Proceedings of the Edinburgh Mathematical Society (2003) **46**, 637–648 ©
DOI:10.1017/S0013091502000883 Printed in the United Kingdom

DUALITY AND CONTACT OF HYPERSURFACES IN $\mathbb{R}^4$ WITH HYPERPLANES AND LINES

A. C. NABARRO

Departamento de Matemática, Universidade Estadual de Maringá, Av. Colombo 5790, Campus Universitário, CEP 87020-900, Maringá, PR, Brazil (acnabarro@uem.br)

(Received 5 September 2002)

Abstract Given an embedded hypersurface M in $\mathbb{R}^4$ we consider families of projections H of M to lines and families of projections P of M to 3-spaces. We characterize generically the singularities of these projections. We also show that there is a duality relation between some strata of the bifurcation sets of H and P, and deduce geometric properties about these sets.

Keywords: singularities; generic geometry; duality

2000 *Mathematics subject classification:* Primary 58K05
Secondary 58K40

1. Introduction

The contact of smooth varieties with degenerate objects (lines, planes, hyperplanes, circles, spheres, etc.) is measured locally by the $\mathcal{K}$-classes of the singularities of some map germs [**11**]. In practice (right–left) $\mathcal{A}$ equivalence classes are sought as they yield a finer classification and more geometric information. This approach allowed the discovery of beautiful geometric results on surfaces in $\mathbb{R}^3$ that are being extended to surfaces and hypersurfaces in $\mathbb{R}^4$ (see [**3**] and [**12**] for references).

In this paper we study the local flat geometry of a generic hypersurface M embedded in $\mathbb{R}^4$ using singularity theory. In [**13**] we considered the contact of M with two-dimensional planes. Here we deal with the contact with lines and hyperplanes, so we study the $\mathcal{A}$-classes of the singularities of projections to lines and to 3-spaces. These projections are parametrized by the 3-sphere $S^3 \subset \mathbb{R}^4$. It follows from the transversality theorem of Looijenga [**8**] that we can expect, in general, (local) singularities of $\mathcal{A}_e$-codimension less than or equal to 3 to occur. (In the presence of moduli, the generic cases are when the union of the orbits along the moduli parameters form a set of $\mathcal{A}_e$-codimension less than or equal to 3.)

In § 2 we deal with the contact of M with hyperplanes. We obtain necessary and sufficient conditions for the height function in a normal direction to have a generic singularity and for these to be versally unfolded by the family of height functions. We adapt the techniques of Bruce in [**1**] to describe the sets in M with a given singularity type of the height function. Finally, we give the geometric characterization of the singularities of the height function.

In § 3 we study the contact of M with lines and give a geometric characterization of the generic singularities of the projection to 3-spaces.

We prove in § 4 a duality result between the families H and P, analogous to that in [**4**], and study the behaviour of the projections P around a flat partial umbilic point.

2. Contact with hyperplanes

Given an embedded hypersurface M in $\mathbb{R}^4$, the family of height functions (projections to lines) is given by $H : M \times S^3 \to \mathbb{R}$, where $H(p,u) = \langle p, u\rangle$ and S^3 is the unit sphere in $\mathbb{R}^4$. For a fixed $u \in S^3$, the height function H_u measures the contact of the surface M with the hyperplane normal to u.

We write M in Monge form in a neighbourhood of a point p, that is, we consider M given locally by the graph of a function $w = f(x,y,z)$ near the origin, with $w = 0$ as the tangent hyperplane at the origin. If we parametrize locally the sphere S^3 by $(a,b,c,1)$ near the normal to the surface at the origin, we obtain the following expression for the family of height functions

$$H(x,y,z,a,b,c) = ax + by + cz + f(x,y,z).$$

In particular, $H_0(x,y,z) = f(x,y,z)$.

In the rest of the paper we will write the Taylor expansion of f of order 5 at the origin as follows (we choose the principal directions as the coordinate axes):

$$\begin{aligned} j^5 f = {} & a_1x^2 + a_2y^2 + a_3z^2 + b_1x^3 + b_2x^2y + b_3xy^2 + b_4y^3 + b_5y^2z + b_6yz^2 \\ & + b_7z^3 + b_8z^2x + b_9zx^2 + b_{10}xyz + c_1x^4 + c_2x^3y + c_3x^2y^2 + c_4xy^3 + c_5y^4 \\ & + c_6y^3z + c_7y^2z^2 + c_8yz^3 + c_9z^4 + c_{10}z^3x + c_{11}z^2x^2 + c_{12}zx^3 + c_{13}x^2yz \\ & + c_{14}xy^2z + c_{15}xyz^2 + d_1x^5 + d_2x^4y + d_3x^3y^2 + d_4x^2y^3 + d_5xy^4 + d_6y^5 \\ & + d_7y^4z + d_8y^3z^2 + d_9y^2z^3 + d_{10}z^4y + d_{11}z^5 + d_{12}z^4x + d_{13}x^2z^3 + d_{14}x^3z^2 \\ & + d_{15}x^4z + d_{16}x^3yz + d_{17}x^2y^2z + d_{18}x^2yz^2 + d_{19}xy^2z^2 + d_{20}xy^3z + d_{21}xyz^3. \end{aligned}$$

We recall some concepts on the geometry of the hypersurfaces in $\mathbb{R}^n$. Let M be an embedded manifold of dimension n in $\mathbb{R}^{n+1}$ with a unit normal vector field N, and consider $p \in M$. The Weingarten function $L_p : T_pM \to T_pM$, given by $L_p(v) = -d_pN(v)$, measures how M is curved in $\mathbb{R}^{n+1}$ in the direction v. When $\|v\| = 1$, the number $k(v) = \langle L_p(v), v\rangle$ is called the *normal curvature* of M at p in the direction v. The eigenvalues $k_1(p), \dots, k_n(p)$ of L_p are called the *principal curvatures* of M at p and the unit eigenvectors of L_p are the *principal directions*.

