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A NOTE ON GRAPHS WITH A PRESCRIBED ADJACENCY PROPERTY

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Let m and n be nonnegative integers and k be a positive integer. A graph G is said to have property $P(m, n, k)$ if for any set of $m + n$ distinct vertices of G there are at least k other vertices, each of which is adjacent to the first m vertices of the set but not adjacent to any of the latter n vertices. The problem that arises is that of characterising graphs having property $P(m, n, k)$. This problem has been considered by several authors and a number of results have been obtained. In this paper, we establish a lower bound on the order of a graph having property $P(m, n, k)$. Further, we show that all sufficiently large Paley graphs satisfy properties $P(1, n, k)$ and $P(n, 1, k)$.

1. INTRODUCTION

For our purposes graphs are finite, loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [7]. Thus G is a graph with vertex set $V(G)$, edge set $E(G)$, $\nu(G)$ vertices, $\varepsilon(G)$ edges, minimum degree $\delta(G)$ and maximum degree $\Delta(G)$. However, we denote the complement of G by $\overline{G}$.

In the application of graph theory to problems arising in network design and analysis, the requirements of the network (such as efficiency and reliability) can often be translated into adjacency restrictions on the graph representing the network. Here we consider graphs having a prescribed adjacency property.

More specifically, a graph G is said to have property $P(m, n, k)$ if for any set of $m + n$ distinct vertices there are at least k other vertices, each of which is adjacent to the first m vertices but not adjacent to any of the latter n vertices. The class of graphs having property $P(m, n, k)$ is denoted by $\mathcal{G}(m, n, k)$. Observe that if $G \in \mathcal{G}(m, n, k)$, then $\overline{G} \in \mathcal{G}(n, m, k)$. The cycle C_ν of length ν is a member of $\mathcal{G}(1, 1, 1)$ for every $\nu \geqslant 5$. The well-known Petersen graph is a member of $\mathcal{G}(1, 2, 1)$ and also of $\mathcal{G}(1, 1, 2)$. Despite these relatively simple examples, few members of $\mathcal{G}(m, n, k)$ have been found. Recently [3] we constructed several classes of graphs in $\mathcal{G}(1, n, k)$.

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The problem that arises is that of characterising the class $\mathcal{G}(m, n, k)$. Two particularly interesting problems that arise concern the functions

$$p(m, n, k) = \min\{\nu(G) : G \in \mathcal{G}(m, n, k)\}$$

and

$$q(\ell, (m, n, k)) = \min\{\varepsilon(G) : \nu(G) = \ell \text{ and } G \in \mathcal{G}(m, n, k)\}.$$

The only result concerning the latter function is due to Erdös and Moser [8] who determined $q(\ell, (m, 0, 1))$. Exoo [9], established bounds on $p(n, n, 1)$. Blass and Harary [5] established, using probabilistic methods, that almost all graphs have property $P(n, n, 1)$. From this it is not too difficult to show that almost all graphs have property $P(m, n, k)$. Despite this result, few graphs have been constructed which exhibit the property $P(m, n, k)$.

Properties of graphs in $\mathcal{G}(m, n, k)$ were given in [1, 2, 9]. In particular, in [2] we proved that

$$p(2, 2, k) \geqslant \begin{cases} 34, & \text{for } k = 1, \\ 8k + 25, & \text{for odd } k \geqslant 3, \\ 8k + 21, & \text{otherwise.} \end{cases}$$

In Section 2, we prove that

$$p(n, n, k) \geqslant 4^{n-1}\left[2(n + k) + \frac{1}{2}\left(3 + (-1)^{n+k+1}\right) + \frac{1}{3}\right] - \frac{1}{3}.$$

This generalises a result of Exoo [9].

