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ON SUBSERIES CONVERGENT SEQUENCES

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In this article some properties on subseries convergent sequences in locally convex topological vector spaces are studied and some open questions of (2) are answered.

The linear spaces we use are defined over the field K of the real or complex numbers. By "space" we understand separated locally convex topological vector space. If $\langle P, Q\rangle$ is a dual pair, $\sigma(P, Q)$ and $\mu(P, Q)$ are the weak and the Mackey topology on P, respectively. Given a space E, E^* is its algebraic dual and E' stands for its topological dual. If A is a bounded absolutely convex set of E then E_A denotes the normed space over the linear hull of A with the gauge of A as norm. A sequence (x_n) in E is subseries convergent if for every strictly increasing sequence (n_h) of positive integers the series $\Sigma\, x_{n_h}$ is convergent. A sequence (y_n) is bounded multiplier convergent if for every bounded sequence (a_n) of K the series $\Sigma\, a_n y_n$ is convergent.

In an article by I. Tweddle (10) the finest locally convex topology $\mathcal{T}_0$ on E, such that every measure μ defined over a σ-algebra with values in E is a $\mathcal{T}_0$-measure, is introduced and characterised by means of considering the space $G' = G'(E)$ of the linear forms v on E so that if (x_n) is any subseries convergent sequence in E then

$$\left\langle \sum x_n, v\right\rangle = \sum \langle x_n, v\rangle$$

and it is proved there that $\mathcal{T}_0$ is the Mackey topology $\mu(E, G')$.

In (2) D. Bucchioni and A. Goldman introduce the finest locally convex topology $\mathcal{T}_1$ such that the topological dual $G'_1 = G'_1(E)$ of $E[\mathcal{T}_1]$ is formed by all those linear forms v on E such that if (x_n) is any subseries convergent sequence in E then

$$\sum |\langle x_n, v\rangle| < \infty$$

The topology $\mathcal{T}_1$ is finer than $\mathcal{T}_0$ and is a bornological and barrelled topology. They ask the following problems:

1) Is $\mathcal{T}_1$ the bornological barrelled associated topology to $\mathcal{T}_0$?

2) Is $\mathcal{T}_1$ ultrabornological?

They answer negatively to 1) leaving 2) open. We shall give also a negative answer to 2).

We write l_0^∞ to denote the subspace of l^∞ generated by the functions defined on $\mathbb{N}$ and which are characteristic of a subset of $\mathbb{N}$ and we consider the dual pair $\langle l^1, l_0^\infty\rangle$ with the bilinear form

$$\sum u_n v_n, (u_n) \in l^1, (v_n) \in l_0^\infty$$

If (x_n) is a subseries convergent sequence in E let T be the mapping of $E'[\sigma(E', E)]$ into $l^1[\sigma(l^1, l_0^\infty)]$ such that

$$T(w) = (\langle x_n, w \rangle) \text{ for every } w \in E'.$$

The mapping T is well defined since

$$\sum |\langle x_n, w \rangle| < \infty$$

On the other hand, if $\{w_i : i \in I, \geqslant\}$ is a net in $E'[\sigma(E', E)]$ converging to the origin and if $M(J)$ is the characteristic function of $J \subset N$ then

$$\lim \{\langle T(w_i), M(J) \rangle : i \in I, \geqslant\} = \lim \left\{ \sum_{n \in J} \langle x_n, w_i \rangle : i \in I, \geqslant \right\} = 0$$

and therefore T is continuous. If B is the closed unit ball of l_0^∞ then $T^*(B)$ is a bounded set of E where T^* denotes the transposed mapping to T and $T^*(B)$ can be represented as

$$\left\{ \sum a_n x_n : (a_n) \in l_0^\infty, |a_n| \leqslant 1, n = 1, 2, \ldots \right\}.$$

We denote by $E[(x_n)]$ the normed space defined over the linear hull of $T^*(B)$ having $T^*(B)$ as unit ball. Considering T^* as a mapping from l_0^∞ onto $E[(x_n)]$ it is obvious that T^* is a topological homomorphism and since l_0^∞ is barrelled (6) then $E[(x_n)]$ is a normed barrelled space.

