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FRACTIONAL INTEGRAL OPERATORS ON α-MODULATION SPACES IN THE FULL RANGE

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Abstract

We use a unified approach to study the boundedness of fractional integral operators on α-modulation spaces and find sharp conditions for boundedness in the full range.

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1. Introduction

Decomposition methods on potential spaces provide a variety of ways to characterise the smoothness of functions and distributions. In addition to Triebel-type spaces, such as classical Sobolev spaces, there are also modulation and Besov spaces. See, for example, [1, 2, 8, 16] for Fourier multipliers on modulation spaces and [3, 26] for partial differential equations on modulation and Besov spaces.

As function spaces associated with uniform and dyadic decompositions, respectively, modulation and (inhomogeneous) Besov spaces can be regarded as special forms of α-modulation spaces. The α-modulation space $M^{s,\alpha}_{p,q}$, introduced by Gröbner [11] in 1992, plays the role of an intermediate space between the modulation space $M^s_{p,q}$ [9] and the inhomogeneous Besov space $B^s_{p,q}$. (Note, however, that it cannot be obtained by complex interpolation between endpoint spaces [13].) More accurately, the modulation space is the special α-modulation space with $\alpha = 0$, and the inhomogeneous Besov space $B^s_{p,q}$ can be regarded as the limit of $M^{s,\alpha}_{p,q}$ as $\alpha \to 1$ (see [11]). For convenience, we use $M^{s,1}_{p,q}$ to denote the inhomogeneous Besov space $B^s_{p,q}$.

We present some properties of α-modulation spaces in Section 2. See also [4, 7, 29] for the study of certain operators on α-modulation spaces and [5, 6, 13–15, 23] for the structure of α-modulation spaces.

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Let $\mathbb{R}^n$ be the Euclidean space of dimension n. The Riesz potential operator of order β is defined as

$$I_\beta f(x) = \mathscr{F}^{-1}(|\xi|^{-\beta}\mathscr{F} f)(x),$$

where $0 < \beta < n$ and $\mathscr{F}$ and $\mathscr{F}^{-1}$ are the Fourier transform and inverse Fourier transform, respectively. I_β can also be represented as a fractional integral operator by

$$I_\beta f(x) = c_{n,\beta} \int_{\mathbb{R}^n} \frac{f(x-y)}{|y|^{n-\beta}}\, dy.$$

From the celebrated Hardy–Littlewood–Sobolev embedding theorem, $1/q = 1/p - \beta/n$ is the sharp conditions for the boundedness of

$$\|I_\beta f\|_{L^q} \lesssim \|f\|_{L^p}, \tag{1.1}$$

where $1 < p < q < \infty$ (see [18]). Recalling that I_β is a potential lifting operator, the inequality (1.1) can be interpreted as trading some of the regularity for integrability.

In this paper, we focus on the boundedness property of I_β for $0 < \beta < n$ in the frame of α-modulation spaces. Sugimoto and Tomita [19, 24] first considered the boundedness of I_β on modulation spaces. With $M_{p,q} = M^{0,0}_{p,q}$, they gave the sharp conditions for boundedness of $I_\beta : M_{p_1,q_1} \to M_{p_2,q_2}$.

Theorem A [19]. *Let $0 < \beta < n$ and $1 < p_i, q_i < \infty$ for $i = 1, 2$. The fractional integral operator $I_\beta : M_{p_1,q_1} \to M_{p_2,q_2}$ is bounded if and only if*

$$\frac{1}{p_2} \leqslant \frac{1}{p_1} - \frac{\beta}{n} \quad \text{and} \quad \frac{1}{q_2} < \frac{1}{q_1} + \frac{\beta}{n}.$$

Observing that the Sugimoto–Tomita theorem does not include the endpoint case $p_2 = \infty$, Zhong and Chen [28] modified the proof to obtain a result for $p_2 = \infty$ (see [28, Theorem 2]). Wu and Chen [27] extended the latter result to α-modulation spaces and obtained the following theorem.