The *second fundamental form* of M at p, $\mathcal{L}_p$, is the quadratic form associated with the function L_p, defined by

$$\mathcal{L}_p(v) = \langle L_p(v), v\rangle = \langle \ddot{\alpha}(t_0), N(p)\rangle,$$

where $\alpha : I \to M$ is any parametrized curve in M with $\alpha(t_0) = p$ and $\dot{\alpha}(t_0) = v$. In particular, when $\|v\| = 1$, $\mathcal{L}_p(v)$ is equal to the normal curvature of M at p in the direction v.

A point p in M is an *umbilic point* if all principal curvatures at p are equal. If they are all 0, then p is a *flat umbilic point*. If at least two principal curvatures at p are 0, then p is called a *partial flat umbilic point*.

Consider the particular case where M is a hypersurface in $\mathbb{R}^4$ and let $p \in M$. As before, we take M in Monge form

$$w = f(x, y, z) \quad \text{with } j^2 f(x, y, z) = a_1 x^2 + a_2 y^2 + a_3 z^2,$$

where $a_i = k_i/2$ (the k_i being the principal curvatures). Then p is a *parabolic point* if $k_i(p) = 0$ for some i. Away from the parabolic set, if the three principal curvatures at p have the same sign, then p is an *elliptic point*, otherwise it is a *hyperbolic point*.

A direction $u \in T_pM$ is an *asymptotic direction* at p if the normal curvature along u is 0. Since $j^2 f(x, y, z) = a_1 x^2 + a_2 y^2 + a3 z^2$ the vector $(u_1, u_2, u_3, 0)$ of the tangent space is an asymptotic direction if and only if $a_1 u_1^2 + a_2 u_2^2 + a_3 u_3^2 = 0$. At a hyperbolic point the set of asymptotic directions form a cone. At an elliptic point there is no asymptotic direction. At a parabolic point when only one a_i is 0 (we can always rearrange variables and assume $a_3 = 0$) the asymptotic directions are given by two planes whose intersection is the u_i-axis or consists of the u_i-axis. In both cases we call the direction of the u_i-axis the *principal asymptotic direction*.

As pointed out in the introduction, the generic local singularities of $H_0 = f(x, y, z)$ are those of $\mathcal{A}_e$-codimension less than or equal to 3. So, locally at the origin, we expect H_0 to have a singularity of type A_k, $1 \leqslant k \leqslant 4$, or D_4 (using Arnold's notation). We identify these singularities in the following result.

Proposition 2.1. *The height function H_0 has one of the following singularities.*

(i) *Away from partial flat umbilic points:*

$$A_1 \Leftrightarrow a_1 a_2 a_3 \neq 0;$$

$$A_2 \Leftrightarrow a_1 a_2 \neq 0,\ a_3 = 0,\ b_7 \neq 0;$$

$$A_3 \Leftrightarrow a_1 a_2 \neq 0,\ a_3 = 0,\ b_7 = 0,\ 4a_1 a_2 c_9 - b_8^2 a_2 - b_6^2 a_1 \neq 0;$$

$$A_4 \Leftrightarrow a_1 a_2 \neq 0,\ a_3 = 0,\ b_7 = 0,\ 4a_1 a_2 c_9 - b_8^2 a_2 - b_6^2 a_1 = 0,$$
$$4a_1 a_2 d_{11} - 4a_2 c_{10} b_8 - 2a_1 c_8 b_6 + b_6 b_8 b_{10} \neq 0.$$

(ii) *At a partial flat umbilic point:*

$$D_4 \Leftrightarrow a_1 \neq 0,\ a_2 = a_3 = 0,\ b_7 \neq 0,\ b_5 \neq 0,$$
$$4b_6^3 b_4 + 27 b_4^2 b_7^2 - 18 b_4 b_7 b_5 b_6 - b_5^2 b_6^2 + 4 b_5^3 b_7 \neq 0.$$

https://doi.org/10.1017/S0013091502000883 Published online by Cambridge University Press

These singularities are versally unfolding by the family of height functions if and only if $A_{\leqslant 2}$*: always,*

$$A_3 \Leftrightarrow b_6 \neq 0 \text{ or } b_8 \neq 0,$$
$$A_4 \Leftrightarrow \varphi(a_i, b_i, c_i, d_i) \neq 0, \text{ where } \varphi \text{ is a polynomial of degree } 13 \text{ in } a_i, b_i, c_i, d_i \text{ (see [12])},$$
$$D_4 \Leftrightarrow 6b_3b_5b_7 - 2b_3b_6^2 - 9b_4b_{10}b_7 + 6b_4b_8b_6 + b_5b_{10}b_6 - 2b_8b_5^2 \neq 0.$$

Proof. The proof follows by relatively straightforward calculations. □

We now describe the sets in M where the height function has one of the singularity types in Proposition 2.1. We follow Bruce's method in [**1**].