An important graph in the study of the class $\mathcal{G}(m, n, k)$ is the so-called *Paley graph* G_q defined as follows. Let $q \equiv 1 \pmod 4$ be a prime power. The vertices of G_q are the elements of the finite field (Galois field) $\mathbb{F}_q$. Two vertices a and b are joined by an edge if and only if their difference is a quadratic residue, that is $a - b = y^2$ for some $y \in \mathbb{F}_q$.

For a prime $p \equiv 1 \pmod 4$, Blass, Exoo and Harary [4] showed that $G_p \in \mathcal{G}(n, n, 1)$ for $p > n^2 2^{4n}$. In [2] we improved this result by showing that for a prime power $q \equiv 1 \pmod 4$, $G_q \in \mathcal{G}(n, n, k)$ for every $q > \{(2n - 3)2^{2n-1} + 2\}\sqrt{q} + (n + 2k - 1)2^{2n-1} - 2n^2 - 1$. Further, we proved that $G_q \in \mathcal{G}(m, n, k)$ for every $q > \{(t - 3)2^{t-1} + 2\}\sqrt{q} + (t + 2k - 1)2^{t-1} - 1$, where $t \geqslant m + n$; and $G_q \in \mathcal{G}(1, 2, k)$ for every $q > \left(1 + 2\sqrt{2k}\right)^2$. Computational results were also presented to establish the smallest Paley graphs in $\mathcal{G}(2, 2, k)$ for small k.

In Section 3, we prove that $G_q \in \mathcal{G}(1, n, k) \cap \mathcal{G}(n, 1, k)$ for every $q > \{(n - 2)2^n + 2\}\sqrt{q} + (n + 2k - 1)2^n - 2n - 1$. Computational results show that this result is best possible for $n = 1$ (all k) and for $n = 2$ (most k), and close to best possible for $n = 3$.

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2. Lower bound on $p(m, n, k)$

We begin with some notation and terminology. For disjoint subsets A and B of $V(G)$ we denote by $N(A/B)$ the set of vertices of G not in $A \cup B$ which are adjacent to each vertex of A and not adjacent to any vertex of B. When $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ we write for convenience $N(A/B)$ as $N(a_1, a_2, \ldots, a_m/b_1, b_2, \ldots, b_n)$. Where appropriate, lower case letters will denote the cardinality of the set defined by the corresponding upper case letters. Thus for example, $n(a/b) = |N(a/b)|$. Finally, for $X \subseteq V(G)$ we let $G[X]$ denote the subgraph of G induced by X,

An r-regular graph of order ν is called *strongly regular* with parameters (ν, r, λ, μ) if G has the property that any two adjacent vertices have exactly λ common neighbours and any two non-adjacent vertices have exactly μ common neighbours.

We make use of the following results:

Lemma 2.1. (Exoo [9]). *If $G \in \mathcal{G}(1, 1, k)$, then $\nu(G) \geqslant 4k + 1$, with equality holding if and only if there exists a strongly regular graph with parameters $(4k + 1, 2k, k - 1, k)$.*

Lemma 2.2. (Ananchuen and Caccetta [1]). *For $1 \leqslant j \leqslant n$, $\mathcal{G}(m, n, k) \subseteq \mathcal{G}(m, n - j, k + j)$.*

The following is a useful lemma in our work.

Lemma 2.3. *Let $G \in \mathcal{G}(n, n, k)$ and $\{u_1, u_2, \ldots, u_{n-1}, v_1, v_2, \ldots, v_{n-1}\}$ be a set of $2n - 2$ vertices of G. Then the subgraph*

$$H = G[N(u_1, u_2, \ldots, u_{n-1}/v_1, v_2, \ldots, v_{n-1})]$$

has $\delta(H) \geqslant n + k$.

