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Kirillov Theory for a Class of Discrete Nilpotent Groups

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Abstract. This paper is concerned with the Kirillov map for a class of torsion-free nilpotent groups G. G is assumed to be discrete, countable and π-radicable, with π containing the primes less than or equal to the nilpotence class of G. In addition, it is assumed that all of the characters of G have idempotent absolute value. Such groups are shown to be plentiful.

1 Introduction

Kirillov [11] obtained an explicit and calculable 1–1 correspondence between the irreducible unitary representations (up to equivalent) of nilpotent Lie groups G and the so-called integral orbits of the co-adjoint action of G on the dual $\mathcal{G}^\wedge$ of its Lie algebra $\mathcal{G}$. This method has been extended to various levels of completeness in many different directions (see [4],[9],[15]). The direction that concerns us here is exemplified by the paper [4] of Carey, Moran and Pearce. Here the group G in question is a divisible (in the sense of [4]) nilpotent discrete group, and the dual objects the primitive ideal space of Prim G of G. That paper shows a Kirillov correspondence between the integral quasi-orbits of the Pontryagin dual of the Lie algebra $\mathcal{G}$ (as defined in Baumslag [2]) of G and the primitive ideal space. Moreover this correspondence is shown to be topological. Interestingly the proof of the correspondence given there has its topological nature as an essential part of the inductive proof, whereas the topological nature of the original Kirillov correspondence was demonstrated only several years after Kirillov's original work (see [11]).

Here we extend the methodology of [4] to handle a wider class of groups. A group G is said to be π-*radicable* for a set of primes π if, for every $n \in \pi$ and every $a \in G$, the equation $x^n = a$ has a solution. We show that, if G is π-radicable where π contains all primes not exceeding the nilpotence class of G, then there is a Kirillov correspondence in the sense of [4] and it is topological.

The methodology consists to some extent of extending the techniques of [4] to this case. The key result of that paper (Theorem 4.2), however, makes full use of the fact that the Lie algebra is a $\mathbb{Q}$-vector space. Since this property is not available in our context we have to find other techniques to overcome the problem. We assume that all groups involved are discrete and countable.

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2 Notation and Terminology

If $a_1, a_2, \dots, a_n$ are elements of a group, $\langle a_1, a_2, \dots, a_n\rangle$ is used to denote the subgroup generated by the a_i's. Let G be a group. We write $H \triangleleft G$ if H is a normal subgroup of G, and $[a, b]$ to denote $a^{-1}b^{-1}ab$, the commutator of a and b. We write 1_G for the identy element of G, $Z(G)$ for the centre of G, and for some $B \triangleleft G$, we let $Z(G/B) := \{x \in G : [x, G] \subseteq B\}$, where $[x, G] := \{[x, g] : g \in G\}$. We denote the primitive ideal space of $C^*(G)$ by Prim G. For a function ψ on G, we write $\psi|_A$ to indicate the restriction of ψ to a subset $A \subseteq G$. If G is abelian, we write $\hat{G}$ for the Pontryagin dual of G, that is the compact group of all complex homomorphisms from G to the circle group $\mathbb{T}$.

We recall that a group G is *nilpotent* if its upper central series defined by $Z^1(G) := Z(G)$, and $Z^{k+1}(G) := \{x \in G : [x, G] \subseteq Z^k(G)\}$ for $k \in \{1, 2, \dots\}$, terminates after a finite number of steps in G. The *nilpotence class* of G is the smallest n such that $Z^n(G) = G$.

A group G is called *torsion-free* if it has no elements of finite order. If G is torsion-free and nilpotent, $\widetilde{G}$ indicates its (Mal'cev) completion (see [2] or [14] for the definition). Let π be a set of primes. An integer is called a π-*number* if its prime divisors lie in π. The group G is called π-*radicable* if, for every π-number n and every $a \in G$, the equation $x^n = a$ has a solution.

Suppose that G is discrete. A positive definite function $\varphi\colon G \to \mathbb{C}$ satisfying $\varphi(g_1g_2) = \varphi(g_2g_1)$ for all $g_1, g_2 \in G$ and $\varphi(1_G) = 1$ (φ is normalized) is called a *trace* of G. The set $\mathrm{Tr}(G)$ of all traces of G equipped with pointwise topology is a compact convex set. The extreme points of this set (*i.e.*, those f which cannot be written as a convex combination of two different traces) are called *characters* of G. We denote the set of all characters of G as $\mathrm{Ch}(G)$. A character $\varphi \in G$ is said to be *faithful* if $k(\varphi) := \{x \in G : \varphi(x) = 1\} = (1_G)$.

We recall that the subgroup G_f of a group G comprises all $x \in G$ such that $x^G := \{g^{-1}xg : g \in G\}$ is finite. Evidently, G_f is normal in G. The group G is said to be *flat* if, for every $x \in G_f$, x^G is a coset of some subgroup of G, or equivalently, $[x, G]$ is a subgroup for all $x \in G_f$. The group is termed a group with *absolutely idempotent characters* (AIC for short) if for every character φ of G, we have $|\varphi|^2 \equiv |\varphi|$, that is $|\varphi(g)| = 1$ or 0 for each $g \in G$ (Bagget, Kaniuth and Moran [1, p. 182]). This is equivalent to say that for every character φ of G, $\varphi \equiv 0$ off $Z(G/k(\varphi))$. It is straightforward to show that every nilpotent group of class 2 is AIC, and it is well known that every radicable nilpotent group is AIC ([3, Theorem 4.2]).

Let H be a normal subgroup of a group G. Suppose that G acts on $\mathrm{Tr}(H)$ by conjugation, that is if $g \in G$ and $\varphi \in \mathrm{Tr}(H)$, $g\cdot\varphi := \varphi^g$ where $\varphi^g(h) := \varphi(g^{-1}hg)$, $h \in H$. A trace $\varphi \in \mathrm{Tr}(H)$ is called G-*invariant* if $\varphi^g = \varphi$ for all $g \in G$. It is well known that, if φ is an G-invariant trace of H, then ψ on G with $\psi|_H = \varphi$ and ψ is 0 off H, is a trace of G (see [17]), and therefore it is easy to show that φ extends to some character of G. We write $\mathrm{Tr}_G(H)$ for the compact convex set (in the pointwise topology) of all such traces, and write $\mathrm{Ch}_G(H)$ for that of their extreme points.