Let p be a point on M and choose three smooth orthogonal tangent vector fields in a neighbourhood U of p, so that together with the normal vector field, they form a system of coordinates at each point in U. The surface can then be given locally at $q \in U$ in Monge form $(x, y, z, f_q(x, y, z))$. We denote f_p by f.

Let V_k denote the set of polynomials in x, y, z of degree greater than or equal to 2 and less than or equal to k. We obtain a smooth map, the *Monge–Taylor map* $\theta : U \to V_k$, where $\theta(p) = j^k f_p$, which associates with each point q in U the k-jet of the functions f_q at the point q. The set V_k has a natural $GL(3, \mathbb{R})$-action given by a linear change of coordinates. It is shown in [**5**] that the flat geometry of smooth manifolds in a Euclidean space is affine invariant. A subset Z (say, representing one of the singularity types in Proposition 2.1) of V_k which is of any geometric significance will be $GL(3, \mathbb{R})$-invariant. Moreover, if Z is furnished with a Whitney regular stratification, then for any generic M the map germ $M, p \to V_k$ will be transverse to the strata of Z (see [**1**] for details). We then determine the diffeomorphism type of $\theta^{-1}(Z)$ at p.

To carry out the calculations explicitly in V_k we need to compute the tangent space to the $GL(3, \mathbb{R})$-orbit of f and the generators of the image of $\mathrm{d}\theta$.

Lemma 2.2. *The generators of the tangent space to the* $GL(3, \mathbb{R})$*-orbit of* f *in* V_k*, at* f*, are*

$$u_1 = xf_x, \qquad u_2 = yf_x, \qquad u_3 = zf_x, \qquad u_4 = yf_y, \qquad u_5 = xf_y,$$
$$u_6 = zf_y, \qquad u_7 = xf_z, \qquad u_8 = yf_z, \qquad u_9 = zf_z.$$

Proof. To obtain the generators, we calculate

$$\frac{\mathrm{d}}{\mathrm{d}t} f(A_t(x, y, z))|_{t=0},$$

where A_t is a path in $GL(3, \mathbb{R})$ with A_0 being the identity. □

Proposition 2 in [**1**] can be extended to cover the case of hypersurfaces in $\mathbb{R}^4$ and give the generators of the image of $\mathrm{d}\theta$ (see [**12**] for the proof).

Proposition 2.3. *The image of $\mathrm{d}\theta(p)$ is generated by v_1, v_2, v_3 such that*

$$\begin{aligned}
v_1 &= j^k\left(\mathrm{d}\theta\left(\frac{\partial}{\partial x}\right)\right)\\
&= j^k(f_x(x,y,z) - f_x(x,y,z)f_{xx}(0,0,0)f(x,y,z)\\
&\qquad - f_y(x,y,z)f(x,y,z)f_{xy}(0,0,0) - f_z(x,y,z)f(x,y,z)f_{xz}(0,0,0)\\
&\qquad\qquad - f_{xx}(0,0,0)x - f_{xy}(0,0,0)y - f_{xz}(0,0,0)z),\\
v_2 &= j^k\left(\mathrm{d}\theta\left(\frac{\partial}{\partial y}\right)\right)\\
&= j^k(f_y(x,y,z) - f_x(x,y,z)f_{xy}(0,0,0)f(x,y,z)\\
&\qquad - f_y(x,y,z)f(x,y,z)f_{yy}(0,0,0) - f_z(x,y,z)f(x,y,z)f_{yz}(0,0,0)\\
&\qquad\qquad - f_{xy}(0,0,0)x - f_{yy}(0,0,0)y - f_{yz}(0,0,0)z),\\
v_3 &= j^k\left(\mathrm{d}\theta\left(\frac{\partial}{\partial z}\right)\right)\\
&= j^k(f_z(x,y,z) - f_x(x,y,z)f_{xz}(0,0,0)f(x,y,z)\\
&\qquad - f_y(x,y,z)f(x,y,z)f_{yz}(0,0,0) - f_z(x,y,z)f(x,y,z)f_{zz}(0,0,0)\\
&\qquad\qquad - f_{xz}(0,0,0)x - f_{yz}(0,0,0)y - f_{zz}(0,0,0)z).
\end{aligned}$$

Proposition 2.4.

(1) *The parabolic set (i.e. the set of points in M where the height function along the normal has an A_2-singularity) is locally a smooth two-dimensional surface.*

(2) *The A_3-singularities of the height function H occur generically on a smooth curve on the parabolic set, labelled the A_3-curve.*

(3) *The A_4-singularities of H occur generically at isolated points on the A_3-curve.*

Proof. (1) Suppose that f has an A_2-singularity at the origin and write, without loss of generality, $j^2f = a_1x^2 + a_2y^2$ with $a_1a_2 \neq 0$. Then a transversal to this orbit is given by the elements $j^2f + \bar{a}_3z^2$, where $\bar{a}_3 \in \mathbb{R}$. Therefore, on this transversal the A_2-stratum is given by $\bar{a}_3 = 0$ (see Proposition 2.1). So it is a smooth manifold of codimension 1 in V_2. It is not hard to show that the tangent space of the A_2-stratum in V_2 is given by the kernel of the differential form $\xi = \mathrm{d}a_3$. On the other hand, the generators v_1, v_2, v_3 of the tangent space to the image of θ in V_2 are given by Proposition 2.3. The image of θ fails to be transverse to the A_2-stratum if and only if v_1, v_2, v_3 belong to the kernel of ξ. Since $\xi(v_3) = b_7 \neq 0$ at an A_2-singularity, we conclude that the image of θ is always transversal to the A_2-stratum. Hence the parabolic set, $\theta^{-1}(A_2)$, is a smooth manifold of codimension 1 in M.

