g}}$ on $\overline{V(\boldsymbol{\pi})} := V_{\mathbf{A}}(\boldsymbol{\pi}) \otimes_{\mathbf{A}} \mathbb{C}$. Moreover, it was proved that $\overline{V(\boldsymbol{\pi})}$ is generated, as a $\widetilde{\mathfrak{g}}$-module, by an element $v_{\overline{\boldsymbol{\pi}}}$ which satisfies the following relations:

$$(x_i^{+} \otimes \mathbb{C}[t, t^{-1}])v_{\overline{\boldsymbol{\pi}}} = 0, \quad (h_i \otimes t^{r})v_{\overline{\boldsymbol{\pi}}} = (\deg \pi_i)v_{\overline{\boldsymbol{\pi}}}, \quad (x_i^{-} \otimes 1)^{\deg \pi_i + 1}v_{\overline{\boldsymbol{\pi}}} = 0,$$

for all $1 \leqslant i \leqslant n$. Since $\overline{V(\boldsymbol{\pi})}$ is a finite-dimensional module for $\widetilde{\mathfrak{g}}$, it follows that the central element acts as zero; i.e., $\overline{V(\boldsymbol{\pi})}$ is a module for $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. Hence, there must exist $f \in \mathbb{C}[t, t^{-1}]$ of minimal degree such that

$$(\mathfrak{g} \otimes f\mathbb{C}[t, t^{-1}])\overline{V(\boldsymbol{\pi})} = 0.$$

Proposition 2.7 of [16] shows that f must be of the form $(t-1)^N$ for some $N \gg 0$. This means that the module $\overline{V(\boldsymbol{\pi})}$ is a module for the quotient Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]/(t-1)^N$. Using the isomorphism

$$\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]/(t-1)^N \cong \mathfrak{g} \otimes \mathbb{C}[t]/(t-1)^N,$$

we see that we can regard $\overline{V(\boldsymbol{\pi})}$ as a module for $\mathfrak{g}[t]$ generated by the element $v_{\overline{\boldsymbol{\pi}}}$ such that

$$(\mathfrak{g} \otimes (t-1)^N\mathbb{C}[t])\overline{V(\boldsymbol{\pi})} = 0.$$

Pulling back $\overline{V(\boldsymbol{\pi})}$ via the automorphism $x \otimes t^r \to x \otimes (t-1)^r$ of $\mathfrak{g}[t]$, we get the module which we have called $L(\boldsymbol{\pi})$. Summarizing, we have the following.

Lemma. *The* $\mathfrak{g}[t]$*-module* $L(\boldsymbol{\pi})$ *is generated by an element* $v_{\overline{\boldsymbol{\pi}}}$ *satisfying*

$$(x_i^+\otimes\mathbb{C}[t])v_{\overline{\boldsymbol{\pi}}}=0,\quad (h_i\otimes t^r)v_{\overline{\boldsymbol{\pi}}}=\delta_{r,0}(\deg\pi_i)v_{\overline{\boldsymbol{\pi}}},\quad (x_i^-\otimes 1)^{\deg\pi_i+1}v_{\overline{\boldsymbol{\pi}}}=0,$$

for all $1\leqslant i\leqslant n$*. In particular, we have*

$$L(\boldsymbol{\pi})_{\mathrm{wt}\,\boldsymbol{\pi}}=\{v\in L(\boldsymbol{\pi}):(h\otimes 1)v=\mathrm{wt}\,\boldsymbol{\pi}(h)v,\,h\in\mathfrak{h}\}=\mathbb{C}v_{\overline{\boldsymbol{\pi}}}.\qquad\square$$

2.2. Restrictions to subalgebras

Suppose that J is a connected subset of I of cardinality m. Then the subalgebra of $\mathfrak{g}$ generated by the elements $x_j^\pm$, $j\in J$, is isomorphic to $\mathfrak{sl}_{m+1}$, and will be denoted by $\mathfrak{g}_J$. We have canonical inclusions of the quantized enveloping algebras, $\mathbf{U}_q(\mathfrak{g}_J)\hookrightarrow\mathbf{U}_q(\mathfrak{g})$ and $\mathbf{U}_q(\widetilde{\mathfrak{g}_J})\hookrightarrow\mathbf{U}_q(\widetilde{\mathfrak{g}})$.

The $\mathbb{Z}_+$-span of the weights ω_j, $j\in J$, is denoted by P_J^+; the subsets Q_J^+ of Q^+ and $\mathcal{P}_J^+$ of $\mathcal{P}^+$ are defined in the obvious way. Define surjective maps $P^+\to P_J^+$ and $\mathcal{P}^+\to\mathcal{P}_J^+$ by

$$\lambda=\sum_{i=1}^n r_i\omega_i\to\lambda_J=\sum_{j\in J}r_j\omega_j,\quad \boldsymbol{\pi}=(\pi_1,\ldots,\pi_n)\to\boldsymbol{\pi}_J=(\pi_j)_{j\in J}.$$

The set P_J^+ (respectively, $\mathcal{P}_J^+$) indexes the isomorphism classes of finite-dimensional irreducible representations of $\mathfrak{g}_J$ and $\mathbf{U}_q(\mathfrak{g}_J)$ (respectively, $\mathbf{U}_q(\widetilde{\mathfrak{g}_J})$). Given $\lambda\in P_J^+$, let $V^J(\lambda)$ be a finite-dimensional irreducible representation of $\mathfrak{g}_J$; the modules $V_q^J(\lambda)$ and $V^J(\boldsymbol{\pi})$, $\boldsymbol{\pi}\in\mathcal{P}_J^+$, are defined in the obvious way. The following result is proved by standard methods (see [14] for details in the quantum case).

Proposition. *Let* $\lambda\in P^+$ *and* $\boldsymbol{\pi}\in\mathcal{P}^+$.

(i) *We have an inclusion of* $\mathfrak{g}_J$*-modules* $V^J(\lambda_J)\to V(\lambda)$*. Analogous statements hold for* $V_q^J(\lambda_J)$ *and for* $V^J(\boldsymbol{\pi}_J)$*.*

(ii) *Suppose that* $\mu\in P^+$ *is such that* $\lambda-\mu\in Q_J^+$*. Then*

$$\mathrm{Hom}_{\mathbf{U}_q(\mathfrak{g})}(V_q(\mu),V(\boldsymbol{\pi}))\neq 0\implies \mathrm{Hom}_{\mathbf{U}_q(\mathfrak{g}_J)}(V_q^J(\mu_J),V^J(\boldsymbol{\pi}_J))\neq 0.\qquad\square$$

2.3. Existence of φ_1

Suppose now that $\boldsymbol{\pi}_1\in\mathcal{P}_{\mathbb{Z}}^+(1)$, and let $\lambda:=\mathrm{wt}\,\boldsymbol{\pi}=\omega_{i_1}+\cdots+\omega_{i_k}$, for some $1\leqslant k\leqslant n$ such that $1\leqslant i_1<\cdots<i_k\leqslant n$. In view of the definition of $M(0,\mathrm{wt}\,\boldsymbol{\pi})$ given in § 1.9 and Lemma 2.1, the existence of the surjective map $\varphi_1:M(0,\mathrm{wt}\,\boldsymbol{\pi})\to L(\boldsymbol{\pi})$ follows, if we prove that

$$(x_i^-\otimes t^{\lambda(h_i)})v_{\overline{\boldsymbol{\pi}}}=0,\quad 1\leqslant i\leqslant n,\quad (x_{i_j,i_{j+1}}^-\otimes t)v_{\overline{\boldsymbol{\pi}}}=0,\quad 1\leqslant j\leqslant k-1.$$

If $\lambda(h_i)=0$ then the first equality is clear from Lemma 2.1. If $\lambda(h_i)=1$, i.e., $i=i_j$ for some $1\leqslant j\leqslant k$, then Lemma 2.1 gives

$$(x_i^-\otimes 1)^2v_{\overline{\boldsymbol{\pi}}}=0.$$

Applying $(x_i^+\otimes t)$, taking commutators, and using the relations in Lemma 2.1 again proves that $(x_i^-\otimes t)v_{\overline{\boldsymbol{\pi}}}=0$.

Suppose for a contradiction that $v := (x^-_{i_j,i_{j+1}} \otimes t)v_{\overline{\boldsymbol{\pi}}} \neq 0$, for some $1 \leqslant j \leqslant k-1$. Then, we claim that $x^+_s v = 0,\ 1 \leqslant s \leqslant n$. This is obviously true except when $s = i_j$ or $s = i_{j+1}$, when we have to prove that

$$(x^-_{i_j+1,i_{j+1}} \otimes t)v_{\overline{\boldsymbol{\pi}}} = 0 = (x^-_{i_j,i_{j+1}-1} \otimes t)v_{\overline{\boldsymbol{\pi}}}.$$

If $i_{j+1} = i_j + 1$, i.e., $x^-_{i_j+1,i_{j+1}} = x^-_{i_{j+1}}$ or $x^-_{i_j,i_{j+1}-1} = x^-_{i_j}$, this was established in the previous paragraph.