Proposition 1. *In real l_0^∞ the absolutely convex hull A of the characteristic functions of all subsets of* $\mathbb{N}$ *contains $\frac{1}{2}B$*

Proof. Let J be a subset of $\mathbb{N}$ and let $M(J)$ be the corresponding characteristic function. If $x \in l_0^\infty$, let $x^{(n)}$ be the value of x at $n \in \mathbb{N}$. If $x \in \frac{1}{2}B$ and takes at most two values different from zero we obtain three disjoint subsets N_1, N_2, N_3 of $\mathbb{N}$ such that $N_1 \cup N_2 \cup N_3 = \mathbb{N}$ and $x^{(n)} = \alpha$, $n \in N_1$, $x^{(n)} = \beta$, $n \in N_2$, $x^{(n)} = 0$, $n \in N_3$. Then

$$x = \alpha M(N_1) + \beta M(N_2), |\alpha| \leqslant \tfrac{1}{2}, |\beta| \leqslant \tfrac{1}{2},$$

and therefore $x \in A$.

Proceeding by recurrence we suppose that for an integer $p \geqslant 2$ all the vectors of $\frac{1}{2}B$ that take at most p different from zero values belong to A. Let z be a vector of $\frac{1}{2}B$ taking exactly $p+1$ different from zero values. We set $\mathbb{N} = N_1 \cup N_2 \cup \ldots \cup N_{p+2}$ such that z takes the value α_j in N_j, $j = 1, 2, \ldots, p+1$, and in N_{p+2} takes value zero and $N_i \cap N_k = \emptyset$, $i \neq k$, i, $k = 1, 2, \ldots, p+2$. Since $p \geqslant 2$, z takes two different values of

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the same sign. Suppose $\alpha_1 < \alpha_2$ and $\alpha_1\alpha_2 > 0$. We consider the vectors u, v which coincide with z in $N_2 \cup N_3 \cup \ldots \cup N_{p+2}$ such that u takes value α_2 in N_1 and v takes value zero in N_1. Then u and v take p non-zero values and since u and v belong to $\frac{1}{2}B$ they belong to A. On the other hand, $0 < \alpha_1/\alpha_2 < 1$ and thus $(\alpha_1/\alpha_2)u + (1-\alpha_1/\alpha_2)v = z$ and therefore $z \in A$.

Proposition 2. *In complex l_0^∞ the absolutely convex hull A of the characteristic functions of all subsets of* $\mathbb{N}$ contains $\frac{1}{4}B$.

Proof. Let z be a vector of $\frac{1}{4}B$. We set $z = z_1 + iz_2$ where z_1, z_2 are real vectors of $\frac{1}{4}B$. According to the former proposition $2z_1$ and $2z_2$ belong to A. Then $z = \frac{1}{2}(2z_1) + \frac{1}{2}(2z_2)$ belongs to A.

Theorem 1. *Given a space E, $E[\mu(E, G_1'(E))]$ is the inductive limit of the family of spaces $\{E[(x_n)]$: (x_n) is subseries convergent sequence$\}$.*

Proof. Let (x_n) be any subseries convergent sequence in E. If $\mathcal{P}(\mathbb{N})$ denotes the σ-algebra of all subsets of $\mathbb{N}$ and if $\mu : \mathcal{P}(\mathbb{N}) \to E$ is defined by

$$\mu(\Delta) = \sum_{n\in\Delta} x_m, \Delta \in \mathcal{P}(\mathbb{N}),$$

then μ is a vector measure. If v is an element of E^* bounded on $E[(x_n)]$ then $v \circ \mu$ is a scalar additive set function defined on $\mathcal{P}(\mathbb{N})$. Obviously, v is bounded in $\{\Sigma_{n\in\Delta} x_n : \Delta \in \mathcal{P}(\mathbb{N})\}$ and therefore (see 2, p. 177)

$$\sum |(v\circ\mu)(\{n\})| = \sum |\langle x_n, v\rangle| < \infty$$

and thus $v \in G_1'(E)$.

Reciprocally, if $v \in G_1'(E)$ then $v\circ\mu$ is exhaustive and therefore bounded (see 2, p. 177), i.e. there is a positive number D such that

$$\sup\left\{\left|v\left(\sum_{n\in\Delta} x_n\right)\right| : \Delta \in \mathcal{P}(\mathbb{N})\right\} \leq D$$

If U is the absolutely convex hull of

$$\left\{\sum_{n\in\Delta} x_n : \Delta \in \mathcal{P}(\mathbb{N})\right\}$$

then, according to Proposition 2,

$$\sup\left|v\left(\sum a_n x_n\right)\right| : (a_n) \in l_0^\infty, |a_n| \leq 1, n = 1, 2, \ldots\} \leq 4 \sup\{|v(x)| : x \in U\}$$

$$= 4 \sup\left\{\left|v\left(\sum_{n\in\Delta} x_n\right)\right| : \Delta \in \mathcal{P}(\mathbb{N})\right\} \leq 4D$$

and thus v is bounded on $E[(x_n)]$.