For (2) and (3) we proceed as above. The only difference here is that the tangent space to the A_3 (respectively, A_4) stratum is given by the intersection of the kernels of 2 (respectively, 3) 1-differential forms. □

Remark 2.5.

(i) The fold/cusp/swallowtail singularities of the Gauss map $M \to S^3$ correspond, respectively, to $A_2/A_3/A_4$ singularities of the height function. However the Monge–Taylor approach yields more information. For instance, it follows from the proof of Proposition 2.4 that the parabolic set is singular if and only if the Monge–Taylor map fails to be transverse to the A_2-stratum in V_2, if and only if the height function has an A_3-singularity not versally unfolded by the family of height functions.

(ii) Following the same labelling as for a surface in $\mathbb{R}^3$, we call the A_3-curve on M the *cuspidal edge curve* and the A_4-points *swallowtail points*.

Proposition 2.6. *The D_4-singularities of H occur generically at isolated points on M. The parabolic set is locally a cone at these singularities. At a D_4^+ the A_3-set is a single line through the cone and at a D_4^- it consists of three lines through this cone.*

Proof. Without loss of generality we set $a_2 = a_3 = 0$ at a D_4 singularity. Then an element of a transversal to the orbit of f in V_2 is given by $a_1x^2 + \bar{a}_2y^2 + \bar{a}_3z^2 + \bar{a}_6yz$, where $\bar{a}_2, \bar{a}_3, \bar{a}_6 \in \mathbb{R}$ and the D_4-stratum on this transversal is given by $\bar{a}_2 = 0$, $\bar{a}_3 = 0$ and $\bar{a}_6 = 0$. We show that the tangent space of the D_4-stratum in V_2 is given by the intersection of the kernel of the differential forms $\xi_1 = \mathrm{d}a_3$, $\xi_2 = \mathrm{d}a_2$ and $\xi_3 = \mathrm{d}a_6$. The generators of the tangent space to the image of θ in V_2 are v_1, v_2 and v_3 with $\bar{a}_2 = \bar{a}_3 = \bar{a}_6 = 0$, given by Proposition 2.3. Since $\ker\xi_1 \cap \ker\xi_2 \cap \ker\xi_3$ has codimension 3, the image of θ fails to be transverse to the D_4-stratum if and only if there is a non-zero vector $v = \lambda v_1 + \mu v_2 + \beta v_3$ such that $\xi_1(v) = 0$, $\xi_2(v) = 0$ and $\xi_3(v) = 0$. It means that the family of height functions fails to versally unfold the D_4-singularity at the origin (see Proposition 2.1). However, for generic embedding of M, the D_4-set consists generically of isolated points on M.

The A_2-stratum is given by $\bar{a}_6^2 - 4\bar{a}_2\bar{a}_3 = 0$ in the above transversal, which is a cone in V_2. Since θ is generically transverse to the D_4-stratum, $\theta(\mathbb{R}^3) \cap A_2$ is a cone in V_2, and therefore $\theta^{-1}(A_2)$ is also a cone in M.

We shall now study the A_3-set on this cone. We need to work in V_3. At a D_4 singularity we can write $j^3f = x^2 + C(x, y, z)$, where the cubic $C(x, y, z) = z^3 \pm y^2z + xa(x, y, z)$. A transversal to the orbit of f in V_3, after a change of coordinates, can be written as

$$j^3\bar{f} = X^2 + \bar{a}_2y^2 + \bar{a}_3z^2 + \bar{a}_6yz + z^3 \pm y^2z.$$

The singularity is A_3 if and only if $\bar{a}_2y^2 + \bar{a}_3z^2 + \bar{a}_6yz = L^2$ (that is, $L = \alpha y + \beta z$) and $L \mid (z^3 \pm y^2z)$. Therefore, $j^3\bar{f} = X^2 + L^2 + LW$, where $W = (z^3 \pm y^2z)/L$. With the change of variable $L = l - \frac{1}{2}W$, we have $j^3\bar{f} = X^2 + l^2$. Therefore we have the following.

(i) If $L \mid (z^3 + y^2z)$, then $L \mid z$ or $L \mid (z^2 + y^2)$. The last case does not hold because L is of degree 1, hence $L = \lambda z$. Now $L^2 = \bar{a}_2y^2 + \bar{a}_3z^2 + \bar{a}_6yz$; therefore $\bar{a}_2 = \bar{a}_6 = 0$. This is a manifold of codimension 2 in V_3, which intersects the cone $\bar{a}_6^2 - 4\bar{a}_2\bar{a}_3 = 0$ in a line. So on M we have a smooth A_3-curve on the parabolic set.

https://doi.org/10.1017/S0013091502000883 Published online by Cambridge University Press

(ii) If $L \mid (z^3 - y^2z)$, then $L \mid z \Rightarrow L^2 = \lambda^2 z^2$, $L \mid (z-y) \Rightarrow L^2 = \lambda^2(z-y)^2$, or $L \mid (z+y) \Rightarrow L^2 = \lambda^2(z+y)^2$. Since $L^2 = \bar{a}_2y^2 + \bar{a}_3z^2 + \bar{a}_6yz$, in the first case we have $\bar{a}_2 = \bar{a}_6 = 0$, in the second case we have $\bar{a}_3 - \bar{a}_2 = 0$ and $\bar{a}_6 + 2\bar{a}_2 = 0$, and in the third case we have $\bar{a}_6 - 2\bar{a}_2 = 0$ and $\bar{a}_3 - \bar{a}_2 = 0$. That is, we have three manifolds of codimension 2 (which intersect the cone $\bar{a}_6^2 - 4\bar{a}_2\bar{a}_3 = 0$ in V_2 in a line). Then the A_3-set on M consists of three smooth curves on the parabolic set passing through the cone point.

