THEORETICAL PEARLS

Representing 'undefined' in lambda calculus

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Abstract

Let ψ be a partial recursive function (of one argument) with λ-defining term $F \in \Lambda^\circ$. This means

$$\psi(n) = m \Leftrightarrow F \ulcorner n \urcorner =_\beta \ulcorner m \urcorner.$$

There are several proposals for what $F \ulcorner n \urcorner$ should be in case $\psi(n)$ is undefined: (1) a term without a normal form (Church); (2) an unsolvable term (Barendregt); (3) an easy term (Visser); (4) a term of order 0 (Statman).

These four possibilities will be covered by one 'master' result of Statman which is based on the 'Anti Diagonal Normalization Theorem' of Visser (1980). That ingenious theorem about precomplete numerations of Ershov is a powerful tool with applications in recursion theory, metamathematics of arithmetic and lambda calculus.

1 Introduction

This paper presents a general theorem of Statman about λ-definability of the partial recursive functions. It analyses, for partial recursive functions that are undefined at some argument n, what is the behaviour of the representing λ-term applied to the corresponding numeral $\ulcorner n \urcorner$. The result is an application of Visser's Anti Diagonal Normalization Theorem for precomplete numerations of Ershov.

The paper is self-contained, except that some elementary facts and notations from recursion theory and λ-calculus are used (see Rogers, 1967, and Barendregt, 1984, if necessary).

Notation

  (i) $\mathbb{N}$ is the set of natural numbers $\{0, 1, 2, \ldots\}$.

  (ii) $\mathscr{PR}$ is the set of unary partial recursive functions from $\mathbb{N}$ to $\mathbb{N}$. If $\psi \in \mathscr{PR}$ then $\psi(n) \downarrow$ denotes that $\psi(n)$ is defined; $\psi(n) \uparrow$ denotes that $\psi(n)$ is undefined.

  (iii) $\mathscr{R}$ is the set of unary total recursive functions.

  (iv) Λ is the set of λ-terms and Λ° is the set of closed λ-terms. $\ulcorner n \urcorner \equiv \lambda fx. f^n x$ are Church's numerals. $M =_\beta N$ means that M and N are β-convertible.

(v) If $M \in \Lambda^\circ$, then $\#M \in \mathbb{N}$ is its code number (according to some effective coding), and $\ulcorner M \urcorner =_{\text{Def}} \ulcorner \#M \urcorner$ is the corresponding numeral. There exists a self-interpreter $\mathbf{E} \in \Lambda^\circ$ such that $\forall M \in \Lambda^\circ \, \mathbf{E} \ulcorner M \urcorner =_\beta M$ (see e.g. Barendregt, 1991).

Theorem 1.1 (λ-definability of the recursive functions; Kleene, 1936)
$$\forall f \in \mathscr{R} \, \exists F \in \Lambda^\circ \, \forall n \in \mathbb{N} \, F \ulcorner n \urcorner =_\beta \ulcorner f(n) \urcorner.$$

Definition 1.2
Let $\psi \in \mathscr{PR}$ and $\mathscr{A} \subseteq \Lambda^\circ$. Then ψ is said to be λ-*definable w.r.t.* $\mathscr{A}$ *as set of undefined elements* if for some $F \in \Lambda^\circ$ one has for all $n \in \mathbb{N}$

$$\psi(n) \downarrow \Rightarrow F \ulcorner n \urcorner =_\beta \ulcorner \psi(n) \urcorner \qquad (1)$$

$$\psi(n) \uparrow \Rightarrow F \ulcorner n \urcorner \in \mathscr{A}. \qquad (2)$$

In this definition the elements of $\mathscr{A}$ are used as a representation of 'undefined'. It is natural to require that $\mathscr{A} \cap \{\ulcorner n \urcorner \mid n \in \mathbb{N}\} = \emptyset$. Then it follows immediately that for all $n, m \in \mathbb{N}$ one has

$$\psi(n) = m \Leftrightarrow F \ulcorner n \urcorner =_\beta \ulcorner m \urcorner \qquad (3)$$

$$\psi(n) \uparrow \Leftrightarrow F \ulcorner n \urcorner \in \mathscr{A}. \qquad (4)$$

Definition 1.3
(i) $M \in \Lambda^\circ$ is called *solvable* $\Leftrightarrow M$ has a head normal form; otherwise M is called *unsolvable*.
(ii) $M \in \Lambda^\circ$ is called *easy* $\Leftrightarrow \forall N \in \Lambda^\circ \, \lambda + M = N$ is consistent.
(iii) $M \in \Lambda$ is called of *order* $0 \Leftrightarrow \forall N \in \Lambda \, M \neq_\beta \lambda x . N$.