Proof: Suppose to the contrary that $d_H(x) = d \leqslant n + k - 1$. Let $y_1, y_2, \ldots, y_d$ be the neighbours of x in H. First we prove that $d \geqslant n$. Suppose $d < n$. Since $G \in \mathcal{G}(n, n, k)$, there exists $w \in N(y_1, y_2, \ldots, y_d/u_1, u_2, \ldots, u_{n-1})$ and a $y \in N(u_1, u_2, \ldots, u_{n-1}, x/v_1, v_2, \ldots, v_{n-1}, w)$. Thus $y \in V(H)$ and, since y is adjacent to x, $y = y_i$ for some i. Hence, y is adjacent to w, a contradiction. Therefore $d \geqslant n$ as required. Now consider a vertex $z \in N(y_1, y_2, \ldots, y_n/u_1, u_2, \ldots, u_{n-1}, x)$; such a z exists since $G \in \mathcal{G}(n, n, k)$. Clearly $n(u_1, u_2, \ldots, u_{n-1}, x/v_1, v_2, \ldots, v_{n-1}, z) \leqslant k - 1$, a contradiction. This proves that $\delta(H) \geqslant n + k$. □

We are now ready to prove the main result in this section.

Theorem 2.1.

$$p(n, n, k) \geqslant 4^{n-1}\left[2(n + k) + \frac{1}{2}\{3 + (-1)^{n+k+1}\} + \frac{1}{3}\right] - \frac{1}{3}. \tag{2.1}$$

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PROOF: Let $G \in \mathcal{G}(n, n, k)$. For $1 \leqslant i \leqslant n$ define

$$\mathcal{I}_i = \{A : A \subseteq V(G) \text{ and } |A| = i\}.$$

Clearly for disjoint sets A and B in $\mathcal{I}_i$, $N(A/B) \neq \emptyset$. Thus the subgraph $H = G[N(A/B)]$ has vertices; in fact $\nu(H) \geqslant k$. For $1 \leqslant i \leqslant n$, let

$$h_i^* = \min\{n(A/B) : A \cap B = \emptyset, A, B \in \mathcal{I}_i\}.$$

Suppose that $h_i^* = n(A^*/B^*)$ and let

$$H_i^* = G[N(A^*/B^*)].$$

Observe that $G \in \mathcal{G}(1, 1, h_1^*)$. We show that $H_i^* \in \mathcal{G}\left(1, 1, h_{i+1}^*\right)$ for each $1 \leqslant i \leqslant n-2$.

Consider the graph $H_i^* = G[N(A^*/B^*)]$. Then for any $a, b \in V(H_i^*)$, the number $n^*(a/b)$ of vertices of H_i^* joined to a but not joined to b satisfies

$$\begin{aligned} n^*(a/b) &= n(A^* \cup \{a\}/B^* \cup \{b\}) \\ &\geqslant h_{i+1}^*. \end{aligned}$$

Thus, as a and b are arbitrary, $H_i^* \in \mathcal{G}\left(1, 1, h_{i+1}^*\right)$. Now, by Lemma 2.1,

$$h_i^* \geqslant 4h_{i+1}^* + 1, \quad 1 \leqslant i \leqslant n-2.$$

Consequently, since $G \in \mathcal{G}(1, 1, h_i^*)$, we have

$$\begin{aligned} \nu(G) &\geqslant 4h_1^* + 1 \\ &\geqslant 4(4h_2^* + 1) + 1 \\ &\geqslant 4(4(4h_3^* + 1) + 1) + 1 \\ &\vdots \\ &\geqslant 4^{n-1}h_{n-1}^* + \frac{4^{n-1} - 1}{3}. \end{aligned} \tag{2.2}$$

We next prove that

$$h_{n-1}^* \geqslant 2(n+k) + \frac{1}{2}[3 + (-1)^{n+k+1}]. \tag{2.3}$$

By Lemma 2.3, $\delta\left(H_{n-1}^*\right) \geqslant n + k$. Further, since $\overline{G} \in \mathcal{G}(n, n, k)$ we also have $\delta\left(\overline{H}_{n-1}^*\right) \geqslant n + k$. Consequently, $h_{n-1}^* \geqslant 2(n+k) + 1$, proving (2.3) when $n + k$