If $i_{j+1} > i_j + 1$, we can write

$$(x^-_{i_j+1,i_{j+1}} \otimes t) = [x^-_{i_j+1,i_{j+1}-1}, x^-_{i_{j+1}} \otimes t].$$

Since $\deg \pi_i = 0$ if $i_j < i < i_{j+1}$, we have $x^-_{i_j+1,i_{j+1}-1} v_{\overline{\boldsymbol{\pi}}} = 0$. Together with the fact that we have proved $(x^-_{i_{j+1}} \otimes t)v_{\overline{\boldsymbol{\pi}}} = 0$, it follows that $(x^-_{i_j+1,i_{j+1}} \otimes t)v_{\overline{\boldsymbol{\pi}}} = 0$. The proof that $(x^-_{i_j,i_{j+1}-1} \otimes t)v_{\overline{\boldsymbol{\pi}}} = 0$ is similar, and we omit the details. This proves the claim, and it follows that we have

$$\dim \operatorname{Hom}_{\mathfrak{g}}(V(\mu), L(\boldsymbol{\pi})) > 0, \quad \mu = \lambda - \sum_{s=0}^{i_{j+1}-i_j} \alpha_{i_j+s},$$

and hence also

$$\dim \operatorname{Hom}_{\mathbf{U}_q(\mathfrak{g})}(V_q(\mu), V(\boldsymbol{\pi})) > 0, \quad \mu = \lambda - \sum_{s=0}^{i_{j+1}-i_j} \alpha_{i_j+s}. \tag{2.1}$$

Setting $J = \{i_j, i_j + 1, \dots, i_{j+1}\}$, we have

$$\mu_J = 0, \quad \boldsymbol{\pi}_J = (\boldsymbol{\omega}_{i_j,a_j}, 1, \dots, 1, \boldsymbol{\omega}_{i_{j+1},a_{j+1}}), \quad a_{i_j} a^{-1}_{i_{j+1}} = q^{\pm(i_{j+1}-i_j+2)}.$$

Using Proposition 2.2 and equation (2.1), we have

$$\dim \operatorname{Hom}_{\mathbf{U}_q(\mathfrak{g}_J)}(V^J_q(0), V^J(\boldsymbol{\pi}_J)) > 0. \tag{2.2}$$

On the other hand, Lemma 1.6 applies to the representation $V^J(\boldsymbol{\pi}_J)$ of $\mathbf{U}_q(\widetilde{\mathfrak{g}}_J)$, and hence we have $V^J(\boldsymbol{\pi}_J) \cong V^J_q(\lambda_J)$ as $\mathbf{U}_q(\mathfrak{g}_J)$-modules. This contradicts equation (2.2), and hence we must have $v = 0$. The existence of the surjective map $\varphi_1 : M(0, \operatorname{wt} \boldsymbol{\pi}) \to L(\boldsymbol{\pi}) \to 0$ is established.

2.4. Tensor products and defining relations

The relations given in Lemma 2.1 are not necessarily the defining relations of $L(\boldsymbol{\pi})$. However, we have the following result, which is a special case of [17, § 4] and the main result of [7].

Theorem 3. *Let $\boldsymbol{\pi} \in \mathcal{P}^+_{\mathbb{Z}}$ be such that the prime factors of $V(\boldsymbol{\pi})$ are*

$$\{V(\boldsymbol{\omega}_{j_s,b_s}) : 1 \leqslant s \leqslant m, 1 \leqslant j_s \leqslant n, b_s \in q^{\mathbb{Z}}\};$$

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i.e.,

$$V(\boldsymbol{\pi}) \cong_{\mathbf{U}_q(\widetilde{\mathfrak{g}})} V(\boldsymbol{\omega}_{j_1,b_1}) \otimes \cdots \otimes V(\omega_{j_m,b_m}).$$

Then $L(\boldsymbol{\pi})$ *is generated by the element* $v_{\overline{\boldsymbol{\pi}}}$ *with the following* defining *relations:*

$$(x_i^+ \otimes \mathbb{C}[t])v_{\overline{\boldsymbol{\pi}}} = 0, \quad (h_i \otimes t^r)v_{\overline{\boldsymbol{\pi}}} = \delta_{r,0}(\deg \pi_i)v_{\overline{\boldsymbol{\pi}}}, \quad (x_i^- \otimes 1)^{\deg \pi_i + 1}v_{\overline{\boldsymbol{\pi}}} = 0,$$

for all $1 \leqslant i \leqslant n$. □

2.5. Simple socle in a tensor product

The hypothesis in Theorem 3 that $V(\boldsymbol{\pi})$ has all its prime factors of the form $\boldsymbol{\omega}_{i,a}$, $1 \leqslant i \leqslant n$, $a \in q^{\mathbb{Z}}$, is generically true. This is seen from the next result, the dual of which is proved in [5, Theorem 4.4 and Corollary 6.2].

Proposition. *Let* $m \geqslant 1$, $1 \leqslant j_1, \ldots, j_m \leqslant n$, *and* $b_1, \ldots, b_m \in q^{\mathbb{Z}}$, *be such that*

$$s > r \implies b_r/b_s \notin \{q^{2p+2-j_s-j_r} : \max\{j_r, j_s\} < p+1 \leqslant \min\{j_r + j_s, n+1\}\}. \tag{2.3}$$

Then $V(\boldsymbol{\omega}_{j_1,b_1} \cdots \boldsymbol{\omega}_{j_m,b_m})$ *is the unique irreducible submodule of* $V(\boldsymbol{\omega}_{j_1,b_1}) \otimes \cdots \otimes V(\boldsymbol{\omega}_{j_m,b_m})$. *Moreover, if equation (2.3) holds for all* $1 \leqslant r, s \leqslant m$, *then we have an isomorphism of* $\mathbf{U}_q(\widetilde{\mathfrak{g}})$*-modules,*

$$V(\boldsymbol{\omega}_{j_1,b_1}) \otimes \cdots \otimes V(\boldsymbol{\omega}_{j_m,b_m}) \cong V(\boldsymbol{\pi}).$$

□

2.6. The elements $\boldsymbol{\pi}^o$ and $\boldsymbol{\pi}^e$

Let $\boldsymbol{\pi} \in \mathcal{P}_{\mathbb{Z}}^+(1)$, in which case we can write

$$\boldsymbol{\pi} = \boldsymbol{\omega}_{i_1,a_1} \cdots \boldsymbol{\omega}_{i_k,a_k}, \quad 1 \leqslant i_1 < \cdots < i_k \leqslant n,$$

where either

$$a_1 = q^m, \quad a_{2j}/a_{2j-1} = q^{i_{2j}-i_{2j-1}+2}, \quad a_{2j+1}/a_{2j} = q^{i_{2j}-i_{2j+1}-2},$$

or

$$a_1 = q^m, \quad a_{2j}/a_{2j-1} = q^{-(i_{2j}-i_{2j-1}+2)}, \quad a_{2j+1}/a_{2j} = q^{-(i_{2j}-i_{2j+1}-2)},$$

for some $m \in \mathbb{Z}$. We shall always assume that we are in the first case. The proof in the other case is identical.

Define

$$\boldsymbol{\pi}^o = \begin{cases} \boldsymbol{\omega}_{i_1,a_1}\boldsymbol{\omega}_{i_3,a_3} \cdots \boldsymbol{\omega}_{i_k,a_k}, & k \text{ odd} \\ \boldsymbol{\omega}_{i_1,a_1}\boldsymbol{\omega}_{i_3,a_3} \cdots \boldsymbol{\omega}_{i_{k-1},a_{k-1}}, & k \text{ even}, \end{cases} \quad \boldsymbol{\pi}^e = \begin{cases} \boldsymbol{\omega}_{i_2,a_2}\boldsymbol{\omega}_{i_4,a_4} \cdots \boldsymbol{\omega}_{i_{k-1},a_{k-1}}, & k \text{ odd} \\ \boldsymbol{\omega}_{i_2,a_2}\boldsymbol{\omega}_{i_4,a_4} \cdots \boldsymbol{\omega}_{i_k,a_k}, & k \text{ even}, \end{cases}$$

and note that $\boldsymbol{\pi}^e\boldsymbol{\pi}^o = \boldsymbol{\pi}$.

Proposition. *Assume that* k *is even. We have an isomorphism of* $\mathbf{U}_q(\widetilde{\mathfrak{g}})$*-modules,*

$$V(\boldsymbol{\pi}^o) = V(\boldsymbol{\omega}_{i_1,a_1}) \otimes \cdots \otimes V(\boldsymbol{\omega}_{i_{k-1},a_{k-1}}), \quad V(\boldsymbol{\pi}^e) \cong V_q(\boldsymbol{\omega}_{i_2,a_2}) \otimes \cdots \otimes V(\boldsymbol{\omega}_{i_k,a_k}).$$

Moreover, $V(\boldsymbol{\pi})$ *is a submodule of* $V(\boldsymbol{\pi}^o) \otimes V(\boldsymbol{\pi}^e)$. *Analogous statements hold if* k *is odd.*

Proof. The first assertion of the proposition is immediate from Theorem 2. However, it is easy to give a proof using Proposition 2.5, and we include it for completeness since it is crucial for this paper.

Define integers r_j, $1 \leqslant j \leqslant k$, by

$$r_1 = 0, \quad r_2 = i_2 - i_1 + 2,$$
$$r_{2s+1} = -i_1 + 2(i_2 - i_3 + \cdots - i_{2s-1} + i_{2s}) - i_{2s+1}, \quad s \geqslant 1,$$
$$r_{2s+2} = -i_1 + 2(i_2 - i_3 + \cdots + i_{2s} - i_{2s+1}) + i_{2s+2} + 2, \quad s \geqslant 1,$$

and note that $a_j = q^{r_j+m}$, for $1 \leqslant j \leqslant k$.

We prove that $V(\boldsymbol{\pi}^o)$ is irreducible. Using Proposition 2.5, it suffices to prove that, for all $s > j$ and $p \in \mathbb{Z}_+$ with $p+1 > i_{2s+1}$,

$$r_{2s+1} - r_{2j+1} \neq \pm(2p + 2 - i_{2j+1} - i_{2s+1}),$$

or equivalently that

$$-i_{2j+1} + 2(i_{2j+2} - i_{2j+3} + \cdots - i_{2s-1} + i_{2s}) - i_{2s+1} \neq \pm(2p + 2 - i_{2j+1} - i_{2s+1}).$$

This amounts to proving that

$$p + 1 \neq i_{2s} - i_{2s-1} + \cdots - i_{2j+3} + i_{2j+2}$$

and

$$p + 1 \neq i_{2s+1} - i_{2s} + \cdots + i_{2j+3} - i_{2j+2} + i_{2j+1}.$$

If equality were to hold, then in the first case we would get $p+1 < i_{2s} < i_{2s+1}$ and in the second case we get $p+1 < i_{2s+1}$, which contradicts our assumptions on p. The proof of the irreducibility of $V(\boldsymbol{\pi}^o)$ is complete. A similar argument proves the result for $V(\boldsymbol{\pi}^e)$.