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3 The Lie Algebra of G and the Space of G-Quasi-Orbits on Its Dual

Let G be a torsion-free nilpotent group of class at most c. Suppose that G is π-radicable, where π is a set of primes containing all primes less than or equal to c. We shall assume the properties of G for remainder of the paper. Let $\mathbb{Q}G$ be a group algebra of G over the rationals $\mathbb{Q}$. Write $\overline{\mathbb{Q}G}$ for the completion of $\mathbb{Q}G$ in the vector space topology given by the powers of the augmentation ideal as a neighbourhood base (see Jacobson [10, p. 27]). As in [4], we note that

$$\log g := -\sum_{n=1}^{\infty} \frac{1}{n}(1-g)^n \in \overline{\mathbb{Q}G}.$$

Here we now define the *Lie algebra* $\mathcal{G}$ of G as the R-module $\{\log g : g \in G\}$, where

$$R := \Big\{ \frac{m}{n} : (m,n) = 1 \text{ and } n \text{ is } a\ \pi\text{-number} \Big\}.$$

As a substructure of the Lie algebra of $\widetilde{G}$ defined in [2], that is the $\mathbb{Q}$-vector space $\{\log g : g \in \widetilde{G}\}$, $\mathcal{G}$ has nilpotence class at most c. We define an action of G on $\mathcal{G}\hat{}$, called *co-adjoint action*, by

$$(h \cdot f)(\log g) := f(\log(h^{-1}gh)).$$

Two elements, f and f' in $\mathcal{G}\hat{}$, are in the same *quasi-orbit* if $\mathrm{Cl}(G \cdot f) = \mathrm{Cl}(G \cdot f')$ (see [5]). We denote by $\mathcal{O}$ *the space of G-quasi-orbits in* $\mathcal{G}\hat{}$ with the topology induced from $\mathcal{G}\hat{}$.

4 The Construction of the Kirillov Map

Assume that G is AIC. We shall construct the so-called *Kirillov map* $\kappa : \mathcal{O} \to \mathrm{Ch}(G)$. Let $f \in \mathcal{G}\hat{}$, and consider

$$\begin{aligned} \ker f &:= \{x \in \mathcal{G} : f(x) = 1\}; \\ \mathcal{I}(f) &:= \text{ the largest ideal contained in } \ker f; \\ \mathcal{Z}(f) &:= \{x \in \mathcal{G} : [x, \mathcal{G}] \subseteq \mathcal{I}(f)\}. \end{aligned}$$

Let χ_f be defined as

$$\chi_f(g) := f(\log g),\ g \in Z(f).$$

As f is a circle-valued homomorphism on $\mathcal{G}$, by the Baker-Campbell-Hausdorff formula, it is easy to see that χ_f is a complex homomorphism on $\exp \mathcal{Z}(f)$.

We call the *length* of a repeated commutator the number of the group elements of it. For example, $[a,b]$, $[\log a, \log b]$ are of length 2, $[[a,b],c]$, $[c,[a,b]]$ and $[\log b, [\log a, \log b]]$ are of length 3, etc.

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Lemma 4.1 *If $g_i \in G$, then*

$$[\log g_n, [\log g_{n-1}, \dots, [\log g_2, \log g_1] \cdots]] = \log[g_n, [g_{n-1}, \dots, [g_2, g_1] \cdots]] + S \tag{4.1}$$

where each term of S is a commutator of higher length in $\log g_i$ and has coefficients in R.

Proof Use the Baker-Campbell-Hausdorff formula. ■

Using Lemma 4.1, it is easy to show that $k(\chi_f) := \{g \in \exp \mathcal{Z}(f) : \chi_f(g) = 1\}$ is normal in G. The first item of the following lemma contains the part (1) of Lemma 3.1 in [4].

Lemma 4.2

(i) $\mathcal{I}(f) = \mathcal{Z}(f) \cap \ker f$.
(ii) $\log k(\chi_f) = \mathcal{I}(f)$.
(iii) *If $N \triangleleft G$, then $\log N$ is an additive subgroup of $\log G$.*

Proof For part (i), we note first that, if $y \in \mathcal{Z}(f)$, then for every $x \in \mathcal{G}$, $[x, y] \in \mathcal{I}(f) \subseteq \mathcal{Z}(f) \cap \ker f$. It follows that $\mathcal{Z}(f) \cap \ker f$ is an ideal of $\mathcal{G}$, so that $\mathcal{Z}(f) \cap \ker f \subseteq \mathcal{I}(f)$, and hence (i) follows. For part (ii), we first note that $\mathcal{I}(f) \subseteq \log k(\chi_f)$. On the other hand we notice that

$$k(\chi_f) \subseteq \exp(\mathcal{Z}(f) \cap \ker f) = \exp \mathcal{I}(f).$$

Hence $\mathcal{I}(f) = \log k(\chi_f)$. For part (iii), we first note that, by the Campbell-Hausdorff formula,

$$\log a + \log b = \log(a \cdot b) - \frac{1}{2}[\log a, \log b] + S, \tag{4.2}$$

where each term in S is a commutator of higher length in $\log a$ and $\log b$ with a coefficient in R. Note that if $\frac{m}{n} \in R$, then for $g \in G$ we have $\frac{m}{n} \log g = \log g^{\frac{m}{n}}$, where $g^{\frac{m}{n}}$ is the nth-root of g^m. Then, by formula (4.2), each term in $-\frac{1}{2}[\log a, \log b] + S$ can be written in the form

$$\log[g_k, [g_{k-1}, \dots, [g_2, g_1] \cdots]]$$

where either g_1 or g_2 is in $\{a, b\}$. Since $\{a, b\} \subseteq N$ and $N \triangleleft G$,

$$[g_k, [g_{k-1}, \dots, [g_2, g_1] \cdots]] \in N.$$

Hence $\log a + \log b = \sum \log h_i$, where $h_1 = a \cdot b$ and, for $i \neq 1$, h_i is a commutator in g_i which is in N. Now apply formulae (4.2) and (4.1) in the summation to obtain a form of $\log x + T$, where $x \in N$ and every term in T is of the form $\log x_i$, with x_i is a commutator of higher length in g_i. Applying these formulae repeatedly, we see that the length will eventually be higher than the nilpotence class of G, so that we can finally obtain that $\log a + \log b = \log y$ with $y \in N$. ■

Proposition 4.3 *If $Z(\chi) := \{g \in G : [g, G] \subseteq k(\chi_f)\}$, then $Z(\chi) = \exp \mathcal{Z}(f)$.*

Proof Let $g \in \exp \mathcal{Z}(f)$, that is $[\log g, \log g'] \in \mathcal{I}(f)$ for every $g' \in G$. According to Lemma 4.1, we have

$$\log[g, g'] = [\log g, \log g'] + S$$

where each term of S can be written as a commutator

$$[\log g_k[\log_{k-1}, \ldots, [\log g_1] \cdots]]$$

where either g_1 or g_2 is equal to g. Then, by Lemma 4.2 (iii), $\log[g, g'] \in \mathcal{I}(f)$. Therefore $[g, g'] \in k(\chi_f)$ for every $g' \in G$, so that $g \in Z(\chi)$. Thus we have proved that $\exp \mathcal{Z}(f) \subseteq Z(\chi)$.

