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ON PROJECTIVE LIFT AND ORBIT SPACES

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By constructing the projective lift of a *dp*-epimorphism, we find a covariant functor E from the category $\mathcal{C}_d$ of regular Hausdorff spaces and continuous *dp*-epimorphisms to its coreflective subcategory $\mathcal{E}_d$ consisting of projective objects of $\mathcal{C}_d$. We use E to show that $E(X/G)$ is homeomorphic to EX/G whenever G is a properly discontinuous group of homeomorphisms of a locally compact Hausdorff space X and X/G is an object of $\mathcal{C}_d$.

1. INTRODUCTION

Throughout the paper all spaces are regular Hausdorff and maps are continuous epimorphisms. By X, Y we denote spaces and for $A \subseteq X$, $\mathrm{Cl}\, A$ and $\mathrm{Int}\, A$ mean the closure of A and the interior of A in X respectively. The complete Boolean algebra of regular closed sets of a space X is denoted by $R(X)$ and the Stone space of $R(X)$ by $S(R(X))$. The pair (EX, h_X) is the projective cover of X, where EX is the subspace of $S(R(X))$ having convergent ultrafilters as its members and h_X is the natural map from EX to X, sending $\mathcal{F}$ to its point of convergence $\bigcap \mathcal{F}$ (a singleton is identified with its member). We recall here that $\{\vartheta(F) \mid F \in R(X)\}$ is an open base for the topology on EX, where $\vartheta(F) = \{\mathcal{F} \in EX \mid F \in \mathcal{F}\}$ [**8**].

The *projective lift* of a map $f\colon X \to Y$ (if it exists) is the unique map $Ef\colon EX \to EY$ satisfying $h_Y \circ Ef = f \circ h_X$. In [4], Henriksen and Jerison showed that when X, Y are compact spaces, then the projective lift Ef of f exists if and only if f satisfies the following condition, hereafter called the *HJ-condition*, holds:

$$\mathrm{Cl}\,\mathrm{Int}\, f^{-1}(F) = \mathrm{Cl}\; f^{-1}(\mathrm{Int}\, F), \text{ for each } F \text{ in } R(Y).$$

A condition for a map to satisfy equivalent to the HJ-condition is that it is a c-map: a map $f\colon X \to Y$ satisfying $\mathrm{Cl}\left(f^{-1}(Y - F)\right) = X$ for all F in $R(Y)$ [**6**, 6G3]. Since the composition of c-maps need not be a c-map, we consider a special class of c-maps and obtain a category for our purpose.

DEFINITION 1.1: A map $f\colon X \to Y$ is called a *density preserving map* (*dp-map*) if $\mathrm{Int}\,\mathrm{Cl}\, f(A)$ remains nonempty whenever $\mathrm{Int}\, A \neq \emptyset$, $A \subseteq X$.

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A *dp*-map is a *c*-map. The converse need not be true. The composition of a *c*-map and a *dp*-map need not be a *c*-map. However, the composition of *dp*-maps is a *dp*-map.

In Section 2, we construct the projective lift of a *dp*-map. Denoting the category of regular Hausdorff spaces and continuous epimorphisms by $\mathcal{C}_d$ and by $\mathcal{E}_d$ its full subcategory having extremally disconnected spaces for objects we find that $\mathcal{E}_d$ is a coreflective subcategory of $\mathcal{C}_d$ and $E\colon \mathcal{C}_d \to \mathcal{E}_d$, sending X to EX and $f\colon X \to Y$ to $Ef\colon EX \to EY$ is a covariant functor.

Mioduszewski and Rudolf call a map $f\colon X \to Y$, where X, Y are Hausdorff and f not necessarily continuous, *skeletal* if $\operatorname{Int} f^{-1}(\operatorname{Cl} V) \subseteq \operatorname{Cl} f^{-1}(V)$ for each open set V of Y [**5**]. It may be noted that the *dp*-ness of a map is equivalent to it being a skeletal map. In [**5**] the authors do talk about the possibility of considering the functor E when the morphisms are restricted to skeletal maps. However, neither any construction of the lift Ef is cited (we give one in Section 2) nor any property or application of the lift is studied like we do here in Section 3.