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Note 1. $E[\mu(E, G_1')]$ is a barrelled topological space as proved in (2). Theorem 1 is a stronger result since $E[\mu(E, G_1'(E)]$ is the inductive limit of a family of normed barrelled spaces.

Note 2. Let E be the space $l_0^\infty[\sigma(l_0^\infty, l^1)]$. If e_n is the function on $\mathbb{N}$ which is characteristic of the set $\{n\}$ then (e_n) is a subseries convergent sequence in E. Then $E[(e_n)]$ coincides with l_0^∞. On the other hand, every subseries convergent sequence in E is contained in λB for a $\lambda > 0$ being B the closed unit ball of l_0^∞, and therefore, according to Theorem 1, $E[\mu(E, G_1'(E))]$ coincides with $E[(e_n)] = l_0^\infty$. According to a result of A. Pelczynski, every separable subspace of l_0^∞ is of countable algebraic dimension and thus the associated ultrabornological space to l_0^∞ has the finest locally convex topology and therefore $E[\mu(E, G_1'(E))] = l_0^\infty$ is not ultrabornological (see **4**). This example gives a negative answer to problem 2).

In (2) the class of spaces having properties (S) is defined. A space E satisfies property (S) if for every subseries convergent sequence (x_n) and every sequence (a_n) of l^1, the sequence $(a_n x_n)$ is subseries convergent in E. It is obvious that if every subseries convergent sequence (x_n) in E is bounded multiplier convergent then E has property (S).

If X is a completely regular topological space let $C(X)$ and $C_s(X)$ be the linear spaces over K of the K-valued continuous functions on X with the compact-open topology and the simple topology, respectively.

The authors of (2) ask the following questions:

a) Has $C_s(X)$ property (S)?

b) If E is an ultrabornological space, has E property (S)?

They say (2, p. 194) that S. and P. Dierolf have given a negative answer to b). We shall give here answers to a) and b). We shall need the following result which we gave in (**11**): *If the dimension of a space F is a non-measurable cardinal number then F is a closed subspace of an ultrabornological space.*

The sequence (e_n) in the space $l_0^\infty[\sigma(l_0^\infty, l^1)]$ mentioned above is subseries convergent. On the other hand, the sequence $(n^{-2}e_n)$ is not subseries convergent in this space and thus $l_0^\infty[\sigma(l_0^\infty, l_1)]$ has not property (S). According to our previous result, $l_0^\infty[\sigma(l_0^\infty, l_1)]$ is a closed subspace of an ultrabornological space E and then it easily follows that E has not property (S) and question b) is thus answered.

Let X be the space l^1 provided with the topology $\sigma(l^1, l_0^\infty)$. Obviously, $l_0^\infty[\sigma(l_0^\infty, l^1)]$ is a closed subspace of $C_s(X)$ and thus $C_s(X)$ has not property (S) which answers question a).

In what follows we shall give answer to questions a) and b) using metrizable spaces. Let f_n be the vector of l^1 with vanishing coordinates except that in position n which has value $1/n^2$. The vector space $l^1[(f_n)]$ is dense in l^1 and the element (a_n) of l^1 with $a_n = 1/n^4$, $n = 1, 2, \dots$ is not in $l^1[(f_n)]$. Then there is a dense hyperplane H of l^1 containing $l^1[(f_n)]$ and not (a_n). Since (f_n) is subseries convergent in H and $(a_n) = \Sigma_{n=1}^\infty (1/n^2) f_n$ it follows that H has not property (S). We set $X = H'[\sigma(H', H)]$. Since H is barrelled as a finite codimensional subspace of a barrelled space (**5**) it follows that H is a closed subspace of $C(X)$. On the other hand, it is obvious that $C(X)$ is metrizable and therefore ultrabornological, (**7**), (**9**) and (3). Since H is closed in $C(X)$ and has not property (S) it follows that the metrizable ultrabornological space $C(X)$

has not property (S). Moreover the subseries convergent sequences of $C(X)$ and $C_s(X)$ are the same and thus $C_s(X)$ has not property (S).