Now suppose that $g \in Z(\chi)$, that is $[g, g'] \in k(\chi_f)$ for every $g' \in G$. We note that

$$[\log g, \log g'] = \log[g, g'] + T$$

where T, again by using Lemma 4.1, can be written as a sum of terms of the form

$$\log[g_\ell, [g_{\ell-1}, \ldots, [g_2, g_1] \cdots]]$$

in which either g_1 or g_2 is equal to g. Since $[g, g'] \in k(\chi_f)$, so is

$$[g_\ell, [g_{\ell-1}, \ldots, [g_2, g_1] \cdots]].$$

By Lemma 4.2 (ii) and (iii), $[\log g, \log g'] \in \mathcal{I}(f)$ for every $g' \in G$, that is $\log g \in \mathcal{Z}(f)$, or $g \in \exp \mathcal{Z}(f)$. Hence $Z(\chi) \subseteq \exp \mathcal{Z}(f)$, and this completes the proof. ■

Observe that the normality of $k(\chi_f)$ implies the normality of $Z(\chi)$. We now define $\phi_f : G \to \mathbb{C}$ as follows:

$$\phi_f(g) := \begin{cases} \chi_f(g) & \text{if } g \in \exp \mathcal{Z}(f) \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

By Proposition 3.3 of [16] and Proposition 4.3, ϕ_f is a character of G.

Lemma 4.4 *If f and f' belong to the same G-quasi-orbit, then $\phi_f = \phi_{f'}$.*

Proof Suppose that we have such f and f'. Suppose that $g_n \cdot f \to f'$ for some sequence (g_n) in G. Let $x \in \mathcal{Z}(f)$. Then we have

$$\begin{aligned} \log(g_n^{-1} \cdot \exp x \cdot g_n) &= \log(\exp x \cdot [\exp x, g_n]) \\ &= x + \log[\exp x, g_n] + \frac{1}{2}[x, \log[\exp x, g_n]] + S \end{aligned}$$

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where each term in S is a commutator of higher length in x and $\log[g_n, \exp x]$ with a coefficient in R, so it is in $\mathcal{I}(f)$. Thus all terms, except x, are in $\mathcal{I}(f)$, so that

$$(g_n \cdot f)(x) = f(\log(g_n^{-1} \cdot \exp x \cdot g_n)) = f(x). \tag{4.4}$$

From (4.4) we first notice that if $g \in \exp \mathcal{Z}(f)$ and $\chi_f(g) = f(\log g) = 1$, then $\chi_{f'}(g) = f'(\log g) = 1$. Hence $k(\chi_f) \subseteq k(\chi_{f'})$. By symmetry, as f and f' are in the same G-quasi-orbit, $k(\chi_f) = k(\chi_{f'})$, so that by Proposition 4.3, $\mathcal{Z}(f) = \mathcal{Z}(f')$. Secondly, from (4.4), again we see that $f(x) = f'(x)$ for $x \in \mathcal{Z}(f)$. We thus conclude that $\phi_f = \phi_{f'}$. ∎

The assignment $f \mapsto \phi_f$ defines the desired Kirillov map $\kappa : \mathcal{O} \to \mathrm{Ch}(G)$.

5 The Bijectivity of the Map

The following proposition uses quite different methods from the proof of the corresponding result in [4].

Proposition 5.1 *The map κ is surjective.*

Proof Let $\phi \in \mathrm{Ch}(G)$ and $Z(\phi) := Z(G/k(\phi))$. Define f_ϕ with

$$f_\phi(\log g) := \phi(g),\ g \in Z(\phi).$$

Notice that by repetitive use of Lemma 4.1 and formula (4.2), we have for $g_1, g_2 \in Z(\phi)$, $\log g_1 + \log g_2 = \log(g_1 \cdot g_2 \cdot h)$, with $h \in k(\phi)$. Since ϕ is a homomorphism on $Z(\phi)$, it follows that f_ϕ is a homomorphism on $\log Z(\phi)$ as an additive group. Thus, we have an homomorphism extension f of f_ϕ on $\log G$ (we use property A.7 in [8]). By routine arguments using Lemma 4.1 and Lemma 4.2 (iii), it follows from the normality of $k(\phi)$ in G that $\log k(\phi)$ is an ideal. Since $\log k(\phi) \subseteq \ker f$, we have $\log k(\phi) \subseteq \mathcal{I}(f)$. Again, using the same argument, it follows that $Z(\phi) \subseteq \exp \mathcal{Z}(f)$. We shall now show that $Z(\phi) = \exp \mathcal{Z}(f)$.

Suppose that $Z(\phi) \subset \exp \mathcal{Z}(f)$. Let ϕ_f be a character obtained from f according to (4.3), so that $\exp \mathcal{Z}(f) = Z(\phi_f)$, that is $Z(G/k(\phi_f))$. Notice that

$$H := \exp \mathcal{Z}(f) \cap Z(G/Z(\phi)) \supset Z(\phi)$$

and H satisfies $[G, H] \subseteq Z(\phi) \subset H$. Observe that $\phi_1 := \phi_f|_H \in \mathrm{Ch}_G(H)$. We also have that $\phi_2 := \phi|_H \in \mathrm{Ch}_G(H)$, for if not, then there exists $\psi_1, \psi_2 \in \mathrm{Tr}_G(H)$ such that $\phi_2 = \alpha\psi_1 + (1-\alpha)\psi_2$ for some α, $0 < \alpha < 1$. If we let $\overline{\psi_1} \equiv \psi_1$ on H and $\overline{\psi_1} \equiv 0$ off H, $\overline{\psi_2} \equiv \psi_2$ on H and $\overline{\psi_2} \equiv 0$ off H, then $\overline{\psi_1}, \overline{\psi_2} \in \mathrm{Tr}(G)$, so that $\phi = \alpha\overline{\psi_1} + (1-\alpha)\overline{\psi_2}$, which contradicts the fact that ϕ is a character of G. Next, noting that $\phi_1|_{Z(\phi)} = \phi_2|_{Z(\phi)}$ and employing Lemma 2.2 of [3], we have $\gamma\phi_1 = \phi_2$ for some $\gamma \in (H/Z(\phi))\hat{}$. We have a contradiction since $\phi_2(g) = \phi(g) = 0$ and $|\phi_1(g)| = |\phi_f(g)| = 1$ whenever $g \in H \setminus Z(\phi)$. Therefore $Z(\phi) = \exp \mathcal{Z}(f)$, and this completes the proof. ∎

The proofs of the following lemmas are modified from those of Lemma 3.4 and Lemma 3.5 of [4]. Parts of the proofs follow the same details as those in [4]. We observe that Lemma 3.7 of [4], which states that quasi-orbits of G in $\mathcal{G}^\wedge$ are closed, works for G in our case here.