By inducing an action of a discrete group G on EX, where X is a G-space, Azad and Agrawal [**1**] showed that if G is finite, then $E(X/G)$ is homeomorphic to the orbit space EX/G. An example to show that the above result may fail if G is infinite, is provided in [**1**]. Here in Section 3, by showing the projective lift Ef of a *dp*-map f from a locally compact space to a space to be an open map, we study the projective cover of an orbit space X/G, when G is an infinite discrete group. We prove that if G is a properly discontinuous group of homeomorphisms [**7**] of a locally compact Hausdorff space X, then $E(X/G)$ is homeomorphic to EX/G, whenever X/G is an object of $\mathcal{C}_d$. For terms and notation not explained we refer to [**2, 6, 7, 8**].

2. Projective lift

Lemma 2.1. *Let $f\colon X \to Y$ be a dp-epimorphism. Then $\operatorname{Cl} f(F) \in R(Y)$ whenever $F \in R(X)$.*

Proof: The result is immediate if $F = \emptyset$. Let $F \neq \emptyset$. Obviously $\operatorname{Cl}\operatorname{Int} f(F) \subseteq \operatorname{Cl} f(F)$. For the reverse inclusion, we assume that $G \equiv Y - \operatorname{Cl}\operatorname{Int}\operatorname{Cl} f(F)$ intersects $\operatorname{Cl} f(F)$. Then $[f^{-1}(G) \cap \operatorname{Int} F] \neq \emptyset$. Set $H = \operatorname{Cl}\left(f^{-1}(G) \cap \operatorname{Int} F\right)$. Since f is a *dp*-map and G is a regular open set we have $\emptyset \neq \operatorname{Int}\operatorname{Cl} f(H) \subseteq G \cap \operatorname{Int}\operatorname{Cl}(F) = \emptyset$, a contradiction. ∎

Note 2.2: The following results follow from the above lemma:

(a) A closed *dp*-map is *RC*-preserving [**3**], that is, maps a regular closed set to a regular closed set.

(b) Let $f\colon X \to Y$ be a *dp*-epimorphism. Then for a regular closed set H of Y we have $\operatorname{Cl} f\left(\operatorname{Cl} f^{-1}(\operatorname{Int} H)\right) = H$ and hence $R(Y) = \{\operatorname{Cl} f(F) \mid F \in R(X)\}$.

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THEOREM 2.3. *Let $f: X \to Y$ be a dp-epimorphism. Then the map $Ef: EX \to EY$ sending $\mathcal{F} \in EX$ to the fixed ultrafilter $f_{\#}(\mathcal{F}) = \{\operatorname{Cl} f(F) \mid F \in \mathcal{F}\}$, is the projective lift of f.*

PROOF: Let $H \in R(Y)$. If $H \cap \operatorname{Cl} f(F) \neq \emptyset$, for all $F \in \mathcal{F}$, then $\operatorname{Cl} f^{-1}(\operatorname{Int} H) \in \mathcal{F}$ and hence by note 2.2(b), $H \in f_{\#}(\mathcal{F})$. Thus $f_{\#}(\mathcal{F}) \in S(R(Y))$. Further $f(\bigcap \mathcal{F}) = \bigcap f_{\#}(\mathcal{F})$ proves that $f_{\#}(\mathcal{F}) \in EY$. Hence we have a map $Ef: EX \to EY$ defined by $Ef(\mathcal{F}) = f_{\#}(\mathcal{F})$. The commutativity of the following diagram establishes that Ef is an epimorphism:

$$\begin{array}{ccc} EX & \longrightarrow & EY \\ \downarrow{\scriptstyle h_X} & & \downarrow{\scriptstyle h_Y} \\ X & \longrightarrow & Y \end{array}$$

For continuity, one may observe that $Ef\big(\vartheta(\operatorname{Cl} f^{-1}(\operatorname{Int} H))\big) \subseteq \vartheta(H)$ for all H in $R(Y)$. Finally, the uniqueness of Ef follows by recalling that a *dp*-map is a *c*-map [**5**, 6G3]. ∎