Examples 1.4
(i) $\mathbf{Y} \equiv \lambda f . (\lambda x . f(xx))(\lambda x . f(xx))$ is solvable; $\mathbf{YI} =_\beta (\lambda x . xx)(\lambda x . xx)$ is unsolvable.
(ii) $\mathbf{K} \equiv \lambda xy . x$ is not easy, because $\lambda + \mathbf{K} = \mathbf{KI} \vdash P = Q$ for arbitrary $P, Q \in \Lambda$.
(iii) $(\lambda x . xx)(\lambda x . xx)$ is of order 0; $\mathbf{YK}$ is not of order 0.

Theorem 1.5
All $\psi \in \mathscr{PR}$ can be λ-defined w.r.t. each of the following sets $\mathscr{A}$ as undefined elements.
(i) $\mathscr{A} = \{M \in \Lambda^\circ \mid M \text{ has no normal form}\}$ (Church, 1941).
(ii) $\mathscr{A} = \{M \in \Lambda^\circ \mid M \text{ is unsolvable}\}$ (Barendregt, 1971).
(iii) $\mathscr{A} = \{M \in \Lambda^\circ \mid M \text{ is easy}\}$ (Visser, 1980).
(iv) $\mathscr{A} = \{M \in \Lambda^\circ \mid M \text{ is of order } 0\}$ (Statman, 1987).

Each of the results (ii)–(iv) of Theorem 1.5 have been proved by the method of the proof of (i), plus some extra work. The main content of this paper is the following master theorem which captures all cases of Theorem 1.5.

Definition 1.6
A set $\mathscr{B} \subseteq \Lambda^\circ$ is called a *Visser set* if $\mathscr{B}$ is r.e. (after coding, i.e. $\{\#M \mid M \in \mathscr{B}\}$ is r.e.) and $\mathscr{B}$ is closed under $=_\beta$, i.e. $M \in \mathscr{B} \ \& \ M =_\beta N \Rightarrow N \in \mathscr{B}$.

Theorem 1.7 (Statman, 1990)
Let $\mathscr{A} \subseteq \Lambda^\circ$ be non-empty and a co-Visser set (i.e. $B = \Lambda^\circ \backslash \mathscr{A}$ is a Visser set). Then all $\psi \in \mathscr{PR}$ can be λ-defined w.r.t. $\mathscr{A}$ as a set of undefined elements.

Indeed, Theorem 1.5 follows from Theorem 1.7, because each $\mathscr{A}$ is non-empty and has a Visser set as complement.

2 Precomplete numerations

The notion of precomplete numeration comes from Ershov (1973). He also formulated for these the fixed-point Theorem 2.5:
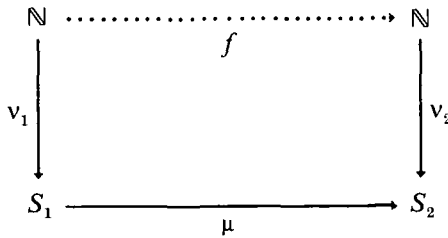
Definition 2.1
   (i) A *numeration* is a pair $\gamma = (v, S)$ with $v : \mathbb{N} \twoheadrightarrow S$ a surjection.
   (ii) Given a numeration $\gamma = (v, S)$ define on $\mathbb{N}$ the following equivalence relation

$$n \sim_\gamma m \Leftrightarrow v(n) = v(m).$$

   (iii) Let $\gamma_1 = (v_1, S_1)$ and $\gamma_2 = (v_2, S_2)$ be two numerations. A map $\mu : S_1 \rightarrow S_2$ is called a *morphism* from γ_1 to γ_2, notation $\mu : \gamma_1 \rightarrow \gamma_2$, if for some $f \in \mathscr{R}$ one has $v_2 \circ f = \mu \circ v_1$; in diagram form:

The intuition behind a numeration $\gamma = (v, S)$ is that the elements of S are somewhat complicated, but have codes in $\mathbb{N}$. If $v(n) = s$, then n is called a code for s. Then $n \sim_\gamma m$ means that n and m code the same object of S. Moreover, $\mu : S_1 \rightarrow S_2$ is a morphism if 'μ can be computed by means of the codes'.