Lemma 5.2 *Let D be the centralizer of $Z^2(G) := Z(G/Z(G))$ in G. Then D is normal in G, G/D is abelian and D has smaller nilpotence class than G. As G is π-radicable, so is D.*

Proof To prove both the normality of D and the commutativity of G/D, we can simply use the formula

$$[[x, y], z]^{y^{-1}}[[y^{-1}, z], x]^{z^{-1}}[[z, x], y^{-1}]^{x^{-1}} = 1.$$

As is mentioned in [4], D has nilpotence class at most $c - 1$. Now, since G is π-radicable, so is $Z^2(G)$ (we use 4.12 from [18] which is given there for radicable groups, but easily extends to π-radicable groups). As is shown in [4], we conclude that D is also π-radicable. ■

Consider $\theta \in \mathcal{O}$. Then, Lemma 4.4 and 4.2 (i) imply that $\mathcal{I}(f)$ is constant for any $f \in \theta$, and we write $\mathcal{I}(f)$ as $\mathcal{I}(\theta)$. We also write

$$D(\theta) := \{g \in G : [g, g'] \in \exp \mathcal{I}(\theta), \forall g' \in Z^2(G/\exp \mathcal{I}(\theta))\}$$

where $Z^2(G/\exp \mathcal{I}(\theta)) := \{g \in G : [g, G] \subseteq Z(G/\exp \mathcal{I}(\theta))\}$.

Lemma 5.3 *Suppose that $\theta \in \mathcal{O}$ such that $\mathcal{I}(\theta) = (0)$, and*

$$r_\theta : \mathcal{G}^\wedge \to \log D(\theta)^\wedge$$

is the restriction map. If θ' is a G-quasi-orbit in $\mathcal{G}^\wedge$, then $r_\theta(\theta')$ is a G-quasi-orbit in $\log D(\theta)^\wedge$, and $\theta = r_\theta^{-1}(r_\theta(\theta))$.

Proof Let $f \in \theta'$. Then it is obvious that $r_\theta(\theta')$ is contained in the G-quasi-orbit containing $r_\theta(f)$. Suppose now that $f_1 \in D(\theta)^\wedge$ such that $\mathrm{Cl}(G \cdot r_\theta(f)) = \mathrm{Cl}(G \cdot f_1)$. Then $g_n r_\theta(f) \to f_1$ for some sequence (g_n) in G. The countability of G implies that $g_{n_k} f \to f_2$ for some subsequence (g_{n_k}) of (g_n), where $f_2 \in \theta'$ as θ' is closed by Lemma 3.7 of [4]. Therefore we have $f_1 = r_\theta(f_2) \in r_\theta(\theta')$. Hence $r_\theta(\theta')$ is a G-quasi-orbit containing $r_\theta(f)$. The proof of $\theta = r_\theta^{-1}(r_\theta(\theta))$ is the same as that of Lemma 3.5 of [4] (it is even simpler in our case since $\mathcal{I}(\theta) = (0)$). ■

Lemma 5.4 *Let D be as in Lemma 5.1 and $\varphi \in \mathrm{Ch}_G(D)$. Assume that φ is faithful. Then the function $\overline{\varphi}$ defined by*

$$\overline{\varphi}(g) := \begin{cases} \varphi(g) & \text{if } g \in D, \\ 0 & \text{otherwise}, \end{cases}$$

is a character of G.

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Proof Note that φ is a restriction of some character ψ of G. As φ is faithful and $Z(G) \subseteq D$, $k(\psi) \cap Z(G) = (1_G)$. Since G is nilpotent, it follows that ψ is faithful. Since G is AIC, it follows that ψ is 0 off $Z(G)$, so that $\psi = \overline{\varphi}$. ∎

We recall that each character ϕ of $C^*(G)$ corresponds to

$$J(\phi) := \{x \in C^*(G) : \phi(x^*x) = 0\},$$

which is a primitive ideal of $C^*(G)$. We note that, as G is AIC, the assignment $\phi \mapsto J(\phi)$ is a 1-1 correspondence (see [3]). Now let $J(\phi)_D := J(\phi) \cap C^*(D)$. We have the following lemma.

Lemma 5.5 *Let D be as before, and $\varphi_1, \varphi_2 \in \mathrm{Ch}_G(D)$. Let $\overline{\varphi_1}$ and $\overline{\varphi_2}$ be characters obtained from φ_1 and φ_2 as in Lemma 5.4. Regarding $\overline{\varphi_1}$ and $\overline{\varphi_2}$ as traces on $C^*(G)$ we obtain that, if $J(\overline{\varphi_1})_D = J(\overline{\varphi_2})_D$, then $J(\overline{\varphi_1}) = J(\overline{\varphi_2})$.*

Proof Let $a = \sum_{i=1}^n \alpha_i g_i$. Suppose that $h_1D, h_2D, \dots, h_{\ell_0}D$ are distinct cosets such that $\bigcup_{k=1}^{\ell_0} h_kD = \bigcup_{i=1}^n g_iD$. Let $D_k = h_kD \cap \{g_1, g_2, \dots, g_n\}$, $k \in \{1, 2, \cdots, \ell_0\}$. Let $\overline{\varphi_1}(a^*a) = 0$. As we have

$$\overline{\varphi_1}(a^*\,a) = \sum_{g_i^{-1}g_j \in D} \overline{\alpha_i}\,\alpha_j\,\overline{\varphi_1}(g_i^{-1}g_j) + \sum_{g_i^{-1}g_j \notin D} \overline{\alpha_i}\,\alpha_j\,\overline{\varphi_1}(g_i^{-1}g_j)$$

with the second term 0, we obtain

$$\sum_{g_i^{-1}g_j \in D} \overline{\alpha_i}\,\alpha_j\,\overline{\varphi_1}(g_i^{-1}g_j) = 0,$$

that is

$$\sum_{k=1}^{\ell_0} \sum_{g_{ik}, g_{jk} \in D_k} \overline{\alpha}_{ik}\,\alpha_{jk}\,\overline{\varphi_1}(g_{ik}^{-1}g_{jk}) = 0.$$