In (2) the following result is proven: *Let* (E_n) *be an increasing sequence of subspaces of a space* E *covering* E. *Let* $\mathcal{T}_n$ *be a topology on* E_n *such that* $E_n[\mathcal{T}_n]$ *is a Banach space and such that the injections* $E_n[\mathcal{T}_n] \to E_{n+1}[\mathcal{T}_{n+1}]$, $n = 1, 2, \ldots$ *are continuous and* E *is the inductive limit of* $(E_n[\mathcal{T}_n])$. *Suppose that the unit ball* B_n *of* $E_n[\mathcal{T}_n]$ *is universally measurable in* E *(which happens if, for instance,* $E_n[\mathcal{T}_n]$ *is separable). Then* E *has property* (S).

The authors of (2) ask if the former result is true whenever B_n is not universally measurable in E, $n = 1, 2, \ldots$. We give an affirmative answer in Theorem 2.

Proposition 3. *Let* E *be a* B_r*-complete space and let* $\mathcal{T}$ *be a separated locally convex topology on* E, *coarser than the original topology. If* (x_n) *is a subseries convergent sequence in* $E[\mathcal{T}]$ *then* (x_n) *is bounded multiplier convergent in* $E[\mathcal{T}]$.

Proof. Let v: $l_0^\infty[\mu(l_0^\infty, l^1)] \to E[\mathcal{T}]$ be the mapping defined by $v((a_n)) = \Sigma\, a_n x_n$, for every $(a_n) \in l_0^\infty$. We know that v is continuous. If F denotes the completion of $E[\mathcal{T}]$ we extend v to a continuous mapping $\bar{v}$ of $l^\infty[\mu(l^\infty, l^1)]$ into F. If U is the closed unit ball of l^∞ it is obvious that $\bar{v}(U)$ is compact in F and since l_0^∞ is dense in l^∞ then $E_{\bar{v}(U)}$ is the completion of $E[(x_n)]$.

Let J be the canonical injection of $E[(x_n)]$ into E. Since $J: E[(x_n)] \to E[\mathcal{T}]$ is continuous then J has closed graph in $E[(x_n)] \times E$ and, according to the fact that $E[(x_n)]$ is barrelled, J is continuous by Ptak's theorem, (8).

Let z be any point of $\bar{v}(U)$. We select a sequence (z_n) in $E[(x_n)]$ converging to z in $E_{\bar{v}(U)}$. The sequence $(J(z_n)) = (z_n)$ is a Cauchy sequence in E and, since E is complete, (z_n) converges to y in E and also in $E[\mathcal{T}]$ and thus $y = z$. Then $\bar{v}(U)$ is contained in E and therefore the element $\Sigma\, a_n x_n$, $(a_n) \in l^\infty$, of F belongs to E which implies that (x_n) is bounded multiplier convergent in E.

Theorem 2. *In a space* E *let* (A_n) *be a sequence of bounded absolutely convex sets such that*:

1. E_{A_n} *is a Banach space*, $n = 1, 2, \ldots$
2. *The linear hull of* $\cup\{A_n : n = 1, 2, \ldots\}$ *coincides with* E.

If (x_n) *is any subseries convergent sequence then* (x_n) *is bounded multiplier convergent.*

Proof. We set

$$C_n = (A_1 + A_2 + \cdots + A_n) \cap E[(x_n)].$$

The sequence (nC_n) of absolutely convex sets covers $E[(x_n)]$ and according to the fact that $E[(x_n)]$ is barrelled and using a theorem of I. Amemiya and Y. Komura (1) there is a positive integer p such that the closure $\bar{C}_p$ of C_p in $E[(x_n)]$ is a neighbourhood of the origin in this space. If F denotes the linear hull of C_p in $E[(x_n)]$, the canonical injection J of F in the Banach space $E_{A_1+A_2+\cdots+A_m}$ is quasi-closed and according to a result of Ptak (8), J is continuous.

If z belongs to $E[(x_n)]$ there is a sequence (z_n) in F converging to z in $E[(x_n)]$. The sequence $(J(z_n)) = (z_n)$ is a Cauchy sequence in $E_{A_1+A_2+\cdots+A_n}$ and therefore converges in this space to an element which has to coincide with z and thus $F = E[(x_n)]$. We apply the former proposition to obtain that the sequence (x_n) is bounded multiplier convergent.

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