To prove that $V(\boldsymbol{\pi}^o) \otimes V(\boldsymbol{\pi}^e)$ contain $V(\boldsymbol{\pi})$ as its unique submodule, we again use Proposition 2.5, and note that it is enough to check that, for all s and j, we have

$$r_{2j-1} - r_{2s} \notin \{2 + 2p - i_{2s} - i_{2j-1} : \max\{i_{2j-1}, i_{2s}\} < p + 1 \leqslant \min\{i_{2j-1} + i_{2s}, n+1\}\}. \tag{2.4}$$

For clarity, we prove this by breaking up the checking into several cases. If $s \geqslant j \geqslant 1$ and $i_{2s} + i_{2j-1} \leqslant n + 1$, we have

$$r_{2s} - r_{2j-1} = i_{2s} + i_{2j-1} + 2 - 2(i_{2j-1} - i_{2j} + \cdots - i_{2s-2} + i_{2s-1});$$

i.e.,

$$r_{2j-1} - r_{2s} = -i_{2s} - i_{2j-1} + 2(-1 + (i_{2j-1} - i_{2j}) + \cdots + (i_{2s-1} - i_{2s}) + i_{2s}).$$

Since $(-1 + (i_{2j-1} - i_{2j}) + \cdots + (i_{2s-1} - i_{2s}) + i_{2s}) < i_{2s}$, we see that equation (2.4) is satisfied. On the other hand, if $j \geqslant s \geqslant 1$ and $i_{2s} + i_{2j-1} \leqslant n+1$, we have

$$r_{2s} - r_{2j-1} = i_{2s} + i_{2j-1} - 2(-1 + (i_{2s} - i_{2s+1}) + \cdots + (i_{2j-2} - i_{2j-1}) + i_{2j-1}).$$

Since the expression in parentheses is less than i_{2j-1}, we see that equation (2.4) is again satisfied. The other two cases are similar, and we omit the details. □

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2.7. Existence of φ_2

Using Proposition 2.6 and the discussion in § 1.13, we see that there exists a map $\varphi_2 : L(\boldsymbol{\pi}) \to L(\boldsymbol{\pi}^o) \otimes L(\boldsymbol{\pi}^e)$. Moreover, since $\operatorname{wt}\boldsymbol{\pi} = \operatorname{wt}\boldsymbol{\pi}^o + \operatorname{wt}\boldsymbol{\pi}^e$, and

$$L(\boldsymbol{\pi})_{\operatorname{wt}\boldsymbol{\pi}} = \mathbb{C}v_{\overline{\boldsymbol{\pi}}}, \quad L(\boldsymbol{\pi}^o)_{\operatorname{wt}\boldsymbol{\pi}^o} = \mathbb{C}v_{\overline{\boldsymbol{\pi}^o}}, \quad L(\boldsymbol{\pi}^e)_{\operatorname{wt}\boldsymbol{\pi}^e} = \mathbb{C}v_{\overline{\boldsymbol{\pi}^e}},$$

we see that

$$\varphi_2(v_{\overline{\boldsymbol{\pi}}}) = v_{\overline{\boldsymbol{\pi}^o}} \otimes v_{\overline{\boldsymbol{\pi}^e}}.$$

Moreover, $V(\boldsymbol{\pi}^o)$ and $V(\boldsymbol{\pi}^e)$ satisfy the conditions of Theorem 3, and hence we have the defining relations of $L(\boldsymbol{\pi}^o)$ and $L(\boldsymbol{\pi}^e)$. We have now established the second step of the proof of Theorem 1, modulo the identification of $L(\boldsymbol{\pi}^o)$ and $L(\boldsymbol{\pi}^e)$ with the level-one Demazure module. This will be done in § 4.1.

3. Level-two Demazure modules in the tensor product of level-one Demazure modules

We establish the third step in the proof of Theorem 1.

3.1. Extended and affine Weyl groups

We use freely the notation established in § 1.1. Let $\mathfrak{n}^\pm$ be the subalgebra spanned by the elements $x^\pm_{i,j}$, $1 \leqslant i \leqslant j \leqslant n$, and set $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Set $h_{i,j} = [x^+_{i,j}, x^-_{i,j}]$, and note that $h_{i,i} = h_i$. Define elements $x_0^\pm$ and h_0 of $\widehat{\mathfrak{g}}$ by

$$x_0^\pm = x^\mp_{1,n} \otimes t^{\pm 1}, \quad h_0 = c - h_{1,n},$$

and set

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d, \quad \widehat{\mathfrak{n}}^+ = \mathfrak{g} \otimes t\mathbb{C}[t] \oplus \mathfrak{n}^+, \quad \widehat{\mathfrak{b}} = \widehat{\mathfrak{n}}^+ \oplus \widehat{\mathfrak{h}}.$$

We shall regard an element of $\mathfrak{h}^*$ as an element of $\widehat{\mathfrak{h}}^*$ by setting it to be zero on c and d. Define elements $\Lambda_i \in \widehat{\mathfrak{h}}^*$, $0 \leqslant i \leqslant n$, by

$$\Lambda_0(\mathfrak{h} \oplus \mathbb{C}d) = 0, \quad \Lambda_0(c) = 1, \quad \Lambda_i = \omega_i + \Lambda_0, \quad 1 \leqslant i \leqslant n,$$

and let $\delta \in \widehat{\mathfrak{h}}^*$ be given by $\delta(\mathfrak{h} \oplus \mathbb{C}c) = 0$, $\delta(d) = 1$. Let $\widehat{P}$ be the $\mathbb{Z}$-span of $\{\delta\}$ and $\{\Lambda_i : 0 \leqslant i \leqslant n\}$, and let $\widehat{P}^+$ be the direct sum of the $\mathbb{Z}$-span of $\{\delta\}$ with the $\mathbb{Z}_+$-span of $\{\Lambda_i : 0 \leqslant i \leqslant n\}$.

Let s_i, $0 \leqslant i \leqslant n$ be the simple reflection of the affine Weyl group $\widehat{W}$; recall that it acts on $\widehat{\mathfrak{h}}^*$ and $\widehat{\mathfrak{h}}$ by

$$s_i(\mu) = \mu - \mu(h_i)\alpha_i, \quad s_i(h) = h - \alpha_i(h)h_i, \quad h \in \widehat{\mathfrak{h}}, \mu \in \widehat{\mathfrak{h}}^*.$$

Note that

$$w \in \widehat{W} \implies w(c) = c \quad \text{and} \quad w(\delta) = \delta.$$

The Weyl group W of $\mathfrak{g}$ is the subgroup of $\widehat{W}$ generated by the elements s_i, $1 \leqslant i \leqslant n$, and is isomorphic to the symmetric group on $n+1$ letters. Let $w_0 \in W$ be the unique element of maximal length. The action of W on $\widehat{\mathfrak{h}}^*$ preserves P and Q, and we have an isomorphism of groups

$$\widehat{W} \cong W \ltimes Q.$$

The extended affine Weyl group $\widetilde{W}$ is the semidirect product $W \ltimes P$. The affine Weyl group is a normal subgroup of $\widetilde{W}$, and, if $\mathcal{T}$ is the cyclic subgroup of the diagram automorphisms of $\widehat{\mathfrak{g}}$, we have

$$\widetilde{W} \cong \mathcal{T} \ltimes \widehat{W}.$$

Here $\mathcal{T}$ is just isomorphic to the cyclic group of order $n+1$. Since $\mathcal{T}$ preserves $\widehat{P}$ and $\widehat{P}^+$, we see that $\widetilde{W}$ preserves $\widehat{P}$. The following formulae make explicit the action of $\mu \in P$ on $\widehat{\mathfrak{h}}^*$:

$$t_\mu(\lambda) = \lambda - (\lambda, \mu)\delta, \quad \lambda \in \mathfrak{h}^* \oplus \mathbb{C}\delta, \quad t_\mu(\Lambda_0) = \Lambda_0 + \mu - \tfrac{1}{2}(\mu, \mu)\delta,$$

where $(\ ,\ )$ is the symmetric bilinear form defined by $(\alpha_i, \omega_j) = \delta_{i,j}$, for all $1 \leqslant i, j \leqslant n$.

3.2. Integrable highest-weight modules

Recall that a weight module V for $\widehat{\mathfrak{h}}$ is one where $\widehat{\mathfrak{h}}$ acts diagonally. Let $\operatorname{wt} V \subset \widehat{\mathfrak{h}}^*$ be the set of eigenvalues for this action, and, given $\mu \in \operatorname{wt} V$, let V_μ be the corresponding eigenspace. For $\Lambda \in \widehat{P}^+$, let $V(\Lambda)$ be the irreducible highest-weight integrable $\widehat{\mathfrak{g}}$-module, which is generated by an element v_Λ with defining relations

$$\widehat{\mathfrak{n}}^+ v_\Lambda = 0, \quad h v_\Lambda = \Lambda(h) v_\Lambda, \quad (x_i^-)^{\Lambda(h_i)+1} v_\Lambda = 0,$$

where $h \in \widehat{\mathfrak{h}}$ and $0 \leqslant i \leqslant n$. Note that $\operatorname{wt} V(\Lambda) \subset \Lambda - \widehat{Q}^+$. (Here $\widehat{Q}^+$ is the $\mathbb{Z}_+$-span of the elements α_i, $0 \leqslant i \leqslant n$, and δ). It is easily seen that, for all $r \in \mathbb{Z}$, we have an isomorphism of $\widetilde{\mathfrak{g}}$-modules,

$$V(\Lambda - r\delta) \cong V(\Lambda). \tag{3.1}$$

The following proposition is well known (see [32, Chapters 10 and 11], for instance).

Proposition. (i) *Let $\Lambda \in \widehat{P}^+$. We have*

$$\dim V(\Lambda)_\mu = \dim V(\Lambda)_{w\mu}, \quad \textit{for all } w \in \widehat{W}, \mu \in \widehat{\mathfrak{h}}^*.$$

In particular, $\dim V(\Lambda)_{w\Lambda} = 1$ *for all* $w \in \widehat{W}$.