We notice that for every i, j and k, if $g_{ik}, g_{jk} \in D_k$, then $g_{ik}^{-1}g_{jk} = d_{ik}^{-1}d_{jk}$ for some $d_{ik}, d_{jk} \in D$. Thus we have $\sum_{k=1}^{\ell_0} \overline{\varphi_1}(b_k^*\,b_k) = 0$, where $b_k = \sum_{i=1}^{|D_k|} \alpha_{ik}\,d_{ik}$. Since $\overline{\varphi_1}(b_k^*\,b_k) \geq 0$, we have $\overline{\varphi_1}(b_k^*\,b_k) = 0$ for every $k \in \{1, 2, \dots, l_0\}$. Since $J(\overline{\varphi_1})_D = J(\overline{\varphi_2})_D$, it follows that

$$\begin{aligned}
\overline{\varphi_2}(a^*\,a) &= \sum_{g_i^{-1}g_j \in D} \overline{\alpha_i}\,\alpha_j\,\overline{\varphi_2}(g_i^{-1}g_j) \\
&= \sum_{k=1}^{\ell_0} \overline{\varphi_2}(b_k^*\,b_k) \\
&= 0.
\end{aligned}$$

Hence $J(\overline{\varphi_1}) \subseteq J(\overline{\varphi_2})$. By symmetry, we have $J(\overline{\varphi_1}) = J(\overline{\varphi_2})$. ∎

Let G be a group acting on Prim A, where A is a C^*-algebra. Let $\mathfrak{I} \in \operatorname{Prim} A$. We recall that $\mathfrak{I}$ is said to be *G-invariant* if $G \cdot \mathfrak{I} \subseteq \mathfrak{I}$. By the *$G$-kernel* of $\mathfrak{I}$ we mean the largest G-invariant ideal contained in $\mathfrak{I}$. The *G-quasi-orbit* of $\mathfrak{I}$ in Prim A is the set of all primitive ideals of A whose G-orbit closures in Prim A coincide with that of $\mathfrak{I}$; or equivalently, whose G-kernels are the same as that of $\mathfrak{I}$. (See Green [7].)

Returning now to the groups G and D we are considering, we have the following lemma.

Lemma 5.6 *Let $k_1 : \varphi \mapsto \overline{\varphi}$ be a map from $\operatorname{Ch}_G(D)$ to $\operatorname{Ch}(G)$ according to Lemma 5.4, $k_2 : \overline{\varphi} \mapsto J(\overline{\varphi})$ be the map from $\operatorname{Ch}(G)$ to* Prim *G as is stated earlier, and let*

$$k_3 : J(\overline{\varphi}) \mapsto \text{a quasi-orbit of } J(\overline{\varphi})_D \text{ in } \operatorname{Prim} D.$$

Then $l = k_3 \circ k_2 \circ k_1$ is a 1-1 correspondence between $\operatorname{Ch}_G(D)$ and G-quasi-orbits in Prim *D if they are restricted to the corresponding faithful characters of G which vanish off D.*

Proof The surjectivity of l is obvious. We now show that l is injective. Suppose that $\varphi_1, \varphi_2 \in \operatorname{Ch}_G(D)$ such that $\ell(\varphi_1) = \ell(\varphi_2)$, that is, $J(\overline{\varphi_1})_D$ and $J(\overline{\varphi_2})_D$ belong to the same quasi-orbit in Prim D. Since $J(\overline{\varphi_1})_D$ and $J(\overline{\varphi_2})_D$ are G-invariant, they must coincide with their G-kernel, so that $J(\overline{\varphi_1})_D = J(\overline{\varphi_2})_D$. This implies, by Lemma 5.5, $J(\overline{\varphi_1}) = J(\overline{\varphi_2})$. Therefore $\overline{\varphi_1} = \overline{\varphi_2}$, and hence $\varphi_1 = \varphi_2$. ■

We now have the following proposition whose proof is based on our claim that the map is a homeomorphism. Recall that, as the group G we are considering is nilpotent and AIC, we can identify $\operatorname{Ch}(G)$ with Prim G through the map J we stated earlier (see also [3]).

Proposition 5.7 *The map κ is injective.*

Proof We prove this by induction on the nilpotence class of G. Suppose that we have the homeomorphism property of the map κ for all groups whose nilpotence class is lower than that of G. Let θ_1 and θ_2 be quasi-orbits in $\mathcal{G}\hat{}$ such that $\theta_1 \neq \theta_2$ and $\kappa(\theta_1) = \kappa(\theta_2)$. Let $f_1 \in \theta_2$ and $f_2 \in \theta_2$. Then $\phi_{f_1} = \phi_{f_2} = \varphi$. This implies that

$$Z(\phi_{f_1}) = \exp \mathcal{Z}(f_1) = Z(\phi_{f_2}) = \exp \mathcal{Z}(f_2).$$

Since $\mathcal{I}(f_i) = \ker f_i \cap \mathcal{Z}(f_i), i \in \{1, 2\}$, and $f_1|_{\mathcal{Z}(f_1)} = f_2|_{\mathcal{Z}(f_2)}$, we have $\mathcal{I}(f_1) = \mathcal{I}(f_2) = \mathcal{I}$, say. Suppose now that $\overline{\varphi}$ is defined on $\overline{G} = G/k(\varphi)$ from φ and $\overline{f_i}$ are defined on $\log G/\mathcal{I}$ from $f_i, i \in \{1, 2\}$, where $\log G/\mathcal{I} = \log \overline{G}$ as $\mathcal{I} = \log k(\varphi)$. Let $\overline{\theta_1}$ and $\overline{\theta_2}$ be $\overline{G}$-quasi-orbits in $\log \overline{G}$ containing respectively $\overline{f_1}$ and $\overline{f_2}$. Then $\kappa(\overline{\theta_1}) = \kappa(\overline{\theta_2})$. Since $\mathcal{I}(f_1) = \mathcal{I}(f_2)$, it follows that $D(\overline{\theta_1}) = D(\overline{\theta_2}) = \overline{D}$, where $\overline{D} = \{\overline{g} \in \overline{G} : [\overline{g}, \overline{g}'] = 1_{\overline{G}}, \forall \overline{g}' \in Z^2(\overline{G})\}$. As $\overline{\theta_1} \neq \overline{\theta_2}$, by Lemma 5.3, $r(\overline{\theta_1}) \neq r(\overline{\theta_2})$. As we have Lemma 5.2, then according to our inductive hypothesis, $\overline{D}$-quasi-orbits in $(\log \overline{D})\hat{}$ is homeomorphic to Prim $\overline{D}$. Since the corresponding homeomorphism is $\overline{G}$-equivariant, it induces a bijection between $\overline{G}$-quasi-orbits in $(\log \overline{D})\hat{}$ and $\overline{G}$-quasi-orbits in Prim $\overline{D}$. By Lemma 5.6, the fact that $r(\overline{\theta_1}) \neq r(\overline{\theta_2})$ would give distinct characters, which contradicts $\kappa(\overline{\theta_1}) = \kappa(\overline{\theta_2})$. ■