□

We give geometric characterizations of the generic singularities of the height function below.

Proposition 2.7. *The generic singularities of the height function in the normal direction u at $p \in M$ occur when*

(A_1) *p is not a parabolic point;*

(A_2) *p is a parabolic point, the principal asymptotic direction is transverse to the parabolic set;*

(A_3) *p is a parabolic point, the principal asymptotic direction is tangent to the parabolic set but transverse to the A_3-curve;*

(A_4) *p is a parabolic point, the principal asymptotic direction is tangent to the A_3-curve;*

(D_4) *the parabolic set is a cone—at a D_4^+, the A_3-set is a line on the cone and at a D_4^-, it consists of three lines on this cone.*

Proof. We write M in Monge form $w = f(x,y,z)$. We choose the principal directions as a coordinate system in the tangent plane so $j^2f = a_1x^2 + a_2y^2 + a_3z^2$. Note that $a_i = \kappa_i/2$ $(i = 1,\dots,3)$, where κ_i is the principal curvature along the corresponding principal direction. It follows that A_1 singularities occur at non-parabolic points and $A_{\geqslant 2}$ and D_4 singularities occur at parabolic points. The parabolic set is also the set of points where the determinant of the Hessian matrix of f vanishes. Assuming that $a_3 = 0$ and $a_1a_2 \neq 0$, the 1-jet of this equation (after scaling) is given by $b_8x + b_6y + 3b_7z$. The principal asymptotic direction is along $(0,0,1)$. Therefore, it is transverse to the parabolic set if and only if $b_7 \neq 0$, that is, if and only if the singularity is A_2 but not A_3. At an A_3-singularity this direction is tangent to the parabolic set.

A calculation shows that the tangent direction to the A_3-curve is along the vector (u_1,u_2,u_3) with

$$u_1 = -b_6(b_8^2a_2 + b_6^2a_1 - 4a_1a_2c_9),$$
$$u_2 = b_8(b_8^2a_2 + b_6^2a_1 - 4a_1a_2c_9)$$

and

$$u_3 = b_6(b_8b_9a_2 + \tfrac{1}{2}b_6a_1b_{10} - a_1a_2c_{10} - a_1b_5b_8) + b_8(a_1a_2c_8 - \tfrac{1}{2}a_2b_8b_{10}).$$

https://doi.org/10.1017/S0013091502000883 Published online by Cambridge University Press

Table 1. *Singularities of $\mathcal{A}_e$-codimension less than or equal to 3*

name	normal form	$\mathcal{A}_e$-codimension
II	(x, y, z^2)	0
3^*	$(x, y, z^3 + h(x, y)z)$	$\mu(h)$
4_1^k	$(x, y, z^4 + xz \pm y^k z^2), k \geqslant 1$	$k-1$
4_2^k	$(x, y, z^4 + (y^2 \pm x^k)z + xz^2), k \geqslant 2$	k
5_1	$(x, y, z^5 + xz + yz^2)$	1
5_2	$(x, y, z^5 + xz + y^2z^2 + yz^3)$	2
5_3	$(x, y, z^5 + xz + yz^3)$	3
5_4	$(x, y, z^5 + yz + x^2z^2 \pm z^6 + az^7)$	3
6_1	$(x, y, z^6 + yz + xz^2 \pm z^8 + az^9)$	2
6_2	$(x, y, z^6 + yz + xz^2 + z^9)$	3

$^*h$ is one of $\pm x^2 \pm y^{k+1}(A_k)$, $x^2y \pm y^{k-1}(D_k)$, $x^3 \pm y^4(E_6)$, $x^3 + xy^3(E_7)$ or $x^3 + y^5(E_8)$; μ denotes the Milnor number.

From the proof of Proposition 2.4 (2) the parabolic set is smooth at an A_3-singularity ($b_7 = 0$) if and only if $b_6 \neq 0$ or $b_8 \neq 0$. Then the principal asymptotic direction $(0, 0, 1)$ is tangent to the A_3-curve if and only if $b_8^2a_2 + b_6^2a_1 - 4a_1a_2c_9 = 0$ (i.e. f has an A_4-singularity).

The characterization of the D_4-singularity follows from Proposition 2.6. □

3. Contact with lines

The family of projections to 3-spaces is given by $P : M \times S^3 \to B$ with $P(p, u) = (u, p - \langle u, p\rangle u)$, where $B = \{(u, y) \in S^3 \times \mathbb{R}^4; \langle u, y\rangle = 0\}$, that is, B is the tangent bundle of S^3. We take M in Monge form $w = f(x, y, z)$ and assume that we are projecting along $u = (a, b, 1, c)$, where (a, b, c) is close to the origin. Then it is not hard to show that the above family of projections can locally be written in the following form: $P_u(x, y, z) = (x - az, y - bz, f(x, y, z) - cz)$. In particular, $P_0(x, y, z) = (x, y, f(x, y, z))$, which is a corank 1 germ. As pointed out in the introduction, the $\mathcal{A}$-classes of the singularities of P_0 that can occur generically are those of $\mathcal{A}_e$-codimension less than or equal to 3.