Examples 2.2
   (i) $\Lambda_\beta = (E, \Lambda^\circ / =_\beta)$ with $E(n) = \mathsf{E}^\ulcorner n \urcorner$ is a numeration.
   (ii) $\mathbf{PR} = (\Phi, \mathscr{PR})$, with $\Phi(n) = \phi_n$, the nth partial recursive function, is a numeration.

Definition 2.3
A numeration γ is said to be *precomplete* if

$$\forall \psi \in \mathscr{PR} \, \exists f \in \mathscr{R} \, \forall n \in \mathbb{N} [\psi(n) \downarrow \Rightarrow f(n) \sim_\gamma \psi(n)].$$

Following Visser (1980), we say that f *totalizes* ψ *modulo* $\sim_\gamma$.

Proposition 2.4
   (i) Λ_β *is precomplete.*
   (ii) $\mathbf{PR}$ *is precomplete.*

Proof

(i) Given $\psi \in \mathscr{PR}$ let $F \in \Lambda°$ be a λ-defining term for ψ. Define $f(n) = \#(\mathbf{E} \circ F^{\ulcorner}n^{\urcorner})$. Then if $\psi(n) \downarrow$ one has

$$\mathbf{E}^{\ulcorner}f(n)^{\urcorner} =_{\beta} \mathbf{E}^{\ulcorner}\mathbf{E} \circ F^{\ulcorner}n^{\urcorner\urcorner}$$
$$=_{\beta} \mathbf{E} \circ F^{\ulcorner}n^{\urcorner}$$
$$=_{\beta} \mathbf{E}(F^{\ulcorner}n^{\urcorner})$$
$$=_{\beta} \mathbf{E}^{\ulcorner}\psi(n)^{\urcorner};$$

hence $f(n) \sim_E \psi(n)$.

(ii) Given $\psi \in \mathscr{PR}$ define

$$\theta(n, m) = \phi_{\psi(n)}(m).$$

By the s–m–n theorem one has for some $f \in \mathscr{R}$ and all $n, m \in \mathbb{N}$

$$\theta(n, m) = \phi_{f(n)}(m).$$

Then $\psi(n) \downarrow \Rightarrow \phi_{f(n)} = \phi_{\psi(n)} \Rightarrow f(n) \sim_{\gamma} \psi(n)$ for all $n \in \mathbb{N}$, and we are done. $\square$

Theorem 2.5 (Fixed-point theorem)
Let $\gamma = (v, S)$ be a precomplete numeration. Then

$$\forall f \in R \exists n \in \mathbb{N} f(n) \sim_{\gamma} n.$$

Proof

Given $f \in \mathscr{PR}$ define $\psi(m) = f(\phi_m(m))$. Then $\psi \in \mathscr{PR}$, so there is an $h \in \mathscr{R}$ that totalizes ψ modulo $\sim_{\gamma}$. Let $h = \phi_e$. Then $\phi_e(e) = h(e) = n$, say, is defined, hence also $\psi(e) \downarrow$. It follows that

$$n = h(e) \sim_{\gamma} \psi(e) = f(\phi_e(e)) = f(n).$$

Therefore n satisfies our requirement. $\square$

Corollary 2.6
Let $\gamma = (v, S)$ be precomplete. Let $\mu: S \to S$ be an endomorphism, i.e. $\mu: \gamma \to \gamma$. Then μ has a fixed-point:

$$\exists s \in S \mu(s) = s.$$

Proof

By the definition of morphism there is an $f \in \mathscr{R}$ such that $v \circ f = \mu \circ v$. By the theorem there is an $n \in \mathbb{N}$ such that $f(n) \sim_{\gamma} n$. Then $s = v(n)$ is a fixed-point of μ:

$$\mu(s) = \mu(v(n))$$
$$= v(f(n))$$
$$= v(n)$$
$$= s.   \square$$

Theorem 2.5 implies both the fixed-point theorem of λ-calculus and the recursion theorem.