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is even. When $n+k$ is odd, then at least one of H^*_{n-1} or $\overline{H}^*_{n-1}$ contains a vertex of degree at least $n+k+1$ and thus (2.3) also holds. Now (2.2) and (2.3) together yield (2.1). This completes the proof of the theorem. □

COROLLARY. *Let* $t = \min\{m, n\}$ *and* $k' = k + |m-n|$. *Then*

$$p(m, n, k) \geqslant 4^{t-1}\left[2(t+k') + \frac{1}{2}\{3 + (-1)^{t+k'+1}\} + \frac{1}{3}\right] - \frac{1}{3}.$$

PROOF: Since $G \in \mathcal{G}(m, n, k)$ implies that $\overline{G} \in \mathcal{G}(n, m, k)$, we have $p(m, n, k) = p(n, m, k)$. Hence

$$\begin{aligned} p(n, m, k) &= p(t, t+|m-n|, k) \\ &\geqslant p(t, t, k') \qquad \text{(by Lemma 2.2)} \\ &\geqslant 4^{t-1}\left[2(t+k') + \frac{1}{2}\{3 + (-1)^{t+k'+1}\} + \frac{1}{3}\right] - \frac{1}{3} \end{aligned}$$

as required. □

REMARK. A consequence of the proof of Theorem 2.1 is that $\mathcal{G}(n, n, k) \subseteq \mathcal{G}(n-1, n-1, h^*_{n-1})$. For the particular case $n = 2$, $k = 1$, we have $\mathcal{G}(2, 2, 1) \subseteq \mathcal{G}(1, 1, 8)$. Thus, by Lemma 2.1, $p(2, 2, 1) \geqslant 33$ with equality possible only if there exists a strongly regular graph with parameters $(33, 16, 7, 8)$. It is well-known that such a graph does not exist. Hence, $p(2, 2, 1) \geqslant 34$. To date, the smallest graph in $\mathcal{G}(2, 2, 1)$ constructed is the Paley graph on 61 vertices.

3. PALEY GRAPHS

The Paley graph G_q of order $q \equiv 1 \pmod 4$, q a prime power, was defined in the introduction. Observe that G_q is self-complementary. Further, see [6], it is strongly regular with parameters $(4t+1, 2t, t-1, t)$ when $q = 4t+1$. In [2] we proved that $G_q \in \mathcal{G}(m, n, k)$ for all $q > \{(t-3)2^{t-1}+2\}\sqrt{q} + (t+2k-1)2^{t-1} - 1$, where t is an integer satisfying $t \geqslant m+n$. In this section we prove a sharper result for the case when one of m or n is 1.

We make use of the following basic notation and terminology. Let $\mathbb{F}_q$ be a finite field of order q, where q is a prime power.

A *character* χ on $\mathbb{F}_q^*$, the multiplicative group of the non-zero elements of $\mathbb{F}_q$, is a map from $\mathbb{F}_q^*$ to the multiplicative group of complex numbers with $|\chi(x)| = 1$ for all x and with

$$\chi(xy) = \chi(x)\chi(y)$$

for any $x, y \in \mathbb{F}_q^*$. Since $\chi(1) = \chi(1)\chi(1)$, we have $\chi(1) = 1$.

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Among the characters of $\mathbb{F}_q^*$ we have the *principal character* χ_0 defined by $\chi_0(x) = 1$ for all $x \in \mathbb{F}_q^*$; all other characters of $\mathbb{F}_q^*$ are called *non-principal.*

It will be convenient to extend the definition of non-principal character χ to the whole $\mathbb{F}_q$ by putting $\chi(0) = 0$.

If χ is a non-principal character on $\mathbb{F}_q$, it is well-known (see [10]) that

$$\sum_{x \in \mathbb{F}_q} \chi(x) = 0. \tag{3.1}$$

It follows that, for $a \in \mathbb{F}_q$

$$\sum_{x \in \mathbb{F}_q} \chi(x - a) = 0. \tag{3.2}$$

The following lemma, due to Schmidt [11], is very useful in our work.