PROPOSITION 2.4. *Let C_d be the category of regular Hausdorff spaces and continuous dp-epimorphisms and let $\mathcal{E}_d$ be the full subcategory of C_d having extremally disconnected spaces for objects. Then $\mathcal{E}_d$ is a coreflective subcategory of C_d such that $E: C_d \to \mathcal{E}_d$ assigning EX to X and $Ef: EX \to EY$ to $f: X \to Y$ is a covariant functor.*

PROOF: By proving $\operatorname{Cl} Ef(\vartheta(F)) = \vartheta(\operatorname{Cl} f(F))$ for all F in $R(X)$ we establish that the projective lift Ef of a *dp*-epimorphism f is a *dp*-epimorphism. The rest of the proof is left to the reader. ∎

REMARK 2.5. (a) A map $f: X \to Y$ is a *dp*-map if and only if the following stronger version of the HJ-condition holds: for every F in $R(Y)$, $\operatorname{Cl} \operatorname{Int} f^{-1}(F) = \operatorname{Cl} f^{-1}(d \operatorname{Int} F)$, where $d \operatorname{Int} F$ denotes an open set of X which is dense in F.
(b) The class of non *dp*-maps satisfying the HJ-condition consists precisely of the composition maps $f \circ g$, where g is a *dp*-map and f is a non *dp*-map satisfying the HJ-condition.

3. PROJECTIVE COVER OF AN ORBIT SPACE

PROPOSITION 3.1. *Let f be a dp-map from a locally compact space X to a space Y. Then the projective lift Ef is an open map.*

PROOF: It is sufficient to show that $Ef(\vartheta(F))$ is open for all $F \in R(X)$. Choose $Ef(\mathcal{F}) \in Ef(\vartheta(F))$. Since X is locally compact, there is a compact regular closed set K in X such that $K \subseteq F$ and $K \in \mathcal{F}$. By Lemma 2.1, $f(K) \in R(Y)$. We complete the proof by showing that $Ef(\mathcal{F}) \in \vartheta(f(K)) \subseteq Ef(\vartheta(F))$. Clearly $Ef(\mathcal{F}) \in \vartheta(f(K))$. Next we choose an $\mathcal{H}$ in $\vartheta(f(K))$ and observe that any finite subfamily of

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$\mathcal{A} \equiv \{K\} \cup \mathrm{Cl}\, f^{-1}(\mathrm{Int}\, H) \mid H \in \mathcal{H}\}$ has nonempty meet. Moreover, compactness of K guarantees that $\bigcap \mathcal{A} \neq \varphi$. Thus there is a $\mathcal{G}$ in EX containing $\mathcal{A}$. Since $\mathcal{G}$ is a filter containing K and F contains K so $\mathcal{G} \in \vartheta(F)$. Finally by maximality of $\mathcal{H}$, we get $Ef(\mathcal{G}) = \mathcal{H}$, this proves $\mathcal{H} \in Ef(\vartheta(F))$. ∎

REMARK 3.2. It is easy to show that the projective lift of the inclusion of the rationals into the reals is not open. This therefore justifies local compactness in the above proposition.

For a G-space X, the orbit $\{g.x \mid g \in G\}$ of x in X is denoted by Gx and for $A \subseteq X$, the set $\{g.a \mid a \in A\}$, where $g \in G$, is denoted by $g.A$. Thus if G is discrete, by defining $g.\mathcal{F} \equiv \{g.F \mid F \in \mathcal{F}\}$ for an $\mathcal{F}$ in EX, one induces an action of G on EX [1]. We denote a member of $E(X/G)$ by $\mathcal{F}_{Gx}$ and assume that the same converges to Gx in X/G.