Corollary 2.7

  (i) (Fixed-point theorem in λ-calculus.)

$$\forall F \in \Lambda \, \exists N \in \Lambda \, FN =_\beta N.$$

  (ii) (Recursion theorem.)

$$\forall f \in \mathcal{R} \, \exists n \in \mathbb{N} \, \phi_{f(n)} = \phi_n.$$

Proof

  (i) Apply Theorem 2.5 to Λ_β. Define $f(n) = \# F(\mathsf{E}^\ulcorner n \urcorner)$. Then $f \in \mathcal{R}$ and by the theorem $n \sim_E f(n)$ for some $n \in \mathbb{N}$. It follows that

$$\mathsf{E}^\ulcorner n \urcorner =_\beta \mathsf{E}^\ulcorner f(n) \urcorner =_\beta \mathsf{E}^\ulcorner F(\mathsf{E}^\ulcorner n \urcorner) \urcorner =_\beta F(\mathsf{E}^\ulcorner n \urcorner);$$

therefore we can take $N \equiv \mathsf{E}^\ulcorner n \urcorner$.

  (ii) By Theorem 2.5 applied to **PR** one has $f(n) \sim_\phi n$ for some $n \in \mathbb{N}$; therefore $\phi_{f(n)} = \phi_n$. $\square$

3 The Anti Diagonal Normalization Theorem

In Visser (1980) the so-called Anti Diagonal Normalization (ADN) Theorem is proved, which is a result about precomplete numerations. Applied to the numeration **PR** it gives a result that roughly satisfies the following equation:

  Gödel sentence : Rosser sentence = recursion theorem : ADN theorem.

Applications of the ADN theorem to metamathematics of arithmetic can be found in Bernardi and Sorbi (1983). Another application is shown in the next section.

The following definition is reminiscent of the construction of Rosser sentences in Peano arithmetic (see, e.g., Kleene, 1952, §42, theorem 29; or Mendelson, 1987, proposition 3.36).

Definition 3.1

Let $Q_1(n)$ and $Q_2(n)$ be r.e. predicates. Then for some binary recursive relations $R_1(n, m)$ and $R_2(n, m)$ one has for $i \in \{1, 2\}$

$$Q_i(n) \Leftrightarrow \exists m \, R_i(n, m).$$

Write $Q_1(n_1) \leqslant Q_2(n_2) \Leftrightarrow \exists m \, R_1(n_1, m) \quad \& \quad \forall m' < m \, \neg \, R_2(n_2, m')$

and $Q_1(n_1) < Q_2(n_2) \Leftrightarrow \exists m \, R_1(n_1, m) \quad \& \quad \forall m' \leqslant m \, \neg \, R_2(n_2, m').$

These definitions are best understood and remembered by noting that

$$Q_1(n_1) \leqslant Q_2(n_2) \Leftrightarrow \mu m . R_1(n_1, m) \leqslant \mu m . R_2(n_2, m)$$

and $$Q_1(n_1) < Q_2(n_2) \Leftrightarrow \mu m . R_1(n_1, m) < \mu m . R_2(n_2, m).$$

Here $\mu m . \ldots$ denotes the operation of taking the least number m such that ... holds. Note that the notions

$$Q_1(n_1) \leqslant Q_2(n_2) \quad \text{and} \quad Q_1(n_1) < Q_2(n_2)$$

are intensional, i.e. they depend on the way the Q_i are given via the R_i.

Lemma 3.2
   (i) $Q_1(n_1) \leqslant Q_2(n_2)$ *and* $Q_1(n_1) < Q_2(n_2)$ *are both r.e. relations in* n_1, n_2.
   (ii) $[Q_1(n_1) \vee Q_2(n_2)] \Rightarrow [Q_1(n_1) \leqslant Q_2(n_2) \vee Q_1(n_1) < Q_2(n_2)]$.