(ii) *Given $\Lambda', \Lambda'' \in \widehat{P}^+$, we have*

$$V(\Lambda') \otimes V(\Lambda'') \cong \bigoplus_{\Lambda \in \widehat{P}^+} \dim(\operatorname{Hom}_{\widehat{\mathfrak{g}}}(V(\Lambda), V(\Lambda') \otimes V(\Lambda''))) V(\Lambda).$$

Moreover,

$$\dim \operatorname{Hom}_{\widehat{\mathfrak{g}}}(V(\Lambda), V(\Lambda') \otimes V(\Lambda'')) = \begin{cases} 1, & \Lambda = \Lambda' + \Lambda'', \\ 0, & \Lambda \notin \Lambda' + \Lambda'' - \widehat{Q}^+. \end{cases} \tag{3.2}$$

□

Corollary. *Suppose that $\Lambda', \Lambda'' \in \widehat{P}^+$, and that $\Lambda = \Lambda' + \Lambda''$. For all $w \in \widehat{W}$, we have*

$$(V(\Lambda') \otimes V(\Lambda''))_{w\Lambda} = V(\Lambda)_{w\Lambda}, \tag{3.3}$$

where we have identified $V(\Lambda)$ with its image in $V(\Lambda') \otimes V(\Lambda'')$.

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Proof. Since the right-hand side is of dimension one by part (i) of the proposition and is clearly contained in the left-hand side, it suffices to prove that

$$\dim(V(\Lambda')\otimes V(\Lambda''))_{w\Lambda} = 1.$$

If not, we get by using part (ii) of the proposition with the first case of equation (3.2) that there exists $\Lambda_1 \in \widehat{P}^+$ with $\Lambda_1 \neq \Lambda$, such that

$$\dim \operatorname{Hom}_{\widehat{\mathfrak{g}}}(V(\Lambda_1), V(\Lambda')\otimes V(\Lambda'')) \neq 0, \quad V(\Lambda_1)_{w\Lambda} \neq 0.$$

Using part (i) of the proposition, this means that $V(\Lambda_1)_\Lambda \neq 0$. But this is impossible, since $\Lambda_1 \subset \Lambda - \widehat{Q}^+$ and $\Lambda_1 \neq \Lambda$, thus proving the corollary. □

3.3. Stable Demazure modules

Given $\Lambda \in \widehat{P}^+$ and $w\tau \in \widetilde{W}$, where $w \in \widehat{W}$ and $\tau \in \mathcal{T}$, the Demazure module $V_w(\tau\Lambda)$ is the $\widehat{\mathfrak{b}}$-submodule of $V(\tau\Lambda)$ given by

$$V_w(\tau\Lambda) = \mathbf{U}(\widehat{\mathfrak{b}})v_{w\tau\Lambda}, \quad 0 \neq v_{w\tau\Lambda} \in V(\tau\Lambda)_{w\tau\Lambda}.$$

The Demazure modules are necessarily finite dimensional since $\operatorname{wt} V(\Lambda) \subset \Lambda - Q^+$. We say that $V_w(\tau\Lambda)$ is a level-ℓ Demazure module if $\Lambda(c) = \ell$. The following is immediate from Corollary 3.2.

Lemma. *Let $w\tau \in \widetilde{W}$ and $\Lambda', \Lambda'' \in \widehat{P}^+$. We have an isomorphism of $\widehat{\mathfrak{b}}$-modules,*

$$V_w(\tau(\Lambda' + \Lambda'')) \cong \mathbf{U}(\widehat{\mathfrak{b}})(v_{w\tau\Lambda'} \otimes v_{w\tau\Lambda''}) \subset V(\tau\Lambda') \otimes V(\tau\Lambda''). \qquad \square$$

In this paper we are only interested in the Demazure modules $V_w(\Lambda)$ satisfying the condition $w\Lambda(h_i) \leqslant 0$, for all $1 \leqslant i \leqslant n$. In this case, we have $\mathfrak{n}^- v_{w\Lambda} = 0$, and $V_w(\Lambda)$ is a module for the parabolic subalgebra $\widehat{\mathfrak{b}} \oplus \mathfrak{n}^-$; i.e.,

$$V_w(\Lambda) = \mathbf{U}(\widehat{\mathfrak{b}} \oplus \mathfrak{n}^-)v_{w\Lambda} = \mathbf{U}(\mathfrak{g}[t])v_{w\Lambda} = \mathbf{U}(\mathfrak{g}[t])v_{w_0^{-1}w\Lambda},$$

where the last equality follows from the fact that $V_w(\Lambda)$ is a finite-dimensional $\mathfrak{g}$-module. Writing $w\Lambda = w_0\lambda + \Lambda(c)\Lambda_0 + r\delta$, for a unique $\lambda \in P^+$ and $r \in \mathbb{Z}$, we see from equation (3.1) that

$$V_w(\Lambda) \cong_{\mathfrak{g}[t]} V_w(\Lambda - r\delta).$$

Hence, we denote these modules as $\tau_r^* D(\ell, \lambda)$, where $\ell = \Lambda(c)$, $r \in \mathbb{Z}$. Notice that the action of d on these modules defines a $\mathbb{Z}$-grading on them which is compatible with the grading on $\mathfrak{g}[t]$; i.e., the modules $\tau_r^* D(\ell, \lambda)$ are graded $\mathfrak{g}[t]$-modules; for a fixed ℓ and λ, these modules are just grade shifts, and we set $\tau_0^* D(\ell, \lambda) = D(\ell, \lambda)$. The eigenspace $D(\ell, \lambda)_\lambda$ for the $\mathfrak{h}$-action is one dimensional, and we shall frequently denote a non-zero element of this space by $v_{\ell,\lambda}$.

3.4. The main result on tensor products of level-one Demazure modules

The main result of this section is the following.

Theorem 4. *Given $\mu \in P^+$, there exist $\mu^o, \mu^e \in P^+$ with $\mu = \mu^o + \mu^e$ such that we have an injective map of graded $\mathfrak{g}[t]$-modules*

$$D(2, \mu) \hookrightarrow D(1, \mu^o) \otimes D(1, \mu^e), \quad v_{2,\mu} \to v_{1,\mu^o} \otimes v_{1,\mu^e}.$$

3.5. The key proposition

Recall that

$$P^+(1) = \{\lambda \in P^+ : \lambda(h_i) \leqslant 1 \text{ for all } 1 \leqslant i \leqslant n\}.$$

Given $\lambda = \sum_{j=1}^k \omega_{i_j} \in P^+(1)$, with $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$, define $\lambda^o, \lambda^e \in P^+(1)$ by

$$\lambda^o = \begin{cases} \omega_{i_1} + \omega_{i_3} + \cdots + \omega_{i_k}, & k \text{ odd}, \\ \omega_{i_1} + \omega_{i_3} + \cdots + \omega_{i_{k-1}}, & k \text{ even}, \end{cases} \qquad \lambda^e = \lambda - \lambda^o.$$

We shall prove the following.

Proposition. *Given $\lambda \in P^+(1)$ and $\nu \in P^+$, there exists $w \in \widetilde{W}$ such that*

$$w(\nu + \lambda^o + \Lambda_0) \in \widehat{P}^+, \quad w(\nu + \lambda^e + \Lambda_0) \in \widehat{P}^+.$$

3.6. Proof of Theorem 4 and the third step of the proof of Theorem 1

Assuming Proposition 3.5, the proof of Theorem 4 is completed as follows. Write $\mu = 2\nu + \lambda$, where $\nu \in P^+$ and $\lambda \in P^+(1)$, and set $\mu^o = \nu + \lambda^o$ and $\mu^e = \nu + \lambda^e$. Choose $w \in \widetilde{W}$ as in Proposition 3.5, and take

$$\Lambda = w(\mu + 2\Lambda_0), \quad \Lambda^o = w(\nu + \lambda^o + \Lambda_0), \quad \Lambda^e = w(\nu + \lambda^e + \Lambda_0).$$

Then $\Lambda^o, \Lambda^e \in \widehat{P}^+$ and $\Lambda = \Lambda^o + \Lambda^e \in \widehat{P}^+$, and

$$D(2, \mu) = V_{w_0 w^{-1}}(\Lambda), \quad D(1, \mu^o) = V_{w_0 w^{-1}}(\Lambda^o), \quad D(1, \mu^e) = V_{w_0 w^{-1}}(\Lambda^e).$$

Theorem 4 is now immediate from Lemma 3.3.

The third step of the proof of Theorem 1 now follows. Given $\boldsymbol{\pi} \in \mathcal{P}^+(1)$, observe that $\mathrm{wt}\,\boldsymbol{\pi}^o = (\mathrm{wt}\,\boldsymbol{\pi})^o$ and $\mathrm{wt}\,\boldsymbol{\pi}^e = (\mathrm{wt}\,\boldsymbol{\pi})^e$, and hence using Theorem 4 we have a map of $\mathfrak{g}[t]$-modules $D(2, \mathrm{wt}\,\boldsymbol{\pi}) \hookrightarrow D(1, \mathrm{wt}\,\boldsymbol{\pi}^o) \otimes D(1, \mathrm{wt}\,\boldsymbol{\pi}^e)$.

3.7. Proof of Proposition 3.5; reduction to $\nu = 0$

In the rest of this section we prove Proposition 3.5. The first step is to show that, for a fixed λ, it suffices to prove the result when $\nu = 0$. Thus, suppose that we have chosen $w \in \widetilde{W}$ such that $w(\lambda^o + \Lambda_0)$ and $w(\lambda^e + \Lambda_0)$ are in $\widehat{P}^+$. Since $(\lambda^o + \Lambda_0)(c) = 1 = (\lambda^e + \Lambda_0)(c)$, we may write

$$w(\lambda^o + \Lambda_0) = \Lambda_i + p^o\delta, \quad w(\lambda^e + \Lambda_0) = \Lambda_j + p^e\delta,$$

for some $p^o, p^e \in \mathbb{Z}$ and $0 \leqslant i, j \leqslant n$. Using the formulae in § 3.1, and the fact that $\Lambda_p = \omega_p + \Lambda_0$ for $0 \leqslant p \leqslant n$, where $\omega_0 = 0$, we get

$$\begin{aligned} t_{-w\nu} w(\lambda^o + \Lambda_0 + \nu) &= (\Lambda_i + (p^o + \tfrac{1}{2}(\nu, \nu) + (\omega_i, w\nu))\delta) \in \widehat{P}^+, \\ t_{-w\nu} w(\lambda^e + \Lambda_0 + \nu) &= (\Lambda_j + (p^e + \tfrac{1}{2}(\nu, \nu) + (\omega_j, w\nu))\delta) \in \widehat{P}^+, \end{aligned}$$

and the claim is established.