THEOREM 3.3. *Let G be a properly discontinuous group of homeomorphisms on a locally compact Hausdorff space X such that X/G is Hausdorff. Then $E(X/G)$ is homeomorphic to EX/G.*

PROOF: Let $p\colon X \to X/G$ and $q\colon EX \to EX/G$ be the orbit maps. Since p is an open map, it is a dp-map and hence the projective lift Ep of p exists. Further, surjectivity of Ep is obvious and hence for $\mathcal{F}_{Gx} \in E(X/G)$ there exists an $\mathcal{F} \in EX$ such that $Ep(\mathcal{F}) = \mathcal{F}_{Gx}$. By claiming that $Ep^{-1}(\mathcal{F}_{Gx}) = \{g.\mathcal{F} \mid g \in G\}$, the orbit $G\mathcal{F}$ of $\mathcal{F}$ in EX, we define a map $\eta\colon E(X/G) \to EX/G$ by $\eta(\mathcal{F}_{Gx}) = q(\mathcal{F})$ and observe that η is a homeomorphism. In fact, fibers of Ep being precisely the orbits gives us the bijectivity of η. The bicontinuity of η is established by noting (i) openness of the maps Ep and q; and (ii) $\eta \circ Ep = q$. ∎

PROOF OF THE CLAIM: $Ep^{-1}(\mathcal{F}_{Gx}) = \{g.\mathcal{F} \mid g \in G\}$.

From the definition of Ep, we obtain $\{g.\mathcal{F} \mid g \in G\} \subseteq Eq^{-1}(\mathcal{F}_{Gx})$. Let $\mathcal{H} \in Ep^{-1}(\mathcal{F}_{Gx})$ be such that $\mathcal{H} \neq g.\mathcal{F}$ for any g in G. Then we complete the proof by obtaining a contradiction.

$Ep(\mathcal{H}) = \mathcal{F}_{Gx} \Rightarrow \bigcap \mathcal{H} \in Gx$. Since an element lies in a fibre of the map Ep if and only if the entire orbit determined by that element is contained in that fibre, we thus have $\{g\mathcal{H} \mid g \in G\} \subseteq Ep^{-1}(\mathcal{F}_{Gx})$ and hence we can assume $\bigcap \mathcal{H} = x$. As G is a properly discontinuous group of homeomorphisms on X, we can choose an open set V containing x such that $\{g.\mathrm{Cl}\, V \mid g \in G\}$ is a pairwise disjoint family of regular closed sets of X. Also, as $\mathcal{F} \neq \mathcal{H}$, there is a K in $R(X)$ such that $K \in F$ and $K^c \in \mathcal{H}$. Now we set $K_1 = K \wedge \mathrm{Cl}\, V$ and $K_2 = K^c \wedge \mathrm{Cl}\, V$ and observe (i) $K_1 \in \mathcal{F}$ and $K_2 \in \mathcal{H}$; and (ii) for all g, h in G, $g.K_1 \wedge h.k_2 = \emptyset$. But this gives us invariant regular closed sets $H_1 \equiv \bigvee_{g \in G} g.K_1$ and $H_2 \equiv \bigvee_{g \in G} g.K_2$. Hence $p(H_1)$ and $p(H_2)$ are in $R(X/G)$. Also, $H_1 \in \mathcal{F}$; $H_2 \in \mathcal{H}$ and $H_1 \wedge H_2 = \emptyset$. Finally, $p^{-1}(\mathrm{Int}\, p(H_i)) = \mathrm{Int}\, H_i$; $i = 1, 2$

gives $p(H_1) \wedge p(H_2) = \emptyset$, and therefore $Ep(\mathcal{H}) \neq Ep(\mathcal{F})$. This contradicts the choice of $\mathcal{H}$. ∎

CORLLLARY 3.4. *Let X and G be as in Theorem 3.3. Then the projective cover of X/G is $(EX/G, h)$, where $h\colon EX/G \to X/G$ is defined by $h(G\mathcal{F}) = G(\bigcap \mathcal{F})$, the orbit in X determined by $\bigcap \mathcal{F}$.*

REMARK 3.5. Let (X, d) be a locally compact metric space and let G be a group of isometries acting on X such that there exists a $\delta > 0$ satisfying $d(g.x, h.x) > \delta$ for all g, h in G, $g \neq h$ and $x \in X$. Then X/G is metrisable and by Theorem 3.3, $E(X/G)$ is homeomorphic to EX/G.

EXAMPLE 3.6. From the above Remark it follows that for an n-torus T^n, $E(T^n)$ is homeomorphic to $E(\mathbb{R}^n)\mathbb{Z}^n$.

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