Theorem 3.1 (see [6] and [9]). *The $\mathcal{A}$-classes of singularities of map germs $\mathbb{R}^3, 0 \to \mathbb{R}^3, 0$ of corank 1 and of $\mathcal{A}_e$-codimension less than or equal to 3 are given in Table 1.*

We now seek to identify geometrically the generic singularities of the projection. The critical set of P_u is denoted by Σ and its image, the discriminant of P_u, by Δ. The curve on Σ whose image is the self-intersection curve of $\Delta(P_u)$ is called the *double point curve*. The curve on Σ that is mapped to the singular set of $\Delta(P_u)$ is called the *cuspidal edge curve*. We say that a line through a point p along a direction u has a k-point contact with M at p if the projection of M along the direction u has an A_k-singularity.

Theorem 3.2. *The generic singularities of the projection P_u at $p \in M$ occur when the following conditions hold.*

(II) *$u \in T_pM$ but is not an asymptotic direction. In particular, only singularities of this type occur at elliptic points.*

(3_{A_0}) *p is a hyperbolic or a parabolic point; u has 2-point contact with M at p; u is an asymptotic direction in the hyperbolic case and not a principal asymptotic direction in the parabolic case; $\Sigma(P)$ is a smooth surface.*

(3_{A_k}) *($1 \leqslant k \leqslant 3$) p is a parabolic point; u has 2-point contact with M at p; u is a principal asymptotic direction and is transverse to the parabolic set; $\Sigma(P)$ is a surface with an A_k-singularity at p.*

(4_1^k) *($1 \leqslant k \leqslant 4$) p is a hyperbolic point; u has 3-point contact with M at p; u is an asymptotic direction; $\Sigma(P)$ is a smooth surface; the cuspidal-edge curve has an A_{k-1}-singularity.*

($4_2^2, 4_2^3$) *p is a parabolic point; u has 3-point contact with M at p; u is a principal asymptotic direction and is tangent to the parabolic set; $\Sigma(P)$ is a surface with an A_1-singularity at p.*

(5_1) *p is a hyperbolic point; u has 4-point contact with M at p; u is an asymptotic direction; $\Sigma(P)$ is a smooth surface; the double point curve has an A_5-singularity; the cuspidal edge curve is smooth.*

(5_k) *($2 \leqslant k \leqslant 4$) p is a hyperbolic point; u has 4-point contact with M at p; u is an asymptotic direction; $\Sigma(P)$ is a smooth surface; the cuspidal edge curve has an A_1 ($k = 2, 3$) or an A_2 ($k = 4$) singularity at p.*

($6_1, 6_2$) *p is a hyperbolic point; u has 5-point contact with M at p; u is an asymptotic direction; $\Sigma(P)$ is a smooth surface; the cuspidal edge curve is smooth at p.*

The proof is straightforward once we write M in Monge form with p the origin and $P_0 = (x, y, f(x, y, z))$. We observe that the singularities 4_2^2 and 4_2^3 (respectively, 5_2 and 5_3) can be distinguished by some algebraic invariants [**7**, **9**].

4. The duality result

A duality relation between the family of height functions and projections of surfaces in $\mathbb{R}^3$ has been established in [**4**] and extended later in [**2**] and [**5**]. In this section we prove a similar result for hypersurfaces in $\mathbb{R}^4$.

We have, as in [**4**], the following duality for surfaces in S^3. Given a smooth surface N in S^3 and a point $a \in N$ there is an unique equatorial 2-sphere in S^3 tangent to N at a, and a corresponding pair of poles a^*. As a varies in N the poles trace out a surface $\check{N}$, which is the dual of N. In S^3 we have two copies of the dual of N, but if we consider N in $\mathbb{P}^3$, we have only one copy of the dual surface.

It follows by Thom's Transversality Theorem that, for most points $u \in S^3$, the height function H_u and the projection P_u are stable. The *bifurcation set* of each family is given by $\mathrm{Bif}(F) = \{u \in S^3 : F_u \text{ is not stable}\}$, where F is the family H or P.

https://doi.org/10.1017/S0013091502000883 Published online by Cambridge University Press

The bifurcation set of the family of height functions H consists of the directions $u \in S^3$ where the function H_u has a local singularity of type A_2 (or worse), or a multi-local singularity of type $2A_1$ (or worse). So the bifurcation set of H is given by two strata where one is given by normal directions at parabolic points and the other by normal directions to bi-tangent hyperplanes. We denote by $\mathrm{Bif}(H, A_2)$ the regular part of the local stratum of $\mathrm{Bif}(H)$ and by $\mathrm{Bif}(H, 2A_1)$ the regular part of the multi-local strata.

The bifurcation set of the family of projections P consists of the directions $u \in S^3$ where the projection P_u has a local singularity of type 3_{A_1} or worse (i.e. 3_{A_2}, 3_{A_3}, 4_2^2 or 4_2^3), of type 4_1^2 or worse (i.e. 4_1^3 or 4_1^4), or has a multi-local singularity of type 2II (or worse). We denote by $\mathrm{Bif}(P, 3_{A_1})$, $\mathrm{Bif}(P, 4_1^2)$ and $\mathrm{Bif}(P, 2\mathrm{II})$ the regular parts of the respective strata in $\mathrm{Bif}(P)$. We observe that the subset $\mathrm{Bif}(H, 4_1^2)$ does not correspond to parabolic points, therefore it bears no relation to the strata of $\mathrm{Bif}(H)$.

