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JOHN ALISON, Esq., Ex-President, in the Chair

Note on a Third Mode of Section of the Straight Line.

By W. WALLACE, M.A.

The rational treatment of Geometry has this important disadvantage, that for want of suitable demonstrations it seems impossible to preserve the natural grouping of the facts developed. The study of Rational Geometry, in fact, should always be supplemented by a systematic attempt to array the facts demonstrated according to their subject-matter; for it will hardly be denied that a direct and systematic knowledge of the Properties of Geometrical Figures has an intrinsic value apart from the knowledge of their demonstrations. In pursuing such a retrospective scheme as this in connection with the Second Book of Euclid, I have found that a very comprehensive view of the subject-matter is obtained by adding a Third Mode of Section of a straight line to the two which are already recognised. This third mode of section, for which I have not been able to find a more suitable name than "Circuitous Section," along with the other two known as Internal and External Section respectively, exhausts the possible modes of section of a line—for three-dimensional space at any rate. From this point of view, the elementary treatment of the subject may be arranged as follows. It will be observed that several important properties of triangles and polygons acquire a new interpretation as cases of circuitous section.

1. A straight line is divided into two parts—

   internally, when the point of section lies on the line, between its extremities;

   externally, when the point of section lies on the line, beyond its extremities;

   circuitously, when the point of section lies outside the line.

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2. Instead of the axiom,

A line is equal to the sum of its parts;

substitute the lemma,

A line is equal to the sum of the projections of its parts upon it.

Then the successive segments are completely represented

by their respective lengths $\lambda_1\lambda_2$, etc.;

and their inclinations $\theta_1\theta_2$, etc.,

reckoned counter-clockwise from the right-hand parallels through the points of section;

and the lemma becomes expressible in the form

$$L = \Sigma(\lambda\cos\theta).$$

3. Proposition I.—The square on a two-part line

is equal to the sum of the rectangles contained by the line and the projections upon it of its two parts.

[sq. AB = rect. AB, AK$\cos\theta_1$ + rect. AB, KB$\cos\theta_2$]

(Particular cases):

$\theta_1 = 0 \quad \theta_2 = 2\pi$	(internal section)	Euc. II. 2.
$\theta_1 = 0 \quad \theta_2 = \pi$	(external section)	Euc. II. 3.

The general theorem is true for an n-part line.

[sq. AB = Σ(rect. AB, $\lambda\cos\theta$)]

4. Proposition II.—The square on a two-part line

is equal to the sum of the squares on the two segments diminished by twice the rectangle contained by either segment and the projection of the other upon it.

[sq. AB = (sq. AK + sq. KB)—2(rect. AK, KB$\cos\phi$)].

(Particular cases):

$\phi = \pi$	Internal section	Euc. II. 4
$\phi = 0$	External section	Euc. II. 7
$\phi = \frac{\pi}{2}$	Circuitous section	Euc. I. 47
$\phi > \frac{\pi}{2}$		Euc. II. 12
$\phi < \frac{\pi}{2}$		Euc. II. 13

5. Proposition III.—The sum of the squares on the segments of a two-part line

is double the sum of the squares on half the line and on the line between the middle and the point of section.

[sq. AK + sq. KB = 2(sq. AM + sq. MK)].

(Particular cases):

Internal section	Euc. II. 9
External section	Euc. II. 10
X Circuitous section	Apollonius' (?) Theorem.

6. PROPOSITION IV.—The rectangle contained by either segment of a two-part line and the projection of the other upon it

is equal to the square on the line between the middle and the point of section, diminished by the square on half the line.

[rect. AK, KB$\cos\phi$ = (sq. MK – sq. AM)].

(Particular cases):

$\phi = \pi$	Internal section	Euc. II. 5
$\phi = 0$	External section	Euc. II. 6
ϕ bet. 0 and π	Circuitous section.	

NOTE.—By combining Propositions III. and IV. a verification of Proposition I. is obtained.

7. PROPOSITION V.—The locus of points which divide a straight line in a given ratio is a circle.

8. PROPOSITION VI.—The joins of the corresponding points of section of two similarly divided parallel straight lines are concurrent.

The point of concurrence is the "centre of similarity" of the two lines.

(Particular cases):

Internal and external section	Employed in the well known method of dividing a line proportionally to a given divided line.
X Circuitous section	Well known theorem of centre of similarity of similar polygons.