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3.8. Proof of Proposition 3.5; the case $\nu = 0$

Consider the partial order on $P^+(1)$ given by $\mu \leqslant \nu$ iff $\nu - \mu \in Q^+$. The minimal elements of this order are 0 and ω_i, $1 \leqslant i \leqslant n$. If $\lambda = 0$, the result is clear: we just take $\Lambda^o = \Lambda^e = \Lambda_0$ and $w = \mathrm{id}$. If $\lambda = \omega_i$, we take $\lambda^o = \omega_i$ and $\lambda^e = 0$. Since $\omega_i + \Lambda_0 \in \widehat{P}^+$, we take $\Lambda^o = \omega_i + \Lambda_0$, $\Lambda^e = \Lambda_0$, and $w = \mathrm{id}$. If $\lambda = \omega_i + \omega_j$ with $i < j$ then we again take $w = \mathrm{id}$, since $\omega_p + \Lambda_0 = \Lambda_p$ for all $0 \leqslant p \leqslant n$.

For the inductive step, let $\lambda \in P^+(1)$ with $\lambda = \sum_{j=1}^{k} \omega_{i_j}$, $i_1 < \cdots < i_k$ and $k > 2$. Suppose that we have proved the result for all elements $\mu \in P^+(1)$ with $\mu < \lambda$. To prove the result for λ, it clearly suffices to show that there exist $w \in \widetilde{W}$ and $\mu \in P^+$ with $\mu < \lambda$, and $p^o, p^e \in \mathbb{Z}$, such that

$$w(\lambda^o + \Lambda_0) = \mu^o + \Lambda_0 + p^o\delta, \quad \text{and} \quad w(\lambda^e + \Lambda_0) = \mu^e + \Lambda_0 + p^e\delta. \tag{3.4}$$

This is done as follows. Take

$$w = \begin{cases} s_{i_3}s_{i_3+1}\cdots s_n s_0, & k = 3, i_1 = 1, \\ s_{i_3}s_{i_3+1}\cdots s_n s_{i_{k-2}-1}s_{i_{k-2}-2}\cdots s_1 s_0, & k > 3, \text{ or } k = 3, \ i_1 > 1, \end{cases}$$

and

$$\mu = \begin{cases} \omega_{i_1-1} + \omega_{i_2} + \omega_{i_3+1}, & k = 3, \\ \omega_{i_1-1} + \omega_{i_2-1} + \omega_{i_3} + \cdots + \omega_{i_{k-2}} + \omega_{i_{k-1}+1} + \omega_{i_k+1}, & k > 3. \end{cases}$$

Note that $\mu < \lambda$, since

$$\lambda - \mu = \begin{cases} \alpha_{i_1} + \cdots + \alpha_{i_3}, & k = 3, \\ \alpha_{i_1} + 2\alpha_{i_2} + \cdots + 2\alpha_{i_{k-1}} + \alpha_{i_k}, & k > 3. \end{cases}$$

We now establish that equation (3.4) is satisfied. For this, it is most convenient to deal with the cases $k = 3, 4$ separately. If $k = 3$ and $i_1 > 1$, or if $k = 4$, a simple calculation gives

$$w(\lambda^o + \Lambda_0) = w(\Lambda_{i_1} + \Lambda_{i_3} - \Lambda_0) = \Lambda_{i_1} + \Lambda_{i_3} - \Lambda_0 + \sum_{j=0}^{i_1-1} \alpha_j + \sum_{j=i_3+1}^{n} \alpha_j = \mu^o + \Lambda_0 + \delta.$$

Moreover, if $k = 3$, we have

$$w(\lambda^e + \Lambda_0) = w(\Lambda_{i_2}) = \Lambda_{i_2} = \mu^e + \Lambda_0,$$

while, if $k = 4$, we have

$$w(\lambda^e + \Lambda_0) = w(\Lambda_{i_2} + \Lambda_{i_4} - \Lambda_0) = \Lambda_{i_2} + \Lambda_{i_4} - \Lambda_0 + \sum_{j=0}^{i_2-1} \alpha_j + \sum_{j=i_4+1}^{n} \alpha_j = \mu^e + \Lambda_0 + \delta.$$

The case $k = 3$ and $i_1 = 1$ is identical, and we omit the details. For $k \geqslant 5$, we write

$$w(\lambda^o + \Lambda_0) = w(\lambda^o + \Lambda_0)(d)\delta + \sum_{j=0}^{n} (w(\lambda^o + \Lambda_0), \alpha_j)\Lambda_j,$$

and similarly for $\lambda^e + \Lambda_0$. We have

$$w(\lambda^o + \Lambda_0)(d) = (\lambda^o + \Lambda_0)(w^{-1}d) = (\lambda^o + \Lambda_0)(d - h_0) = \lambda^o(h_\theta) - 1.$$

To prove that $(w(\lambda^o + \Lambda_0), \alpha_j) = (\mu^o + \Lambda_0, \alpha_j)$, it is enough to prove that

$$(\lambda^o + \Lambda_0)(w^{-1}h_j) = (\mu^o + \Lambda_0)(h_j),$$

and this is done by using the following easily established formulae:

$$\begin{gathered} w^{-1}(\alpha_0) = \alpha_0 + \alpha_1 + \alpha_n, \quad w^{-1}(\alpha_j) = \alpha_j, \quad i_3 < j < i_{k-2}, \\ w^{-1}(\alpha_j) = \alpha_{j+1}, \quad 0 < j < i_3 - 1, \quad w^{-1}(\alpha_j) = \alpha_{j-1}, \quad j > i_{k-2} + 1, \\ w^{-1}(\alpha_{i_3-1}) = \alpha_{i_3} + \cdots + \alpha_n + \alpha_0, \quad w^{-1}(\alpha_{i_{k-2}+1}) = \alpha_{i_{k-2}} + \cdots + \alpha_1 + \alpha_0, \\ w^{-1}(\alpha_{i_{k-2}}) = -(\alpha_0 + \alpha_1 + \cdots + \alpha_{i_{k-2}-1}), \end{gathered}$$

and

$$w^{-1}(\alpha_{i_3}) = \begin{cases} -(\alpha_{i_3+1} + \cdots + \alpha_n + \alpha_0), & k > 5, \\ -(\alpha_1 + \cdots + \alpha_n + 2\alpha_0), & k = 5. \end{cases}$$

The case of $w(\lambda^e + \Lambda_0)$ is identical, and the proof of Proposition 3.5 is complete.

4. A presentation of level-two $\mathfrak{g}$-stable Demazure modules

An infinite set of defining relations of the Demazure module $V_w(\Lambda)$ was given in [31, 38] for all $w \in \widetilde{W}$ and $\Lambda \in \widehat{P}^+$. In the case when $w\Lambda(h_i) \leqslant 0$, for all $1 \leqslant i \leqslant n$, these relations can be used (see [24, 44]) to give (a still infinite set of) the defining relations of $V_w(\Lambda)$ (or equivalently of $\tau_r^* D(\ell, \lambda)$) as $\mathfrak{g} \otimes \mathbb{C}[t]$-modules. In the case of the level-one Demazure modules, it was shown in [7] for $\mathfrak{sl}_{n+1}$ that the relations could be reduced to a finite number of relations. This was later extended for arbitrary levels in [19]. In this section we show that, in the case of A_n and $\ell = 2$, one can further whittle down the set of defining relations, and as a consequence we prove Proposition 1.11. Along the way, we shall see that, if $\boldsymbol{\pi} \in \mathcal{P}^+(1)$, then $L(\boldsymbol{\pi}^o)$ and $L(\boldsymbol{\pi}^e)$ are level-one Demazure modules, which establishes the missing piece (see § 2.7) of the second step of Theorem 1.

4.1. A refined presentation of Demazure modules

We recall the presentation of $D(\ell, \mu)$, $\mu \in P^+$, given in [19, Theorem 2].

Proposition. *For $\ell \in \mathbb{N}$, $\mu \in P^+$, the $\mathfrak{g}[t]$-module $D(\ell, \mu)$ is generated by an element w_μ satisfying the following defining relations:*

$$\begin{gathered} (x_i^+ \otimes 1)w_\mu = 0, \quad (h_i \otimes t^r)w_\mu = \mu(h_i)\delta_{r,0}w_\mu, \quad (x_i^- \otimes 1)^{\mu(h_i)+1}w_\mu = 0, \\ (x_{i,j}^- \otimes t^{s_{i,j}})w_\mu = 0, \quad 1 \leqslant i \leqslant j \leqslant n, \\ (x_{i,j}^- \otimes t^{s_{i,j}-1})^{m_{i,j}+1}w_\mu = 0, \end{gathered}$$

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where $r \in \mathbb{Z}_+$, $1 \leqslant i \leqslant j \leqslant n$, *and* $s_{i,j}, m_{i,j} \in \mathbb{Z}_+$ *are uniquely defined by requiring* $\mu(h_{i,j}) = (s_{i,j} - 1)\ell + m_{i,j}$ *with* $0 < m_{i,j} \leqslant \ell$. *Moreover, if* $m_{i,j} = \ell$, *the final relation is a consequence of the preceding relations.* □

Corollary. *For all* $\ell \in \mathbb{N}$ *and* $\mu \in P^+$, *the assignment* $w_\mu \to w_\mu$ *defines a canonical surjective map* $D(\ell, \mu) \to D(\ell + 1, \mu) \to 0$ *of* $\mathfrak{g}[t]$*-modules.* □

Remark. It is clear from Lemma 2.1 that, for all $\boldsymbol{\pi} \in \mathcal{P}_{\mathbb{Z}}^+$, the module $L(\boldsymbol{\pi})$ is a quotient of $D(1, \mathrm{wt}\,\boldsymbol{\pi})$. If $\boldsymbol{\pi} \in \mathcal{P}^+(1)$, it follows from Theorem 3 and Proposition 2.6 that

$$L(\boldsymbol{\pi}^o) \cong D(1, \mathrm{wt}\,\boldsymbol{\pi}^o), \quad L(\boldsymbol{\pi}^e) \cong D(1, \mathrm{wt}\,\boldsymbol{\pi}^e),$$

which establishes the missing piece (see § 2.7) of the proof of the second step of Theorem 1.