Proof
(i), (ii). These follow immediately from the intuitive description following Definition 3.1.   □

Definition 3.3
Let $\gamma = (\nu, S)$ be a numeration. An *anti diagonal function* (w.r.t. γ) is a $\delta \in \mathcal{PR}$ such that for all $n \in \mathbb{N}$ one has

$$\delta(n) \downarrow \Rightarrow \delta(n) \not\sim_\gamma n.$$

Theorem 3.4 (Anti Diagonal Normalization Theorem; Visser, 1980)
Let $\gamma = (\nu, S)$ *be a precomplete numeration with* $\delta \in \mathcal{PR}$ *an anti diagonal function. Then for all* $\psi \in \mathcal{PR}$ *there exists an* $f \in \mathcal{R}$ *such that for all* $n \in \mathbb{N}$ *one has*

$$\psi(n) \downarrow \Rightarrow f(n) \sim_\gamma \psi(n)$$
$$\psi(n) \uparrow \Rightarrow f(n) \notin \mathrm{dom}\,(\delta).$$

We say that f *totalizes* ψ *modulo* $\sim_\gamma$ *avoiding* $\mathrm{dom}\,(\delta)$.

Proof
Let $\psi \in \mathcal{PR}$ be given. Define $\theta(n) = \phi_n(n)$. Then also $\theta \in \mathcal{PR}$. Since γ is precomplete, there is a $g \in \mathcal{R}$ such that

$$\phi_n(n) \downarrow \Rightarrow g(n) \sim_\gamma \phi_n(n)$$

for all $n \in \mathbb{N}$. By the s–m–n theorem there exists an $s \in \mathcal{R}$ such that

$$\phi_{s(n)}(m) = \begin{cases} \psi(n) & \text{if } \psi(n) \downarrow \leqslant \delta(g(m)) \downarrow, \\ \delta(g(m)) & \text{if } \delta(g(m)) \downarrow < \psi(n) \downarrow, \\ \uparrow & \text{else.} \end{cases}$$

Claim: $\phi_{s(n)}(s(n)) \downarrow \Rightarrow \phi_{s(n)}(s(n)) = \psi(n)$. Indeed, suppose towards a contradiction that $\phi_{s(n)}(s(n)) \downarrow$ but $\phi_{s(n)}(s(n)) = \delta(g(s(n)))$. Then

$$g(s(n)) \sim_\gamma \phi_{s(n)}(s(n)) = \delta(g(s(n)) \not\sim_\gamma g(s(n)),$$

which is impossible. Therefore we have

$$\psi(n) \downarrow \Rightarrow \phi_{s(n)}(s(n)) = \psi(n) \downarrow, \quad \text{by the claim,}$$
$$\Rightarrow g(s(n)) \sim_\gamma \phi_{s(n)}(s(n)) = \psi(n);$$

and on the other hand,

$$\psi(n) \uparrow \Rightarrow \phi_{s(n)}(s(n)) \uparrow, \quad \text{by the claim,}$$
$$\Rightarrow \delta(g(s(n))) \uparrow$$
$$\Rightarrow g(s(n)) \notin \mathrm{dom}\,(\delta).$$

Therefore we can take $f = g \circ s$.   □

Corollary 3.5
Let $\gamma = (\nu, S)$ be a precomplete numeration. Let $B \subseteq \mathbb{N}$ be a non-trivial (i.e. $B \neq \mathbb{N}$) r.e. set closed under $\sim_\gamma$, i.e. such that

$$n \in B \quad \& \quad n \sim_\gamma m \Rightarrow m \in B.$$

Then $\qquad \forall \psi \in \mathscr{PR} \exists f \in \mathscr{R} [f \text{ totalizes } \psi \text{ modulo } \sim_\gamma \text{ avoiding } B].$

Proof
Let $n_0 \notin B$. Define

$$\delta(n) = n_0 \quad \text{if } n \in B,$$
$$= \uparrow \quad \text{otherwise.}$$

Then for all $n \in \mathbb{N}$

$$\delta(n) \downarrow \Rightarrow \delta(n) = n_0 +_\gamma n.$$

Hence δ is an anti diagonal function. Therefore the theorem applies and $\text{dom}(\delta) = B$ can be avoided. $\quad\square$

4 Notions of 'undefined'

Now we can prove Statman's result.