We establish below the duality relation between the regular strata of $\mathrm{Bif}(H)$ and $\mathrm{Bif}(P)$. (The relation between the local strata is also proved in [**10**] using a different approach.)

Theorem 4.1. *Let M be a generic embedded hypersurface in $\mathbb{R}^4$. Then*

$$\mathrm{Bif}(\check{H}, A_2) = \mathrm{Bif}(P, 3_{A_1}) \quad \textit{and} \quad \mathrm{Bif}(\check{H}, 2A_1) = \mathrm{Bif}(P, 2\mathrm{II}).$$

Similarly,

$$\mathrm{Bif}(H, A_2) = \mathrm{Bif}(\check{P}, 3_{A_1}) \quad \textit{and} \quad \mathrm{Bif}(H, 2A_1) = \mathrm{Bif}(\check{P}, 2\mathrm{II}).$$

Proof. We work as before with M given in Monge form at the origin. If H_0 has a singularity more degenerate than A_1, then the origin is a parabolic point. The degenerated hyperplane at the origin is then given by $w = 0$ with $(0, 0, 0, 1)$ the unit normal. We write, without loss of generality, $j^2 f = a_2 y^2 + a_3 z^2$, with $a_2 a_3 \neq 0$.

We need to identify the dual direction of $u_0 = (0, 0, 0, 1)$, that is, we need to find the tangent plane to the A_2-stratum in S^3. We can parametrize normals near u_0 by $u = (R, S, T, 1)$, The surface M has a singular contact with the hyperplane normal to u if $R + f_x = 0$, $S + f_y = 0$ and $T + f_z = 0$. Furthermore, this contact is degenerated when the determinant of the Hessian of H vanishes, and the 1-jet of this equation is given by $8a_1a_2b_8x + 8a_1a_2b_6y + 24a_1a_2b_7z$. Since $b_7 \neq 0$ (we have a genuine A_2-singularity, Proposition 2.1), we can write z as a function of (x, y), and thus R, S, T are also functions of x, y: $R = -2a_1x + O_2(x, y)$, $S = -2a_2y + O_2(x, y)$ and $T = O_3(2)$. Hence the stratum A_2 is a smooth surface ($a_1a_2 \neq 0$). The tangent hyperplane to this stratum is given by $T = 0$, therefore the dual direction of $u_0 = (0, 0, 0, 1)$ is $u_0^* = (0, 0, 1, 0)$.

We now consider the projection in the dual direction u_0^* that belongs to the tangent hyperplane, so $P_{u_0^*}$ is singular. We have the 3-jet of $P_{u_0^*}(x, y, z)$ equivalent to $(x, y, b_5y^2z + b_6yz^2 + b_7z^3 + b_8xz^2 + b_9x^2z + b_{10}xyz)$. As $b_7 \neq 0$, we change coordinates and set $P_{u_0^*} = (X, Y, (AX^2 + BXY + CY^2)Z + b_7Z^3)$. This is equivalent to $(X, Y, (\pm X^2 \pm Y^2)Z + Z^3)$ if and only if $B^2 - 4AC \neq 0$, that is $3b_{10}^2b_7 - 4b_{10}b_6b_8 + 4b_6^2b_9 - 12b_5b_7b_9 + 4b_5b_8^2 \neq 0$, and this means that $P_{u_0^*}$ has a singularity of the type 3_{A_1}.

We now consider the set $\mathrm{Bif}(P, 2\mathrm{II})$ (multi-local case). Let $u(x, y)$ be a parametrization of this stratum. Let $p_1 = \alpha(x, y)$ and $p_2 = \beta(x, y)$ denote the points in M where the height function has an A_1-singularity at the same level. Projecting M locally at p_1 and p_2 in a direction $u(x,y) \in T_{p_1}M = T_{p_2}M$, which is along $p_1 - p_2$, yields tangential surfaces in $\mathbb{R}^3$. Since $\alpha(x,y) - \beta(x,y)$ is a multiple of $u(x,y)$ we have $\langle \alpha(x,y) - \beta(x,y), N(\alpha(x,y))\rangle = 0$, where $N(\alpha(x,y))$ is the normal vector to M at $\alpha(x,y)$. We also have $N(\alpha(x,y)) = \pm N(\beta(x,y))$. The tangent space to the stratum $\mathrm{Bif}(P, 2\mathrm{II})$ is generated by $\alpha - \beta$, $\alpha_x - \beta_x$ and $\alpha_y - \beta_y$. These vectors generate the same space as $u(x,y)$, $u_x(x,y)$ and $u_y(x,y)$, as $\alpha(x,y) - \beta(x,y) = \lambda(x,y)u(x,y)$ with $\lambda(x,y) \neq 0$. Now α_x, β_x, α_y and β_y belong to $T_{p_i}M$, $i = 1,2$, so

$$\begin{aligned} \langle \alpha(x,y) - \beta(x,y), N(\alpha(x,y))\rangle &= 0, \\ \langle \alpha_x(x,y) - \beta_x(x,y), N(\alpha(x,y))\rangle &= 0, \\ \langle \alpha_y(x,y) - \beta_y(x,y), N(\alpha(x,y))\rangle &= 0. \end{aligned}$$

Therefore, the normal vector $N(\alpha(x,y))$ is dual to $u(x,y)$, so $\mathrm{Bif}(\check{P}, 2\mathrm{II}) = \mathrm{Bif}(H, 2A_1)$.