4.2. Evaluation modules

Let $\mathrm{ev}_0 : \mathfrak{g}[t] \to \mathfrak{g}$ be the map of Lie algebras given by setting $t = 0$, i.e., $a \otimes f \to f(0)a$, for all $a \in \mathfrak{g}$ and $f \in \mathbb{C}[t]$. Given a $\mathfrak{g}$-module V, let $\mathrm{ev}_0^* V$ denote the corresponding $\mathfrak{g}[t]$-module. The following is straightforward.

Lemma. *For all* $\ell \in \mathbb{Z}_+$ *and* $\mu \in P^+$, *we have* $\mathrm{wt}\, D(\ell, \mu) \subset \mu - Q^+$. *Moreover,*

$$\dim \mathrm{Hom}_{\mathfrak{g}}(V(\mu), D(\ell, \mu)) = \dim \mathrm{Hom}_{\mathfrak{g}[t]}(D(\ell, \mu), \mathrm{ev}_0^* V(\mu)) = 1,$$

and

$$D(\ell, \lambda) \cong_{\mathfrak{g}[t]} \mathrm{ev}_0^* V(\lambda), \quad \text{if } \lambda(h_{1,n}) \leqslant \ell. \qquad \square$$

4.3. The $\mathfrak{sl}_2$ case

In the case of $n = 1$, i.e., when $\mathfrak{g}$ is of type $\mathfrak{sl}_2$, we identify P^+ with $\mathbb{Z}_+$ freely, and let $x^\pm$ be the root vectors $x_1^\pm$. Given a partition, $\xi = (\xi_1 \geqslant \cdots \geqslant \xi_m \geqslant 0)$, $r \in \mathbb{N}$, define a $\mathfrak{sl}_2[t]$-module $V(\xi)$ as follows: it is the cyclic module generated by an element v_ξ with defining relations

$$(x^+ \otimes t)v_\xi = 0, \quad (h \otimes t^r)v_\xi = |\xi|\delta_{r,0}v_\xi, \quad (x^- \otimes 1)^{|\xi|+1}v_\xi = 0, |\xi| = \sum_{k \geqslant 1} \xi_k, \tag{4.1}$$

and

$$(x^+ \otimes t)^s (x^- \otimes 1)^{s+r} v_\xi = 0, \tag{4.2}$$

for all s, r satisfying the condition that there exists $k \in \mathbb{Z}_+$ with $s + r \geqslant 1 + rk + \sum_{p \geqslant k+1} \xi_k$.

In this paper, we shall only be interested in the case when the maximum value of a part is 2, i.e., ξ is of the form $2^a 1^b$ for some $a, b \in \mathbb{Z}_+$. We summarize for this special case, the results of [19, Theorem 2, Theorem 5] that we shall need.

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Proposition. *Let $\xi = 2^a 1^b$, $a, b \in \mathbb{Z}_+$ be a partition.*

(i) *We have $V(1^b) \cong D(1, b)$ as $\mathfrak{sl}_2[t]$-modules.*

(ii) *If $b \in \{0, 1\}$, then $V(2^a 1^b) \cong D(2, 2a+b)$ as $\mathfrak{sl}_2[t]$-modules.*

(iii) *If $b \geqslant 2$, then we have a short exact sequence of graded $\mathfrak{g}[t]$-modules,*

$$0 \to V(2^a 1^{b-2}) \xrightarrow{\iota} V(2^a 1^b) \xrightarrow{\pi} V(2^{a+1} 1^{b-2}) \to 0, \tag{4.3}$$

such that

$$\iota(v_{2^a 1^{b-2}}) = (x \otimes t^{a+b-1}) v_{2^a 1^b}, \quad \pi(v_{2^a 1^b}) = v_{2^{a+1} 1^{b-2}}. \qquad \square$$

Corollary. *The module $V(2^a 1^b)$ is generated by the element $v_{2^a 1^b}$ with the relations given in (4.1) and the single additional relation $(x^- \otimes t^{a+b}) v_{2^a 1^b} = 0$.*

Proof. If $a = 0$, then the corollary follows from Proposition 4.1 and part (i) of Proposition 4.3. A straightforward induction on a together with the short exact sequence in (4.3) establishes the corollary. $\square$

4.4. A further refinement of the presentation of level-two Demazure modules

Proposition. *Assume that $\mathfrak{g}$ is of type $\mathfrak{sl}_{n+1}$, and let $\mu \in P^+$. For $1 \leqslant i \leqslant j \leqslant n$, write $\mu(h_{i,j}) = 2(s_{i,j} - 1) + m_{i,j}$ with $m_{i,j} \in \{1, 2\}$. The relation*

$$(x_{i,j}^- \otimes t^{s_{i,j}-1})^{m_{i,j}+1} w_\mu = 0$$

in $D(2, \mu)$ is redundant. $\square$

Proof. It follows from Proposition 4.1 that it suffices to prove the proposition when $m_{i,j} = 1$. If $n = 1$, then the result follows from Proposition 4.3(ii) and Corollary 4.3. Otherwise, set $\alpha = \alpha_{i,j}$, and consider the subalgebra $\mathfrak{s}_\alpha[t]$ of $\mathfrak{g}[t]$ spanned by the elements $x_\alpha^\pm \otimes \mathbb{C}[t]$. Clearly $\mathfrak{s}_\alpha[t]$ is isomorphic to $\mathfrak{sl}_2[t]$ with the element $x_\alpha^\pm$ mapping to $x^\pm$. Moreover, if we denote by $D_{\mathfrak{sl}_2[t]}(2, \mu(h_\alpha))$ the level-two Demazure module for $\mathfrak{sl}_2[t]$, then we have a non-zero map of $\mathfrak{sl}_2[t]$-modules,

$$D_{\mathfrak{sl}_2[t]}(2, \mu(h_\alpha)) \to \mathbf{U}(\mathfrak{s}_\alpha[t]) w_\mu \subset D(2, \mu).$$

The proposition is now immediate from the $n = 1$ case. $\square$

4.5. Proof of Proposition 1.11

We prove Proposition 1.11. Let $\lambda = \omega_{i_1} + \cdots + \omega_{i_k} \in P^+(1)$, where $1 \leqslant i_1 < \cdots < i_k \leqslant n$ and $\nu \in P^+$. For $\alpha \in R^+$, write $(2\nu + \lambda)(h_\alpha) = 2(s_\alpha - 1) + m_\alpha$ with $s_\alpha \geqslant 1$ and $m_\alpha \in \{1, 2\}$. Note that

$$\nu(h_i) + \lambda(h_i) = s_i, \quad 1 \leqslant i \leqslant n, \quad \nu(h_{i_p, i_{p+1}}) + 1 = s_{i_p, i_{p+1}}, \quad 1 \leqslant p \leqslant k-1.$$

It is clear from the definition of $M(\nu, \lambda)$ (see § 1.9) and Proposition 4.1 that there exists a surjective morphism

$$M(\nu, \lambda) \to D(2, 2\nu + \lambda) \to 0$$

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of $\mathfrak{g}[t]$-modules. The proposition follows if we prove that the preceding map is an isomorphism. By Corollary 4.4, it suffices to show that

$$(x^-_{i_p,i_{p+1}} \otimes t^{s_{i_p,i_{p+1}}})v_{\nu,\lambda} = 0, \quad 1 \leqslant p \leqslant k-1 \implies (x^-_{i,j} \otimes t^{s_{i,j}})v_{\nu,\lambda} = 0, 1 \leqslant i \leqslant j \leqslant n. \tag{4.4}$$

We proceed by induction on $j-i$, with induction beginning at $i=j$ by hypothesis (see (1.6)). Assume that we have proved the result for all $\alpha_i + \cdots + \alpha_j$ with $j-i<s$, and consider the case when $j=i+s$. Suppose first that there does not exist $1 \leqslant p \leqslant k$ such that $i \leqslant i_p \leqslant i+s$. In this case, we have $\mu(h_{i,j}) = 2\nu(h_{i,j}) = 2\nu(h_i) + 2\nu(h_{i+1,j})$. The induction hypothesis implies that

$$(x^-_i \otimes t^{\nu(h_i)})v_{\nu,\lambda} = 0 = (x^-_{i+1,j} \otimes t^{\nu(h_{i+1,j})})v_{\nu,\lambda}.$$

Since $[x^-_{\alpha_i}, x^-_{\alpha_{i+1,j}}] = x^-_{\alpha_{i,j}}$, we get (4.4) in this case. We consider the other case, when we can choose $1 \leqslant p \leqslant k$ minimal and $1 \leqslant r \leqslant k$ maximal so that $i \leqslant i_p \leqslant i_r \leqslant j$. If $i < i_p$ we have again by the inductive hypothesis that

$$(x^-_i \otimes t^{\nu(h_i)})v_{\nu,\lambda} = 0 = (x^-_{i+1,j} \otimes t^{(\nu+\lambda)(h_{i+1,j})})v_{\nu,\lambda},$$

and the inductive step is completed as before. If $j > i_r$, then the proof is similar: we write $\alpha_{i,j} = \alpha_{i,j-1} + \alpha_j$. Finally, suppose that $i = i_p$ and $j = i_r$. If $r = p+1$, then the inductive step is the hypothesis in equation (1.7). If $r \geqslant p+2$, then we write $\alpha_{i,j} = \alpha_{i_p,i_{p+1}} + \alpha_{i_{p+1}+1,i_r}$. This time, the induction hypothesis gives,

$$(x^-_{i_p,i_{p+1}} \otimes t^{\nu(h_{i_p,i_{p+1}})+1})v_{\nu,\lambda} = 0 = (x^-_{i_{p+1}+1,i_r} \otimes t^{(\nu+\lambda)(h_{i_{p+1}+1,i_r})})v_{\nu,\lambda},$$

and the inductive step is completed as before.