Three Parabolas connected with a Plane Triangle.

By R. Tucker, M.A.

1. The parabolas considered in the present Note are obtained in the following manner :

From any point in one side of a triangle perpendiculars are let fall on the other two sides. The join of the feet of these perpendiculars envelopes a parabola.

Let AD, BE, CF be the perpendiculars from the angles on the opposite sides and let DK, DL be the perpendiculars on AC, AB. Then the envelope of KL is the parabola P_a.

It is evident that BE, CF are particular positions of KL, as also are AC, AB: hence P_a is also the envelope of the analogue of KL for the triangle BHC.

Now since the circumcircles of AEB, AFC intersect in D, D is the focus of P_a; and as E, F are the orthocentres of the above triangles, EF is the directrix of the curve.

2. Draw the perpendicular DX to EF cutting KL in V, then since KL bisects DX, V is the vertex of P_a.

Now $$DV = DK\cos B = b\sin C\cos C\cos B,$$

hence the Latus-Rectum $(L_a) = 2b\sin 2C\cos B,$

$$= 2R\sin 2B\sin 2C.$$

If L_b, L_c are the corresponding lines for P_b, P_c, we have

$$a'.\,L_a = 8\Delta\cos A\cos B\cos C = b'.\,L_b = c'.\,L_c,$$

and $$L_a\,.\,L_b\,.\,L_c = 16\Delta'^2/R',$$

where a', Δ', R' refer to the Pedal triangle DEF.

3. If we draw Db, Dc' $\perp$ to DF, DE respectively to meet AB, AC produced in b, c', then these are the points where P_a touches AB, AC.

Now $Ac' = b\sin A\sin C/\cos B$, $Ab = c\sin A\sin B/\cos C$,

$$\therefore \qquad \frac{Ac'}{Ab} = \frac{\cos C}{\cos B} = \frac{DK}{DL},$$

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and $\quad Bb = c\cos A\cos B/\cos C, \quad Cc' = b\cos C\cos A/\cos B,$

$$Bb \,.\, Cc' = bc\cos^2 A\,;$$

also $\quad Db \,.\, Dc' = DA^2, \quad \angle DbA = 90^\circ - C, \quad \angle Dc'A = 90^\circ - B.$

4. If the productions of $c'D$, bD, meet HB, HC in b_1, c_2 then these are the points where P_a touches HB, HC.

5. The diameter through A, being parallel to DX, passes through the circumcentre of ABC.

Since $\quad LK = AL\sin A/\sin B = 2R\sin A\sin B\sin C,$

$\therefore$ the intercepts on the vertical tangents of P_a, P_b, P_c by pairs of sides of ABC are equal, *i.e.*, they are equally distant from the centre of the "Taylor" circle of ABC.

6. Since $\quad \angle KLA = C = \angle DFL,$

$\therefore$ the portions intercepted on KL between the sides of the Pedal triangle and the sides of ABC are equal half of the adjacent sides of DEF.

7. If c_1, a_2; a_1, b_2; are points obtained as in § 4, we have the relations

$$Aa_1 = Ha_2 = b\cos C\cos A/\sin A,$$
$$Aa_2 = Ha_1 = c\cos A\cos B/\sin A,$$

hence $\quad Ha_1 \,.\, Hb_1 \,.\, Hc_1 = Ha_2 \,.\, Hb_2 \,.\, Hc_2 = 8R^3\cos^2 A\cos^2 B\cos^2 C$

$$= HD \,.\, HE \,.\, HF\,.$$

Also the triangles DEF, $a_1b_1c_1$, $a_2b_2c_2$, are in Perspective with ABC, having the orthocentre for Perspective centre.

Plainly the Nine-point Circle bisects a_1a_2, b_1b_2, c_1c_2.

If δ be the area of $a_1b_1c_1$, or $a_2b_2c_2$, then

$$8\Delta \,.\, \delta = \cos A\cos B\cos C(a^4 + b^4 + c^4).$$

8. Let x_1, x_1' be the projections of b_1 on HF, HD, and x_2, x_2' be the projections of b_2 on the same lines, then

$$x_1 + x_1' = a\cos B\cos C,$$
$$x_2 + x_2' = c\cos A\cos B,$$

hence their sum $\quad = b\cos B = DF.$

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9. If Cb_2, Bc_1, meet in X and HX cut BC in R, then

$$Hb_2 \,.\, Hc_2 \,.\, BR = Hb_1 \,.\, Hc_1 \,.\, CR,$$

whence $BR = CD$ and HR is the isotomic of HD and X lies on AO, where O is the circumcentre.