LEMMA 3.1. *Let χ be a non-principal character on $\mathbb{F}_q$. If $a_1, a_2, \ldots, a_s$ are distinct elements of $\mathbb{F}_q$, then*

$$\left| \sum_{x \in \mathbb{F}_q} \chi\{(x - a_1)(x - a_2)\ldots(x - a_s)\} \right| \leqslant (s - 1)\sqrt{q}.$$

Let q be a power of an odd prime. We define a *quadratic (residue) character* η on $\mathbb{F}_q$ by

$$\eta(a) = a^{(q-1)/2}, \qquad \text{for all } a \in \mathbb{F}_q.$$

Equivalently, η is 1 on squares, 0 at 0, and -1 otherwise. Therefore, η is a non-principal character.

The following lemma was proved in [2].

LEMMA 3.2. *If $a_1, a_2, \ldots, a_s$ are distinct elements of $\mathbb{F}_q$ and s is even, then*

$$\sum_{x \in \mathbb{F}_q} \eta\{(x - a_1)(x - a_2)\ldots(x - a_s)\} = -1 \pm \sum_{x \in \mathbb{F}_q} \eta\{(x + b_1)(x + b_2)\ldots(x + b_{s-1})\}$$

for some distinct elements $b_1, b_2, \ldots, b_{s-1}$ of $\mathbb{F}_q$.

Using (3.2) and Lemma 3.1 we have the following corollaries to Lemma 3.2.

COROLLARY 1. *For $a, b \in \mathbb{F}_q$ with $a \neq b$*

$$\sum_{x \in \mathbb{F}_q} \eta\{(x - a)(x - b)\} = -1.$$

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COROLLARY 2. *Let $a_1, a_2, \ldots, a_s$ be distinct elements of $\mathbb{F}_q$. Then for even s*

$$\left|\sum_{x\in\mathbb{F}_q}\eta\{(x-a_1)(x-a_2)\ldots(x-a_s)\}\right| \leqslant 1+(s-2)\sqrt{q}.$$

In addition to the above, we need the following lemma.

LEMMA 3.3. *Let $a \in V(G_q)$ and $B = \{b_1, b_2, \ldots, b_n\}$ a subset of n vertices of $G-a$. Put*

$$g = \sum_{x\in\mathbb{F}_q}\{1+\eta(x-a)\}\prod_{i=1}^{n}\{1-\eta(x-b_i)\}.$$

As usual, an empty product is defined to be 1. Then

$$g \geqslant q-\{(n-2)2^n+2\}\sqrt{q}-\{2^n-2n-1\}.$$

PROOF: We can write

$$\begin{aligned} g = &\sum_{x\in\mathbb{F}_q}1+\sum_{x\in\mathbb{F}_q}\left\{\eta(x-a)-\sum_{i=1}^{n}\eta(x-b_i)\right\}\\ &+\sum_{x\in\mathbb{F}_q}\left\{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\eta\{(x-b_i)(x-b_j)\}-\sum_{i=1}^{n}\eta\{(x-a)(x-b_i)\}\right\}\\ &+\ldots+\sum_{x\in\mathbb{F}_q}\prod_{i=1}^{n}\eta\{(x-a)(b-b_i)\}. \end{aligned}$$

Observe that the first term of the expression is equal to q and from (3.2) the second term is 0. Using Corollary 1 of Lemma 3.2 the third term of the above expansion is equal to $n-\binom{n}{2} = (3n-n^2)/2$. Hence,

$$\begin{aligned} \left|g-q-\frac{1}{2}(3n-n^2)\right| \leqslant &\left|\sum_{x\in\mathbb{F}_q}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{k=j+1}^{n+1}\eta\{(x-c_i)(x-c_j)(x-c_k)\}\right|\\ &+\left|\sum_{x\in\mathbb{F}_q}\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\sum_{\ell=k+1}^{n+1}\eta\{(x-c_i)(x-c_j)(x-c_k)(x-c_\ell)\}\right| \end{aligned}$$

(3.3)

$$+\ldots+\left|\sum_{x\in\mathbb{F}_q}\prod_{i=1}^{n+1}\eta(x-c_i)\right|,$$

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where $\{c_1, c_2, \ldots, c_{n+1}\} = B \cup \{a\}$.