5. A characterization of $\mathfrak{g}$-stable level-two Demazure modules

We establish the final step of the proof of Theorem 1.

5.1.

We shall prove the following result in the rest of the section.

Proposition. *Let $\mu \in P^+$, and let V be a (not necessarily graded) $\mathfrak{g}[t]$-module which is isomorphic to $D(2,\mu)$ as a $\mathfrak{g}$-module. Assume that $\varphi : D(1,\mu) \to V \to 0$ is a surjective map of $\mathfrak{g}[t]$-modules. Then V is isomorphic to $D(2,\mu)$ as $\mathfrak{g}[t]$-modules.*

Let $\nu \in P^+$ and $\lambda = \omega_{i_1} + \cdots + \omega_{i_k} \in P^+(1)$, $1 \leqslant i_1 < \cdots < i_k \leqslant n$, be the unique elements such that $\mu = 2\nu + \lambda$. Setting $\varphi(w_\mu) = v_\mu \in V$, we see that v_μ satisfies the relations in (1.5). Since $\dim V = \dim D(2,\mu)$ by hypothesis, it follows from Proposition 1.11 that it suffices to prove that the element v_μ satisfies (1.6) and (1.7).

5.2.

Lemma. *For $1 \leqslant i \leqslant n$, with $(\nu+\mu)(h_i) > 0$, we have $\dim V_{\mu-\alpha_i} \leqslant (\nu+\lambda)(h_i)$.*

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Proof. Since

$$V \cong_{\mathfrak{g}} D(2,\mu) \implies V_{\mu-\alpha_i} \cong D(2,\mu)_{\mu-\alpha_i},$$

it suffices to prove that $\dim D(2,\mu)_{\mu-\alpha_i} \leqslant (\nu+\lambda)(h_i)$. A simple application of the Poincare–Birkhoff–Witt (PBW) theorem, along with the fact that $(x_i^- \otimes t^{(\nu+\lambda)(h_i)})w_\mu = 0$ in $D(2,\mu)$, shows that $D(2,\mu)_{\mu-\alpha_i}$ is spanned by the elements $\{(x_i^- \otimes t^r)w_\mu : 0 \leqslant r < (\nu+\lambda)(h_i)\}$, and hence proves the lemma. □

5.3.

We now prove that v_μ satisfies equation (1.6). Proposition 4.1 implies that the element $w_\mu \in D(1,\mu)$ satisfies $(x_i^- \otimes t^{\mu(h_i)})w_\mu = 0$, $1 \leqslant i \leqslant n$, and hence we also have

$$(x_i^- \otimes t^{\mu(h_i)})v_\mu = 0, \quad 1 \leqslant i \leqslant n. \tag{5.1}$$

If $(\nu+\lambda)(h_i) \in \{0,1\}$, then $\mu(h_i) = (\nu+\lambda)(h_i)$, and there is nothing to prove.

Suppose that $\mu(h_i) \geqslant 2$, or equivalently that $\nu(h_i) > 0$. Using the second relation in (1.5), we get

$$(h_i \otimes t)(x_i^- \otimes t^s)v_\mu = -2(x_i^- \otimes t^{s+1})v_\mu,$$

and hence

$$(x_i^- \otimes t^s)v_\mu = 0 \implies (x_i^- \otimes t^{s+1})v_\mu = 0.$$

We now proceed by contradiction to prove that v_μ satisfies (1.6). Thus, if $(x_i^- \otimes t^{(\nu+\lambda)(h_i)})v_\mu \neq 0$, then the elements of the set $\{(x_i^- \otimes t^s)v_\mu : 0 \leqslant s \leqslant (\nu+\lambda)(h_i)\}$ are all non-zero, and, by Lemma 5.2, they must be linearly dependent. It follows that there exists $0 \leqslant m < (\nu+\lambda)(h_i)$ such that we have a non-trivial linear combination,

$$\sum_{s=m}^{(\nu+\lambda)(h_i)} z_s(x_i^- \otimes t^s)v_\mu = 0, \quad z_s \in \mathbb{C}, m \leqslant s \leqslant (\nu+\lambda)(h_i), \quad z_m \neq 0. \tag{5.2}$$

Since $\mu(h_i) - m - 1 \geqslant \nu(h_i) > 0$, we apply $(h_i \otimes t^{\mu(h_i)-m-1})$ to the preceding equation and use (1.5), (5.1), along with the fact that $z_m \neq 0$, to get

$$(x_i^- \otimes t^{\mu(h_i)-1})v_\mu = 0.$$

If $\mu(h_i) \in \{2,3\}$, then we have $\nu(h_i) = 1$ and $(\nu+\lambda)(h_i) = \mu(h_i) - 1$, and we have the desired contradiction. Otherwise, we have $\mu(h_i) \geqslant 4$, i.e., $\nu(h_i) \geqslant 2$, and hence we get $\mu(h_i) - m - 2 \geqslant \nu(h_i) - 1 > 0$. Hence, applying $(h_i \otimes t^{\mu(h_i)-m-2})$ to the expression in equation (5.2), we now get $(x_i^- \otimes t^{\mu(h_i)-2})v_\mu = 0$. If $\mu(h_i) \in \{4,5\}$, we would have $(\nu+\lambda)(h_i) = \mu(h_i) - 2$, which would again contradict our assumptions. Further iterations of this argument give a contradiction for all values of $\mu(h_i)$. Hence, we have $(x_i^- \otimes t^{(\nu+\lambda)(h_i)})v_\mu = 0$, as required.

5.4.

We need some additional notation to complete the proof that v_μ satisfies (1.7). Recall that $\alpha_{i,j} = \alpha_i + \cdots + \alpha_j$. Let

$$\mathbf{U}(\mathfrak{n}^-[t])_{\alpha_{i_p,i_{p+1}}} = \{u \in \mathbf{U}(\mathfrak{n}^-[t]) : [h,u] = -\alpha_{i_p,i_{p+1}}(h)u, \text{ for all } h \in \mathfrak{h}\}.$$

The elements

$$(x_{\beta_s}^- \otimes t^{\ell_s}) \cdots (x_{\beta_1}^- \otimes t^{\ell_1}), \quad s \geqslant 1, \ell_s \in \mathbb{Z}_+,$$

and $\beta_j \in R^+$, $\ell_j \in \mathbb{Z}_+$ satisfying $\beta_1 = \alpha_{i_p} + \alpha_{i_p+1} + \cdots + \alpha_m$, for some $m \geqslant i_p$, $\beta_1 + \cdots + \beta_r \in R^+$, for all $1 \leqslant r \leqslant s$, $\beta_1 + \cdots + \beta_s = \alpha_{i_p,i_{p+1}}$, are a basis for $\mathbf{U}(\mathfrak{n}^-[t])_{\alpha_{i_p,i_{p+1}}}$. Let $S \subset \mathbf{U}(\mathfrak{n}^-[t])$ be the subset of this basis with the additional restriction that $0 \leqslant \ell_j \leqslant N_{\beta_j}$, where

$$N_{\beta_j} = \begin{cases} \nu(h_{i_p,i_{p+1}}) + 1, & s = 1, \\ \nu(h_{\beta_j}), \quad j \in \{1, s\}, & s > 1, \\ \nu(h_{\beta_j}) - 1, & 1 < j < s. \end{cases} \tag{5.3}$$

Then S is a finite linearly independent subset of this space, and we let $\mathbf{S}$ be the span of S. Notice also that $\mathbf{S}$ is graded, and that

$$\mathbf{S}[p] = 0, \quad p > \nu(h_{i_p,i_{p+1}}) + 1 \quad \text{and} \quad \mathbf{S}[\nu(h_{i_p,i_{p+1}}) + 1] = \mathbb{C}(x_{i_p,i_{p+1}}^- \otimes t^{\nu(h_{i_p,i_{p+1}})+1}). \tag{5.4}$$

Moreover, if $h \in \mathfrak{h}$ and $r \in \mathbb{Z}_+$, then

$$\mathbf{m} \in \oplus_{p \in \mathbb{Z}_+} \mathbf{S}[p] \implies [h \otimes t^r, \mathbf{m}] \in \oplus_{p \in \mathbb{Z}_+} \mathbf{S}[p + r].$$

5.5.

We turn to the proof that v_μ satisfies (1.7). Denote by $\tilde{V}$ the graded quotient of $D(1, \mu)$ by the $\mathfrak{g}[t]$-submodule generated by the set

$$\{(x_i^- \otimes t^{(\nu+\lambda)(h_i)}) w_\mu : 1 \leqslant i \leqslant n\},$$

and let $\tilde{v}_\mu$ be the image of w_μ in $\tilde{V}$. The results in § 5.3 show that V is a $\mathfrak{g}[t]$-quotient of $\tilde{V}$; the definition of $D(2, \mu)$ given in Proposition 4.1 shows that $D(2, \mu)$ is a graded $\mathfrak{g}[t]$-quotient of $\tilde{V}$. Since

$$(x_i^- \otimes t^{\nu(h_i)})\tilde{v}_\mu = 0, \quad i_p < i < i_{p+1}, \quad (x_i^- \otimes t^{\nu(h_i)+1})\tilde{v}_\mu = 0, \quad i \in \{i_p, i_{p+1}\},$$

taking repeated commutators gives

$$(x_{i_p,i_{p+1}}^- \otimes t^{\nu(h_{i_p,i_{p+1}})+2})\tilde{v}_\mu = 0 \quad \text{and}$$
$$(x_{i_p,i}^- \otimes t^{\nu(h_{i_p,i})+1})\tilde{v}_\mu = (x_{j,i_{p+1}}^- \otimes t^{\nu(h_{j,i_{p+1}})+1})\tilde{v}_\mu = (x_{i,j}^- \otimes t^{\nu(h_{i,j})})\tilde{v}_\mu = 0, \tag{5.5}$$

for all $i_p < i \leqslant j < i_{p+1}$.

The following is a straightforward consequence of the relations in equation (5.5) and the PBW theorem.