The lines Cb_1, Bc_2 intersect on HD.

Numerous other properties of these points can be readily obtained, but I have not succeeded in getting simple equations to the circles $a_1b_1c_1$, $a_2b_2c_2$, or to the parabolas through $b_1b_2c_1c_2$, etc.

10. Parallels through D to AC, AB, being perpendicular to BE, CF, meet those lines where KL meets them, *i.e.*, in y_1, y_2 suppose.

The triangles Hy_1y_2, HFE are inversely similar. Again, the triangle Dy_1y_2 is similar to AFE and to ABC and is in Perspective with them.

Since $y_1y_2 = a\cos B\cos C$, the modulus of similarity of Dy_1y_2, with regard to ABC, is $\cos B\cos C$, and

$$y_1y_2 + EF = KL.$$

11. Let y_1', y_1'' ; y_2', y_2'' be the analogous points for the angles B, C, then

$$\begin{aligned} Hy_1 &= 2R\cos B\cos^2 C, & Hy_2 &= 2R\cos^2 B\cos C,\\ Hy_1' &= 2R\cos C\cos^2 A, & Hy_2' &= 2R\cos^2 C\cos A,\\ Hy_1'' &= 2R\cos A\cos^2 B, & Hy_2'' &= 2R\cos^2 A\cos B.\end{aligned}$$

Hence $\qquad \Pi \,.\, Hy_1 = 8R^3\cos^3 A\cos^3 B\cos^3 C = \Pi \,.\, Hy_2.$

Again
$$\begin{aligned} Hy_1 \,.\, Hy_2'' &= 4R^2\cos^2 A\cos^2 B\cos^2 C\\ &= \lambda^2(\text{suppose}) = Hy_1' \,.\, Hy_2\,;\end{aligned}$$

hence H is the radical centre of the three circles which pass through y_1, y_2'', y_1', y_2, etc.; and their orthotomic circle is the incircle of the Pedal triangle.

12. Taking the lines HB, HC as axes, the equation to the circle $y_1y_2''y_1'y_2$ is

$$x^2 + y^2 - 2xy\cos A - 2gx - 2fy + \lambda^2 = 0,$$

and to the parabolas through them is

$$(x \pm y)^2 - 2gx - 2fg + \lambda^2 = 0,$$

where $$g \equiv R\cos B(\cos^2 C + \cos^2 A),$$

and $$f \equiv R\cos C(\cos^2 A + \cos^2 B)\,;$$

hence axes of the parabolas are known as to direction.

If ρ_a, ρ_b, ρ_c are the radii of the circles, then, since

$$y_2 y_2'' = 2\rho_a \cos B,$$

$$\rho_a^2 = R^2[\sin^2 A(\cos_2 B - \sin^2 C) + \sin^2 B \sin^2 C],$$

and $$2\Sigma(\rho^2) = 3 - [\Sigma\sin^2 A\sin^2 B + \Sigma\cos^2 A\cos^2 B].$$

13. If we suppose P to be the point in BC from which perpendiculars Pq, Pr are drawn to AC, AB, and if we take BP $= ma$, CP $= na$, $m + n \equiv 1$, then the equation of qr, in trilinear co-ordinates is

$$-(mb + nc\cos A)(mb\cos A + nc)\alpha + mb\cos B(mb + nc\cos A)\beta$$
$$+ nc\cos C(mb\cos A + nc)\gamma = 0,$$

i.e., $$am^2\cos B\cos C\,.\,\Sigma(a\alpha)$$
$$+ cm[a\alpha(\cos A\cos B - \cos C) + b\beta\cos A\cos B - \gamma\cos C(c + a\cos B)]$$
$$+ c^2(\gamma\cos C - \alpha\cos A) = 0,$$

the envelope of which is

$$[a\alpha(\cos A\cos B - \cos C) + b\beta\cos A\cos B - \gamma\cos C(c + a\cos B)]^2$$
$$= 4a\cos B\cos C\Sigma(a\alpha)\,.\,(\gamma\cos C - \alpha\cos A),$$

i.e., a parabola.