Theorem 4.1
Let $\mathscr{B} \subseteq \Lambda^\circ$ be a non-trivial Visser set. Then

$$\forall \psi \in \mathscr{PR} \exists F \in \Lambda^\circ \forall n \in \mathbb{N} \begin{cases} \psi(n) \downarrow \Rightarrow F^\ulcorner n \urcorner =_\beta \ulcorner \psi(n) \urcorner; \\ \psi(n) \uparrow \Rightarrow F^\ulcorner n \urcorner \notin \mathscr{B}. \end{cases}$$

Proof
We will apply 3.5 to $\Lambda_\beta = (E, \Lambda^\circ/\beta)$ which is a precomplete numeration. Define $B = \{n \mid \mathsf{E}^\ulcorner n \urcorner \in \mathscr{B}\}$. Then B is non-trivial, r.e. and $\sim_E$-closed. Given $\psi \in \mathscr{PR}$ define

$$\psi_1(n) = \#^\ulcorner \psi(n) \urcorner, \quad \text{if } \psi(n) \downarrow;$$
$$= \uparrow, \qquad \text{otherwise.}$$

Then $\mathsf{E}^\ulcorner \psi_1(n) \urcorner =_\beta \ulcorner \psi(n) \urcorner$. There exists an $f_1 \in \mathscr{R}$ that totalizes ψ_1 modulo $\sim_E$ avoiding B, i.e.

$$\psi_1(n) \downarrow \Rightarrow f_1(n) \sim_E \psi(n);$$
$$\psi_1(n) \uparrow \Rightarrow f_1(n) \notin B.$$

Let F_1 λ-define f_1. Then

$$\psi(n) \downarrow \Rightarrow \psi_1(n) \downarrow$$
$$\Rightarrow \mathsf{E} \circ F_1^\ulcorner n \urcorner =_\beta \mathsf{E}^\ulcorner f_1(n) \urcorner =_\beta \mathsf{E}^\ulcorner \psi_1(n) \urcorner =_\beta \ulcorner \psi(n) \urcorner;$$
$$\psi(n) \uparrow \Rightarrow \psi_1(n) \uparrow$$
$$\Rightarrow f_1(n) \notin B$$
$$\Rightarrow \mathsf{E}^\ulcorner f_1(n) \urcorner \notin \mathscr{B}$$
$$\Rightarrow \mathsf{E} \circ F_1^\ulcorner n \urcorner \notin \mathscr{B}.$$

So we can take $F \equiv \mathsf{E} \circ F_1$. $\quad\square$

Corollary 4.2
Let $\mathscr{A} \subseteq \Lambda^\circ$ be one of the following sets:
   (i) *$\mathscr{A} = \{M \in \Lambda^\circ \mid M$ has no normal form$\}$;*
   (ii) *$\mathscr{A} = \{M \in \Lambda^\circ \mid M$ is unsolvable$\}$;*
   (iii) *$\mathscr{A} = \{M \in \Lambda^\circ \mid M$ is easy$\}$;*
   (iv) *$\mathscr{A} = \{M \in \Lambda^\circ \mid M$ is of order 0$\}$.*

Then every $\psi \in \mathscr{PR}$ can be λ-defined by an $F \in \Lambda^\circ$ such that

$$\psi(n) \downarrow \Rightarrow F^\ulcorner n \urcorner =_\beta {}^\ulcorner \psi(n) \urcorner$$
$$\psi(n) \uparrow \Rightarrow F^\ulcorner n \urcorner \in \mathscr{A}.$$

Proof
Let $\mathscr{B} = \Lambda^\circ \backslash \mathscr{A}$. Then $\mathscr{B}$ satisfies the requirements of the theorem.   ☐

The results in this paper on the unary partial recursive functions can be generalized to the k-ary ones. The reason is that k-tuples of numbers or λ-terms can be effectively coded as a single number or λ-term, respectively. Also, decoding is an effective process.

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