Lemma. *The space $\tilde{V}_{\lambda - \alpha_{i_p,i_{p+1}}}$ is spanned by the elements $\{\mathbf{x}\tilde{v}_\mu : \mathbf{x} \in S\}$, and hence*

$$\dim \tilde{V}_{\lambda - \alpha_{i_p,i_{p+1}}} \leqslant |S|. \qquad \square$$

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5.6.

We now prove the following.

Lemma. *We have* $\dim V_{\lambda-\alpha_{i_p,i_{p+1}}} < |S|$*; i.e., there exists a non-zero element* $\mathbf{m} \in \mathbf{S}$ *such that* $\mathbf{m}v_\mu = 0$.

Proof. Assume for a contradiction that $\dim V_{\lambda-\alpha_{i_p,i_{p+1}}} = |S|$. Then we have

$$\dim \tilde{V}_{\lambda-\alpha_{i_p,i_{p+1}}} = |S| = \dim D(2,\mu)_{\lambda-\alpha_{i_p,i_{p+1}}},$$

where the first equality follows because V is a quotient of $\tilde{V}$, and the second equality follows because $V \cong D(2,\mu)$ as $\mathfrak{g}$-modules. Since $D(2,\mu)$ is a quotient of $\tilde{V}$, this means that, using Lemma 5.5, the elements $\{\mathbf{x}\tilde{v}_\mu : \mathbf{x} \in S\}$, and hence the elements $\{\mathbf{x}w_\mu : \mathbf{x} \in S\}$ are linearly independent subsets of $\tilde{V}$ and $D(2,\mu)$, respectively. But the latter is impossible, since

$$(x^-_{i_p,i_{p+1}} \otimes t^{\nu(h_{i_p,i_{p+1}})+1}) \in S,$$

and $(x^-_{i_p,i_{p+1}} \otimes t^{\nu(h_{i_p,i_{p+1}})+1})w_\mu = 0$ is a defining relation in $D(2,\mu)$. The lemma is proved. □

5.7.

Proposition. *Let* $0 \neq \mathbf{m} \in \mathbf{S}$ *be such that* $\mathbf{m}v_\mu = 0$*, and assume that the minimal graded component of* $\mathbf{m}$ *is* k*, for some* $k < \nu(h_{i_p,i_{p+1}})+1$*. There exists* $0 \neq \mathbf{m}' \in \mathbf{S}$ *with* $\mathbf{m}'v_\mu = 0$ *whose minimal graded component is at least* $k+1$*. In particular, we have*

$$\mathbf{S}[\nu(h_{i_p,i_{p+1}})+1]v_\mu = \mathbb{C}(x^-_{i_p,i_{p+1}} \otimes t^{\nu(h_{i_p,i_{p+1}})+1})v_\mu = 0,$$

and hence v_μ *satisfies (1.7).*

Proof. For $1 \leqslant i \leqslant n$, let $h_{\omega_i} \in \mathfrak{h}$ be the unique element such that $\alpha_j(h_{\omega_i}) = \delta_{i,j}$, for all $1 \leqslant j \leqslant n$. Write $\mathbf{m} = \sum_{\mathbf{x}\in S} z_{\mathbf{x}}\mathbf{x}$, $z_{\mathbf{x}} \in \mathbb{C}$, and note that $[h \otimes t^r, \mathbf{m}]v_\mu = 0$, for all $h \in \mathfrak{h}$, $r \in \mathbb{Z}_+$. Moreover, we have

$$[h_{\omega_1} \otimes t, \mathbf{x}] = (x^-_{\beta_s} \otimes t^{\ell_s}) \cdots (x^-_{\beta_1} \otimes t^{\ell_1+1}), \quad \mathbf{x} \in S.$$

Suppose that the set

$$S_1(\mathbf{m}) = \{\mathbf{x} = (x^-_{\beta_s} \otimes t^{\ell_s}) \cdots (x^-_{\beta_1} \otimes t^{\ell_1}) \in S : z_{\mathbf{x}} \neq 0, \ell_1 < N_{\beta_1}\} \neq \emptyset.$$

Then, the elements $\{[h_{\omega_1} \otimes t, \mathbf{x}] : \mathbf{x} \in S_1(\mathbf{m})\}$ are all distinct elements of S, and, by (5.5) and the definition of N_{β_1}, we have

$$\mathbf{x} \notin S_1(\mathbf{m}) \implies [h_{\omega_1} \otimes t, \mathbf{x}]v_\mu = 0.$$

Set

$$\mathbf{m}_1 = \sum_{\mathbf{x}\in S_1(\mathbf{m})} z_{\mathbf{x}}(x^-_{\beta_s} \otimes t^{\ell_s}) \cdots (x^-_{\beta_1} \otimes t^{\ell_1+1}),$$

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and note that $\mathbf{m}_1 \in \mathbf{S}$ is non-zero. Moreover,

$$[h_{\omega_1} \otimes t, \mathbf{m}]v_\mu = \mathbf{m}_1 v_\mu = 0.$$

Since the minimal grade of $\mathbf{m}_1$ is at least $k+1$, the proposition is proved when $S_1(\mathbf{m}) \neq \emptyset$.

Suppose now that $S_1(\mathbf{m}) = \emptyset$; i.e.,

$$z_{\mathbf{x}} \neq 0 \implies \mathbf{x} = (x^-_{\beta_s} \otimes t^{\ell_s}) \cdots (x^-_{\beta_2} \otimes t^{\ell_2})(x^-_{\beta_1} \otimes t^{N_{\beta_1}}).$$

If $\mathbf{m} = x^-_{i_p, i_{p+1}} \otimes t^{\nu(h_{i_p, i_{p+1}})+1}$, there is nothing to prove. Otherwise, there exists $i_p < j \leqslant i_{p+1}$ minimal so that the set

$$S_2(\mathbf{m}) = \{\mathbf{x} = (x^-_{\beta_s} \otimes t^{\ell_s}) \cdots (x^-_{\beta_2} \otimes t^{\ell_2})(x^-_{\beta_1} \otimes t^{N_{\beta_1}}) \in S : z_{\mathbf{x}} \neq 0, \beta_1 = \alpha_{i_p, j-1}\} \neq \emptyset.$$

Then,

$$[h_{\omega_j} \otimes t, \mathbf{x}] = (x^-_{\beta_s} \otimes t^{\ell_s}) \cdots (x^-_{\beta_2} \otimes t^{\ell_2+1})(x^-_{\beta_1} \otimes t^{N_{\beta_1}}), \quad \mathbf{x} \in S_2(\mathbf{m}),$$

and, using (5.5), we have

$$[h_{\omega_j} \otimes t, \mathbf{x}]v_\mu = 0, \quad \mathbf{x} \notin S_2(\mathbf{m}), \quad z_{\mathbf{x}} \neq 0.$$

Let

$$\mathbf{m}_2 = \sum_{\mathbf{x} \in S_2(\mathbf{m})} z_{\mathbf{x}} (x^-_{\beta_s} \otimes t^{\ell_s}) \cdots (x^-_{\beta_2} \otimes t^{\ell_2+1})(x^-_{i_p, j-1} \otimes t^{N_{\alpha_{i_p, j-1}}}).$$

The preceding discussion proves that

$$[h_{\omega_j} \otimes t, \mathbf{m}]v_\mu = \mathbf{m}_2 v_\mu = 0,$$

and that $\mathbf{m}_2$ has minimal grade at least $k+1$. However, it need not be true that $\mathbf{m}_2 \in S$: an instance is if $\mathbf{x} \in S_2(\mathbf{m})$ is such that $\ell_2 = N_{\beta_2}$. To address this issue, we define a further subset $S_2'(\mathbf{m})$ of $S_2(\mathbf{m})$ consisting of elements $\mathbf{x}$ with $\ell_2 + 1 \leqslant N_{\beta_2}$. Let $\mathbf{m}' \in \mathbf{S}$ be defined by

$$\mathbf{m}' = \left(\sum_{\mathbf{x} \in S_2'(\mathbf{m})} z_{\mathbf{x}} (x^-_{\beta_s} \otimes t^{\ell_s}) \cdots (x^-_{\beta_2} \otimes t^{\ell_2+1}) \right) (x^-_{i_p, j-1} \otimes t^{N_{\alpha_{i_p, j-1}}}).$$

Setting $S_2''(\mathbf{m}) = S_2(\mathbf{m}) \setminus S_2(\mathbf{m}')$, we note that

$$(\mathbf{m}_2 - \mathbf{m}')v_\mu = \left(\sum_{\mathbf{x} \in S_2''(\mathbf{m})} z_{\mathbf{x}} (x^-_{\beta_s} \otimes t^{\ell_s}) \cdots (x^-_{\beta_3} \otimes t^{\ell_3})(x^-_{\beta_2+\beta_1} \otimes t^{N_{\beta_1+\beta_2}}) \right) v_\mu.$$

The expression in parentheses on the right-hand side is an element of $\mathbf{S}$, and we denote it by $\mathbf{m}''$. Moreover, the elements $\mathbf{m}'$ and $\mathbf{m}''$ are clearly linearly independent elements of $\mathbf{S}$ assuming that at least one of them is non-zero. To see that this is in fact the case, assume that $\mathbf{m}' = 0$; i.e., $S_2'(\mathbf{m}) = \emptyset$ and $S_2''(\mathbf{m}) = S_2(\mathbf{m})$. This means that the elements

$$(x^-_{\beta_s} \otimes t^{\ell_s}) \cdots (x^-_{\beta_3} \otimes t^{\ell_3})(x^-_{\beta_2+\beta_1} \otimes t^{N_{\beta_1+\beta_2}}), \quad \mathbf{x} \in S_2(\mathbf{m}),$$

are all distinct, and thus linearly independent (recall that $\beta_1 = \alpha_{i_p, j-1}$), and, hence, $\mathbf{m}'' \neq 0$. Therefore

$$0 \neq \tilde{\mathbf{m}} := \mathbf{m}' + \mathbf{m}'' \in \mathbf{S}, \quad \tilde{\mathbf{m}} v_\mu = 0,$$

and the minimal graded component of $\tilde{\mathbf{m}}$ is at least $k+1$. The proposition is proved. □

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