The other two equalities in the theorem follow in the same way as above. □

We have the following consequences of the main duality result.

Proposition 4.2. *Let $u \in \mathrm{Bif}(H, A_2)$ and let u^* be its dual direction that determines a projection $P : M, p \to \mathbb{R}^3$. Suppose that the parabolic set is smooth. Then H_u has*

(i) *an A_2-singularity if and only if P_{u^*} has a singularity 3_{A_k}, $k \geqslant 1$;*

(ii) *an A_k-singularity ($k \geqslant 3$) if and only if P_{u^*} has a singularity 4_2^2 at p.*

We now study the projection in the dual direction at a partial flat umbilic point p, that is, where H_u has a singularity D_4. Then $j^3 f(x,y,z) = a_1 x^2 + C(x,y)$ with $a_1 \neq 0$ and

$$C(x,y,z) = b_1x^3 + b_2x^2y + b_3xy^2 + b_8xz^2 + b_9x^2z + b_{10}xyz + y^3 \pm yz^2.$$

Hence $(0,\beta,\gamma,0)$ in the tangent hyperplane to M are asymptotic directions and they are parametrized by the unit circle $\mathcal{C}$ in the (y,z)-plane. If $\gamma \neq 0$, the 3-jet of the projection P along $(0,\beta,1,0)$ is $\mathcal{A}$-equivalent to

$$\begin{aligned} (X, Y, (\beta^3 \pm \beta)Z^3 &+ (b_9 + b_2\beta)X^2Z + (b_{10} + 2b_3\beta)XYZ \\ &+ (b_8 + b_{10}\beta + b_3\beta^2)XZ^2 + (3\beta)Y^2Z + (3\beta^2 \pm 1)YZ^2). \end{aligned}$$

In the case where $\beta^3 \pm \beta = C(0,\beta,1) \neq 0$, we have j^3P $\mathcal{A}$-equivalent to $(X, Y, (\pm X^2 \pm Y^2)Z + Z^3)$ (a 3_{A_1}-singularity) if and only if $\lambda_0 + \lambda_1\beta + \lambda_2\beta^2 + \lambda_3\beta^3 = 0$. If $\gamma = 0$, then projecting along $(0,1,0,0)$ yields generically a singularity of type 3_{A_1}. Therefore, we have the following result.

Proposition 4.3. *Let p be a partial flat umbilic point on M. There is a circle of directions in T_pM where the projection has generically a singularity of type 3_{A_1}. There are three (at a D_4^-) or one (at a D_4^+) directions on this circle where the singularity is of type 3_{A_2} and three or one directions where it is of the type 4_2^2.*

Acknowledgements. This work is part of my PhD thesis and I thank FAPESP for financial support. I am also very grateful to J. W. Bruce for his useful comments and especially to Farid Tari for his stimulating supervision.

References

1. J. W. Bruce, Projections and reflections of generic surfaces in $\mathbb{R}^3$, *Math. Scand.* **54** (1984), 262–278.
2. J. W. Bruce, Generic geometry, transversality and projections, *J. Lond. Math. Soc.* **49** (1994), 183–194.
3. J. W. Bruce, Generic geometry and duality, in *Singularities, Lille, 1991*, London Mathematical Society Lecture Note Series (ed. J. P. Brasselet), vol. 201, pp. 29–60 (Cambridge University Press, 1994).
4. J. W. Bruce and M. C. Romero-Fuster, Duality and projections of curves and surfaces in 3-space, *Q. J. Math.* **42** (1991), 433–441.
5. J. W. Bruce, P. J. Giblin and F. Tari, Families of surfaces: height functions and projections to planes, *Math. Scand.* **82** (1998), 165–185.
6. W. Hawes, Multi-dimensional motions of the plane and space, PhD thesis, University of Liverpool (1994).
7. K. Houston, Calculating generalised image and discriminant Milnor numbers in low dimensions, *Glasgow Math. J.* **43** (2001), 165–175.
8. E. J. N. Looijenga, Structural stability of smooth families of C^∞-functions, PhD thesis, University of Amsterdam (1974).
9. W. L. Marar and F. Tari, On the geometry of simple germs of co-rank 1 maps from $\mathbb{R}^3$ to $\mathbb{R}^3$, *Math. Proc. Camb. Phil. Soc.* **119** (1996), 469–481.
10. D. K. H. Mochida, Geometria genérica de subvariedades em codimensão maior que um em $\mathbb{R}^n$, PhD thesis, University of São Paulo, ICMC–São Carlos (1993).
11. J. A. Montaldi, On contact between submanifolds, *Michigan Math. J.* **33** (1986), 195–199.
12. A. C. Nabarro, Sobre a geometria local de hipersuperfícies em $\mathbb{R}^4$, PhD thesis, University of São Paulo, ICMC–São Carlos (2000) (http://gauss.dma.uem.br/~nabarro).
13. A. C. Nabarro, Projections of hypersurfaces in $\mathbb{R}^4$ to planes, in *Proc. 6th Workshop on Real and Complex Singularities, 17–21 July 2000, ICMC–USP São Carlos, Brazil* (ed. D. Mond and M.Saia), Lecture Notes in Pure and Applied Mathematics, vol. 232, pp. 283–300 (Marcel Dekker, New York, 2003).