Now Lemma 3.1 and Corollary 2 of Lemma 3.2 together imply that

$$\left|\sum_{x\in\mathbb{F}_q}\sum_{i_1<i_2<\ldots<i_s}\eta\{(x-c_{i_1})(x-c_{i_2})\ldots(x-c_{i_s})\}\right| \leqslant \begin{cases}\binom{n+1}{s}(s-1)\sqrt{q}, & \text{if } s \text{ is odd},\\ \binom{n+1}{s}\{1+(s-2)\sqrt{q}\}, & \text{if } s \text{ is even}.\end{cases} \tag{3.4}$$

Making use of (3.4) we get from (3.3)

(3.5)

$$\begin{aligned}\left|g-q-\frac{1}{2}(3n-n^2)\right| &\leqslant \sum_{\substack{s=3\\ s \text{ odd}}}^{n+1}\binom{n+1}{s}(s-1)\sqrt{q}+\sum_{\substack{s=4\\ s \text{ even}}}^{n+1}\binom{n+1}{s}\{1+(s-2)\sqrt{q}\}\\ &= \left\{\sum_{s=3}^{n+1}\binom{n+1}{s}s-\sum_{s=3}^{n+1}\binom{n+1}{s}-\sum_{\substack{s=4\\ s \text{ even}}}^{n+1}\binom{n+1}{s}\right\}\sqrt{q}+\sum_{\substack{s=4\\ s \text{ even}}}^{n+1}\binom{n+1}{s}\\ &= \{(n-2)2^n+2\}\sqrt{q}+2^n-\frac{1}{2}(n^2+n)-1.\end{aligned}$$

Hence

$$\begin{aligned}g &\geqslant q+\frac{1}{2}(3n-n^2)-\{(n-2)s^n+2\}\sqrt{q}-\{2^n-\frac{1}{2}(n^2+n)-1\}\\ &= q-\{(n-2)2^n+2\}\sqrt{q}-\{2^n-2n-1\},\end{aligned}$$

as required. □

Before stating and proving our main result of this section we make the following observation. If a and b are any vertices of G_q, $q \equiv 1 \pmod 4$ a prime power, then

$$\eta(a-b) = \begin{cases}1, & \text{if } a \text{ adjacent to } b,\\ 0, & \text{if } a = b,\\ -1, & \text{otherwise.}\end{cases}$$

Further, $\eta(-a) = \eta(a)$ for any $a \in \mathbb{F}_q$.

THEOREM 3.1. *Let $q \equiv 1 \pmod 4$ be a prime power and k be a positive integer. If*

$$g > \{(n-2)2^n+2\}\sqrt{q}+(n+2k-1)2^n-2n-1, \tag{3.6}$$

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then $G_q \in \mathcal{G}(1, n, k) \cap \mathcal{G}(n, 1, k)$.

PROOF: Since G_q is a self-complementary graph, it is sufficient to prove that $G_q \in \mathcal{G}(1, n, k)$. Let a be any vertex of G_q and $B = \{b_1, b_2, \ldots, b_n\}$ a set of n vertices of G_q so that $a \notin B$. Then $n(a/b_1, b_2, \ldots, b_n) \geqslant k$ if and only if

$$f = \sum_{\substack{x \in \mathbb{F} \\ x \notin \{a\} \cup B}} \{1 + \eta(x - a)\} \prod_{i=1}^{n} \{1 - \eta(x - b_i)\} \geqslant k2^{n+1}.$$