6 The Kirillov Homeomorphism

We need the following definition for the next lemma. A positive definite function φ is said to be *associated with* the subgroup-representation $\langle\pi, K\rangle$ (see Fell [6] for the definition) if φ is defined on K and there exists a cyclic vector ξ in $\mathcal{H}_\pi$ such that $\varphi(g) = \langle\pi(g)\xi, \xi\rangle$ for all $g \in K$. If, in addition to that, φ is a trace, then every φ_1, with $\varphi_1(g) = \langle\pi(g)\xi_1, \xi_1\rangle$ for some ξ_1, is also a trace, and hence $\varphi_1 = \varphi$. Let $\mathcal{R}(G)$ denote the space of subgroup-representations.

Lemma 6.1 *Let $(\langle\pi_n, K_n\rangle)$ be a sequence of elements of $\mathcal{R}(G)$ and $\langle\pi, K\rangle$ be an element of $\mathcal{R}(G)$. Let ϕ be a positive definite function associated with $\langle\pi, K\rangle$. If $\langle\pi_n, K_n\rangle \to \langle\pi, K\rangle$, then for each subsequence $(\langle\pi_{n_i}, K_{n_i}\rangle)$ there exists a subsequence $(\langle\pi_{i_k}, K_{i_k}\rangle)$ of this subsequence, and a sequence (ϕ_k), such that $\phi_k \to \phi$ (pointwise), where for every k, ϕ_k is a finite sum of positive definite functions associated with $\langle\pi_{i_k}, K_{i_k}\rangle$.*

Proof (This lemma is a special case of Theorem 3.1′ in [6].) ■

If in the lemma above, all φ_n associated with π_n are traces, then for every k, the finite sum of positive definite functions ϕ_k is of the form $n_k\varphi_k$, where n_k is a positive integer and φ_k is a unique trace associated with $\langle\pi_{i_k}, K_{i_k}\rangle$. If ϕ is a trace, then $n_k\varphi_k \to \phi$ implies that $n_k = 1$ eventually as all traces are normalized.

Recall that if ϕ is a character of G, we write $Z(\phi) := Z(G/k(\phi))$ and $\chi_\phi := \phi|_{Z(\phi)}$. Then we notice that χ_ϕ is 1-dimensional character, so that it can be considered as a (1-dimensional) representation of $Z(\phi)$ with which χ_ϕ is associated. We have the following lemma.

Lemma 6.2 (Carey, Moran and Pearce [4, Theorem 4.1].) *Let (ϕ_n) be a sequence in* $\mathrm{Ch}(G)$. *Suppose that $\phi_n \to \phi$ in the hull-kernel topology on* $\mathrm{Ch}(G)$ *identified through the bijection with* Prim G. *Then there exists a subsequence (ϕ_{n_i}) of (ϕ_n) such that*

(i) *$\phi_{n_i} \to \phi$ pointwise on some $Z_0 \subseteq Z(\phi)$; and*
(ii) $\langle\chi_{\phi_{n_i}}, Z(\phi_{n_i})\rangle \to \langle\chi_\phi, Z_0\rangle$,

where the convergence in (ii) *is in the sense of the inner hull-kernel topology of $\mathcal{R}(G)$. Conversely, let the sequence (ϕ_n) and a character ϕ in* $\mathrm{Ch}(G)$*be given such that $\langle\chi_{\phi_n}, Z(\phi_n)\rangle \to \langle\chi_\phi, Z_0\rangle$. Then $\phi_n \to \phi_0$ in the hull-kernel topology on* $\mathrm{Ch}(G)$, *where ϕ_0 is any character whose associated representation is weakly contained in the representation of G induced by the character χ_ϕ of Z_0.*

Proof (See the proof of Theorem 4.1 of [4].) ■

The next theorem is the key result. Its proof differs significantly from that of the corresponding result of [4].

Theorem 6.3 *The map κ is a homeomorphism.*

Proof We shall first show the continuity of κ. Let (θ_n) be a sequence of quasi-orbits such that $\theta_n \to \theta$. Consider a sequence (f_n), $f_n \in \theta_n$ such that $f_n \to f$ for some $f \in \theta$. Then, it suffices to show that for any subsequence (f_{n_i}) of (f_n), there exists a subsequence (h_j) of (f_{n_i}) such that $\kappa(h_j) \to \kappa(f)$. Let the arbitrary subsequence (f_{n_i}) be given. Since the space of subgroup representations is compact, there exists a subsequence (g_k) of (f_{n_i}) such that $\langle \chi_{\kappa(g_k)}, Z(\kappa(g_k))\rangle$ converges to some $\langle \chi_0, Z_0\rangle$, where $\chi_{\kappa(g_k)} = g_k \circ \log$. Suppose that ϕ_0 is the character of G induced from χ_0 on Z_0. By Lemma 6.2, $(\kappa(g_k))$ converges to any character ϕ which is weakly contained in ϕ_0. As all characters weakly contained in ϕ_0 are identified with the same element in Prim G, we need to check that $\kappa(f)$ is weakly contained in ϕ_0. In fact, since $\langle \chi_{\kappa(g_k)}, Z(\kappa(g_k))\rangle$ converges to $\langle \chi_0, Z_0\rangle$, it follows from Lemma 6.1 (see comments following this lemma) that $\chi_{\kappa(h_j)} \to \chi_0$ (pointwise) for some subsequence (h_j) of (g_k). Hence $\chi_0 \circ \exp = f$ as $\chi_{\kappa(h_j)} \circ \exp = h_j$ and $h_j \to f$. Thus $\kappa(f)$ is weakly contained in ϕ_0.