This equation can be put into the two forms

$$(\alpha\sin^2 A + \beta\cos A\cos B + \gamma\cos C\cos A)^2 = 4\beta\gamma\cos B\cos C, \qquad \text{(A)}$$

$$[-\alpha(1 + \cos^2 A) + \beta\cos A\cos B + \gamma\cos C\cos A]^2$$
$$= 4(\alpha\cos A - \beta\cos B)(\alpha\cos A - \gamma\cos C). \qquad \text{(B)}$$

Hence we see that the equation to bc' (§ 3) is

$$\alpha\sin^2 A + \beta\cos A\cos B + \gamma\cos C\cos A = 0, \qquad \text{(C)}$$

and to $b_1 c_2$ (§ 4) is

$$-\alpha(1 + \cos^2 A) + \beta\cos A\cos B + \gamma\cos C\cos A = 0. \qquad \text{(D)}$$

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From (A) we see that the curve cuts BC in

$$\frac{\beta\cos B}{\gamma\cos C} = \tan^2\left(\frac{\pi}{4} \pm \frac{A}{2}\right).$$

If we take P′ the isotomic of P on BC, *i.e.*, write

m for n and n for m in the equation to qr.

we see that the two lines intersect on

$$a\alpha(b\cos B - c\cos C) + b^2\beta\cos B - c^2\cos C\gamma = 0,$$

which is the diameter through H, ($\perp$ to EF).

14. If V′, V″ are the vertices of P_b, P_c, then AV, BV′, CV″ meet in

$$\alpha/a^2\sec A = \beta/b^2\sec B = \gamma/c^2\sec C.$$

15. The equation to the conic, through b, c', and the four analogous points, is

$$\cos A\cos B\cos C\Sigma(\alpha^2\sin^2 A) + \Sigma[\beta\gamma\cos A(\cos^2 B\cos^2 C + \sin^2 B\sin^2 C)] = 0.$$

16. If the lines BK, CL; etc.; intersect in t_1, t_2, t_3, then At_1, Bt_2, Ct_3 cointersect in the point

$$\alpha\operatorname{cosec}A\cos^2 A = \beta\operatorname{cosec}B\cos^2 B = \gamma\operatorname{cosec}C\cos^2 C.$$

17. The equation to bc' (C), on the supposition of $\angle$ A constant, and writing θ for B, is

$$\gamma\cos A\sin A\sin\theta + (\beta - \gamma\cos A)\cos A\cos\theta + \alpha\sin^2 A = 0,$$

hence the envelope of bc' is

$$\cos^2 A(\beta^2 + \gamma^2 - 2\beta\gamma\cos A) = \alpha^2\sin^4 A\ ;$$

which is a hyperbola, if A is acute,

and an ellipse, if A is obtuse.

18. The equation to the circle Abc is

$$\Sigma(\alpha\beta\gamma) + (\Sigma a\alpha)\,.\,\cot A(\gamma\tan B\cos C + \beta\tan C\cos B) = 0.$$

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19. From (C) and (D) we see that bc' and b_1c_2 cut BC where the harmonic conjugate to AD with regard to AB_1AC_1 cuts it, as they should do by the geometry of the figure.

20. If we take a point P″ on BC such that BP″. BP = BC², *i.e.*, in the equation in § 13 write $1/m$ for m, we get for the equation to the tangent

$$c^2(\gamma\cos C - a\cos A)m^2 + cm[\ldots] + a\cos B\cos C\Sigma(a\alpha) = 0\ ;$$

hence the locus of the intersection of the two lines is

$$a\cos B\cos C\Sigma(a\alpha) = c^2(\gamma\cos C - a\cos A),$$

a straight line parallel to BE and cutting BC in a point D′ given by $BD' = a^2\cos B/c$.

21. The polars of B and C are

$$\begin{aligned} &\alpha\sin^2 A\cos A + \beta\cos^2 A\cos B - \gamma(1+\cos^2 A) &&= 0,\\ &\alpha\sin^2 A\cos A - \beta\cos B(1+\cos^2 A) + \gamma\cos^2 A\cos C &&= 0\ ; \end{aligned}$$

hence they intersect on AD; in fact, in the point where AD cuts EF.

The polar of H is

$$-\alpha(1+\cos^2 A) + (\beta+\gamma)\cos A = 0.$$