To show that $f \geqslant k2^{n+1}$, it is clearly sufficient to establish that $f > (k-1)2^{n+1}$. Let

$$g = \sum_{x \in \mathbb{F}_q} \{1 + \eta(x - a)\} \prod_{i=1}^{n} \{1 - \eta(x - b_i)\}.$$

From Lemma 3.3 we have

$$g \geqslant q - \{(n-2)2^n + 2\}\sqrt{q} - \{2^n - 2n - 1\}.$$

Now consider

$$g - f = \sum_{x \in \{a\} \cup B} \{1 + \eta(x - a)\} \prod_{i=1}^{n} \{1 - \eta(x - b_i)\}. \tag{3.7}$$

If $g - f \neq 0$, then for some y the product

$$\{1 + \eta(y - a)\} \prod_{i=1}^{n} \{1 - \eta(y - b_i)\} \neq 0. \tag{3.8}$$

If $y = a$, then for (3.8) to hold we must have $\eta(a - b_i) = -1$ for all i. Hence, the term in (3.7) with $x = b_i$ contributes zero to the sum. Therefore, $g - f = 2^n$, since each factor is 2 and one factor is 1. If $y = b_j$ for some j, then for (3.8) to hold we must have $\eta(b_j - a) = 1$. Hence, the term in (3.7) with $x = a$ contributes zero to the sum and the term with $x = b_j$ contributes 2^n. Thus we conclude that $g - f \leqslant n2^n$. So

$$\begin{aligned} f &\geqslant g - n2^n \\ &\geqslant q - \{(n-2)2^n + 2\}\sqrt{q} - \{2^n - 2n - 1\} - n2^n \\ &= q - \{(n-2)2^n + 2\}\sqrt{q} - \{(n+1)2^n - 2n - 1\}. \end{aligned}$$

Now if inequality (3.6) holds, then $f > (k-1)2^{n+1}$ as required. Since a and B are arbitrary, this completes the proof. □

We have the following three corollaries to Theorem 3.1.

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COROLLARY 1. *Let $q \equiv 1 \pmod 4$ be a prime power and k a positive integer. Then $G_q \in \mathcal{G}(1, 1, k)$ for every $q \geqslant 4k+1$.*

PROOF: Inequality (3.6) shows that q satisfies $q > 4k-3$. Since $q \equiv 1 \pmod 4$, $q \geqslant 4k+1$. □

COROLLARY 2. *Let $q \equiv 1 \pmod 4$ be a prime power and k be a positive integer. If $q > \left(1+2\sqrt{2k}\right)^2$, then $G_q \in \mathcal{G}(1, 2, k) \cap \mathcal{G}(2, 1, k)$.*

COROLLARY 3. *Let $q \equiv 1 \pmod 4$ be a prime number and k be a positive integer. If $q > \left(5+\sqrt{16k+34}\right)^2$, then $G_q \in \mathcal{G}(1, 3, k) \cap \mathcal{G}(3, 1, k)$.*

REMARK 1. From Lemma 2.1 and Corollary 1 it follows that Theorem 3.1 is best possible for $n = 1$.

REMARK 2. We have verified, by computer, that if $q \equiv 1 \pmod 4$ is a prime power less than or equal to 1009 and k is a positive integer with $q < \left(1+2\sqrt{2k}\right)^2$, then $G_q \notin \mathcal{G}(1, 2, k)$. We conjecture that this is true for all q. We can choose a, b_1 and b_2 in the proof of Theorem 3.1 so that $g - f = 8$ and hence, by (3.5)

$$f = g - 8 \leqslant q + 2\sqrt{q} + 1 - 8.$$

Consequently, $f < 8k$ for $q < \left(-1+2\sqrt{2(k+1)}\right)^2$. So the problem is to look at $\left(-1+2\sqrt{2(k+1)}\right)^2 \leqslant q \leqslant \left(1+2\sqrt{2k}\right)^2$.

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