We shall now prove the continuity of κ^{-1}. Let (ϕ_n) be a sequence in $\mathrm{Ch}(G)$ with $\phi_n \to \phi$ (hull-kernel). Let $\kappa(f_n) = \phi_n$, $\kappa(f) = \phi$. We will show that for any subsequence (f_{n_i}) of (f_n), there exists a subsequence (k) of (n_i) and a sequence (h_k) such that $h_k \to f$, where for every k, h_k and f_k belong to the same quasi-orbit. Now let the subsequence (f_{n_i}) be given. By Lemma 6.2, there exists a subsequence (k) of (n_i) such that for some $Z_0 \subseteq Z(\phi)$, $Z(\phi_k) \to Z_0$ and $\phi_k \to \phi_0$ pointwise for some ϕ_0 on Z_0. Since $f_k = \chi_{\phi_k} \circ \exp$ and $f = \chi_\phi \circ \exp$, it follows that $f_k \to f$ on $\mathcal{Z}_0 := \log Z_0$. It suffices to find a sequence (h_k) in $\mathcal{G}\hat{}$ such that for every k, $h_k = f_k$ on $\mathcal{Z}(f_k)$ and $h_k \to f$ on $\mathcal{G}$.

For any $a_1, a_2, \dots, a_m \in \mathcal{G}$ we shall show that for every $\epsilon > 0$, there exists $h \in \mathcal{G}$ such that $h = f_{k_0}$ on $\mathcal{Z}(f_{k_0})$ for sufficiently large k_0 and $|h(a_i) - f(a_i)| < \epsilon$ for $i \in \{1, 2, \dots, m\}$. Let $W := \langle a_1, a_2, \dots, a_m\rangle$ and $T_0 := W \cap \mathcal{Z}_0$. Then W/T_0 is a direct sum of finitely many cyclic subgroups with generators

$$\overline{b_1}, \overline{b_2}, \dots, \overline{b_l}, \overline{b_{l+1}}, \dots, \overline{b_r}$$

where for $i \in \{1, 2, \dots, l\}$, $\overline{b_i} := b_i + T_0$ is of order n_i and for $i \in \{l+1, l+2, \dots, r\}$, $\overline{b_i}$ is torsion-free. It is evidently enough to prove our assertion with b_i's in place of a_i's. For given $\epsilon > 0$, we shall choose $\gamma \in (W/T_0)\hat{}$ such that by choosing k_0 large, we have for all $i \in \{1, 2, \dots, r\}$,

$$\left|\gamma(\overline{b_i}) - \frac{f(b_i)}{f_{k_0}(b_i)}\right| < \frac{\epsilon}{2}. \tag{6.1}$$

Consider first for b_i, $i \in \{1, 2, \dots, l\}$. As $f_k \to f$ on $\mathcal{Z}_0$, since $n_i b_i \in \mathcal{Z}_0$, we have

$$\frac{f(n_i b_i)}{f_k(n_i b_i)} = \left(\frac{f(b_i)}{f_k(b_i)}\right)^{n_i} \to 1.$$

This implies that, by passing successively to subsequences and relabelling if necessary, we may assume $f(b_i)/f_k(b_i)$ tends to some n_i-th root of unity as k tends to infinity. Hence by choosing k_0 large enough, we can define γ such that (6.1) holds for $i \in$

$\{1, 2, \ldots, l\}$. Now let N be a multiple of the torsion of W/T_0. Then we can consider $(W/T_0)/N(W/T_0)$ as having generators $\overline{b_{l+1}}, \overline{b_{l+2}}, \ldots, \overline{b_r}$, each of which has order N. As $\langle \overline{b_{l+1}}, \overline{b_{l+2}}, \ldots, \overline{b_r} \rangle \cong \mathbb{Z}(N)^{r-l}$, we can choose N large enough so that we can define γ on $(W/T_0)/N(W/T_0)$ (and hence on W/T_0) such that γ satisfies (6.1) for $i \in \{l+1, l+2, \ldots, r\}$. Therefore we have the required γ. Next let $T_k := W \cap \mathcal{Z}(f_k)$. Note that we can write $W/T_0 = S_{\mathrm{t}} \oplus S_{\mathrm{tf}}$, where $S_{\mathrm{t}} := \langle \overline{b_1}, \overline{b_2}, \ldots, \overline{b_l} \rangle$ and $S_{\mathrm{tf}} := \langle \overline{b_{l+1}}, \overline{b_{l+2}}, \ldots, \overline{b_r} \rangle$. We also note that γ can be written as $\gamma = \gamma_{\mathrm{t}}\gamma_{\mathrm{tf}}$, where

$$\gamma_{\mathrm{tf}}(\overline{b_i}) = \begin{cases} \gamma(\overline{b_i}) & \text{if } \overline{b_i} \in S_{\mathrm{tf}} \\ 1 & \text{if } \overline{b_i} \in S_{\mathrm{t}}, \end{cases} \tag{6.2}$$

and

$$\gamma_{\mathrm{t}}(\overline{b_i}) = \begin{cases} \gamma(\overline{b_i}) & \text{if } \overline{b_i} \in S_{\mathrm{t}} \\ 1 & \text{if } \overline{b_i} \in S_{\mathrm{tf}}. \end{cases} \tag{6.3}$$

Since $S_{\mathrm{tf}} \cong \mathbb{Z}^{r-l}$, arguing as in [4], there exists a character γ_0 of S_{tf} such that for k_0 large, $\gamma_0(b + T_0) = 1$ for $b \in T_{k_0}$ and for $i \in \{l+1, l+2, \ldots, r\}$,

$$|\gamma_0(\overline{b_i}) - \gamma_{\mathrm{tf}}(\overline{b_i})| < \frac{\epsilon}{2}. \tag{6.4}$$

Then we have $\gamma_1 \in (W/T_0)^\wedge$ such that

$$\gamma_1(\overline{b_i}) = \begin{cases} \gamma_0(\overline{b_i}) & \text{if } \overline{b_i} \in S_{\mathrm{tf}} \\ 1 & \text{if } \overline{b_i} \in S_{\mathrm{t}}. \end{cases} \tag{6.5}$$

Now set $\gamma_2 := \gamma_1\gamma_{\mathrm{t}}$. Since S_{t} is finite and $T_k \to T_0$, we can choose k_0 such that $T_{k_0} \cap S_{\mathrm{t}} = (0)$. In addition to that, as W is finitely generated, we can also make k_0 large enough such that $T_0 \subseteq T_k$ for all $k \geq k_0$. Note that, by (6.3) and (6.5), we have

$$|\gamma_2(\overline{b_i}) - \gamma(\overline{b_i})| = 0 \tag{6.6}$$

if $\overline{b_i} \in S_{\mathrm{t}}$, while by (6.2), (6.3), (6.4) and (6.5), we have

$$|\gamma_2(\overline{b_i}) - \gamma(\overline{b_i})| = |\gamma_1(\overline{b_i}) - \gamma_{\mathrm{tf}}(\overline{b_i})| = |\gamma_0(\overline{b_i}) - \gamma_{\mathrm{tf}}(\overline{b_i})| < \frac{\epsilon}{2} \tag{6.7}$$

if $\overline{b_i} \in S_{\mathrm{tf}}$. Since we see that $\gamma_2 \equiv 1$ on T_{k_0}/T_0, γ_2 determines the character γ_2' of $(W/T_0)/(T_{k_0}/T_0)$ defined by $\gamma_2'((b + T_0) + T_{k_0}/T_0) := \gamma_2(b + T_0)$ for all $b \in W$. Notice that $(W/T_0)/(T_{k_0}/T_0) \cong W/T_{k_0}$ with the corresponding map

$$(b + T_0) + T_{k_0}/T_0 \mapsto b + T_{k_0}.$$

We then have the character γ_2'' of W/T_{k_0} defined by

$$\gamma_2''(b + T_{k_0}) := \gamma_2'((b + T_0) + T_{k_0}/T_0).$$

It follows that $\gamma_2''(b+T_{k_0}) = \gamma_2(b+T_0)$ for all $b \in W$, and hence from (6.1), (6.6) and (6.7) we have

$$\left|\gamma_2''(b_i+T_{k_0}) - \frac{f(b_i)}{f_{k_0}(b_i)}\right| < \epsilon$$

for all $i \in \{1,2,\ldots,r\}$. Note that $(W/T_{k_0}) \cong (W+\mathcal{Z}(f_{k_0}))/\mathcal{Z}(f_{k_0})$. Let

$$\pi : W + \mathcal{Z}(f_{k_0}) \longrightarrow (W+\mathcal{Z}(f_{k_0}))/\mathcal{Z}(f_{k_0})$$

be the quotient projection. We can extend $\gamma_2'' \circ \pi$ to a character $\overline{\gamma_2''}$ of $\mathcal{G}$. Now set $h := \overline{\gamma_2''} f_{k_0}$. Then we have $h \in \mathcal{G}\hat{}$ which satisfies the required conditions, that is $h = f_{k_0}$ on $\mathcal{Z}(f_{k_0})$ and $|h(b_i) - f(b_i)| < \epsilon$ for all $i \in \{1,2,\ldots,r\}$. Therefore, enumerating $\mathcal{G}$ as $(a_n)_{n=1}^{n=\infty}$, we can choose $h_{k_i} \in \mathcal{G}\hat{}$ for each i such that $h_{k_i} = f_{k_i}$ on $\mathcal{Z}(f_{k_i})$, and for each n,

$$|h_{k_i}(a_n) - f(a_n)| < \frac{1}{i}$$

for $i \geq n$, that is $h_{k_i} \to f$. This completes the proof. ■

7 Example

We conclude this paper with examples of a 3-step nilpotent group which is torsion-free, π-radicable, AIC, but, in general, not complete. Let $\pi := \{2,3\}$ and

$$\mathbb{Q}_\pi := \{\frac{m}{n} : m \in \mathbb{Z}, n \text{ is a } \pi\text{-number}\}.$$

(Note that we may choose any other π as long as $\pi \supseteq \{2,3\}$.) Consider the group

$$G = \left\{ \begin{bmatrix} 1 & a & d & f \\ 0 & 1 & b & e \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} : \ a,d,f \in \mathbb{Q}, \ b,c,e \in \mathbb{Q}_\pi \right\},$$

which is 3-step and π-radicable (but not complete). We shall show that G is AIC. In view of Theorem 2.4 of [1], it suffices to show that for all $N \triangleleft G$, $[xN, G/N] = [x,G]N/N$ is a subgroup of G/N for all elements xN of finite conjugacy class of G/N.

For convenience, write an arbitrary $x \in G$ as $(a,b,c,d,e,f) := x$. Now let $x_1 := (a_1,b_1,c_1,d_1,e_1,f_1) \in G$. Then we have the following formula:

$$\begin{aligned}[x_1,x] = (0,0,0,&ba_1 - ab_1, cb_1 - bc_1,\\ &- c_1a_1b + ac_1b_1 - bca_1 + ea_1 + cd_1 + cab_1 - ae_1 - dc_1).\end{aligned}$$

Now consider the set

$$[x_1, G] := \{[x_1,x] : x \in G\}.$$

Suppose that either $a_1 \neq 0$ or $c_1 \neq 0$. Substituting $x = (0,0,0,d,e,f)$, we have $[x_1,x] = (0,0,0,0,0,ea_1 - dc_1)$. Then we see that $[x_1,G]$ contains the centre $Z(G)$

of G. As G is 3-step, it is easy to observe that $[x_1, G]$ is a subgroup of G, and hence for all $N \triangleleft G$, $[x, G]N/N$ is a subgroup of G/N. Now suppose that both $a_1 = 0$ and $c_1 = 0$. Then

$$[x_1, x] = (0, 0, 0, -ab_1, cb_1, cd_1 + cab_1 - ae_1). \tag{7.1}$$

If $b_1 = 0$, then it is obvious that $[x_1, G]N/N$ is a subgroup of G/N for all $N \triangleleft G$. Now let $b_1 \neq 0$. Let $N \triangleleft G$. If $N \supseteq Z(G)$, then it is straightforward that $[x_1, G]N/N$ is a subgroup. Next, let $N \not\supseteq Z(G)$. As the set of all components $cd_1 + cab_1 - ae_1$ in (7.1) is equal to $\mathbb{Q}$, and we know that $\mathbb{Q}$ has no proper subgroup of finite index (see Kurosh [12, pp. 61–62], or more generally in [13, p. 234]), it follows that $[x_1, G]N/N$ is an infinite subset of G/N, and hence x_1N is not an element of finite conjugacy class of G/N. This completes the proof that G is AIC.

Symmetrically, it is easy to see that another combination for the matrix G which gives an example is $G := \{(a, b, c, d, e, f) : a, b, d \in \mathbb{Q}_\pi,\ c, e, f \in \mathbb{Q}\}$.

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