Theorem B [27]. *Let $\alpha n/p_1 < \beta < n$ and $1 < p_1 < \infty$, $0 < q_1, q_2 \leqslant \infty$. The fractional linear operator $I_\beta : M^{0,\alpha}_{p_1,q_1} \to M^{0,\alpha}_{\infty,q_2}$ is bounded when*

$$\frac{1}{p_1} > \frac{\beta}{n} \quad \text{and} \quad \frac{1}{q_2} < \frac{1}{q_1} + \frac{\beta - n\alpha/p_1}{n(1-\alpha)}.$$

Conversely, if $1 < p_1 < \infty$, $0 < q_1, q_2 \leqslant \infty$ and $I_\beta : M^{0,\alpha}_{p_1,q_1} \to M^{0,\alpha}_{\infty,q_2}$ is bounded, then

$$\frac{1}{p_1} \geqslant \frac{\beta}{n} \quad \text{and} \quad \frac{1}{q_2} < \frac{1}{q_1} + \frac{\beta - n\alpha/p_1}{n(1-\alpha)}.$$

However, there is still a gap between the conditions for sufficiency and necessity at the endpoint $1/p_1 = \beta/n$ and the assumption $n\alpha/p_1 < \beta < n$ seems unnatural. Taking into account the potential indexes of the function spaces (denoted by s_1 and s_2), we gave a complete answer for the boundedness of I_β on α-modulation spaces [30] as follows. (We have corrected a minor error in the statement in [30].)

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THEOREM C [30]. *Let $0 \leqslant \alpha \leqslant 1$, $0 < \beta < n$, $1 \leqslant p_i, q_i \leqslant \infty$ and $s_i \in \mathbb{R}$ for $i = 1, 2$. The fractional integral operator $I_\beta : M^{s_1,\alpha}_{p_1,q_1} \to M^{s_2,\alpha}_{p_2,q_2}$ is bounded if and only if*

$$\frac{1}{p_1} \neq \frac{\beta}{n}, \quad \frac{1}{p_2} \neq \frac{n-\beta}{n},$$

$$n\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big) \geqslant \beta,$$

$$n\alpha\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big) \leqslant s_1 - s_2 + \beta, \tag{1.2}$$

and

$$n\alpha\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big) - n(1-\alpha)\Big(\frac{1}{q_1} - \frac{1}{q_2}\Big) \leqslant s_1 - s_2 + \beta, \tag{1.3}$$

with strict inequality in (1.3) when (1.2) is strict.

In [30], we also consider $I_\beta f(x) = \mathscr{F}^{-1}(|\xi|^{-\beta}\mathscr{F}f)(x)$ for $\beta \leqslant 0$. We note that all the previous results concern only the case of $p_i, q_i \in [1, \infty]$ for $i = 1, 2$ and function spaces with the same scale ($\alpha_1 = \alpha_2 = \alpha$). The aim of this paper is to give sharp conditions for the boundedness of

$$I_\beta : M^{s_1,\alpha_1}_{p_1,q_1} \to M^{s_2,\alpha_2}_{p_2,q_2}$$

in the full range $\alpha_i \in [0, 1]$, $p_i, q_i \in (0, \infty]$, $s_i \in \mathbb{R}$, $i = 1, 2$. Our main theorem follows. It is a substantial extension of the known results, even in the case $\alpha_1 = \alpha_2$.

THEOREM 1.1. *Let $0 \leqslant \alpha_i \leqslant 1$, $0 < \beta < n$, $0 < p_i, q_i \leqslant \infty$ and $s_i \in \mathbb{R}$ for $i = 1, 2$. Then the fractional integral operator $I_\beta : M^{s_1,\alpha_1}_{p_1,q_1} \to M^{s_2,\alpha_2}_{p_2,q_2}$ is bounded, if and only if*

$$\frac{1}{p_1} \neq \frac{\beta}{n}, \quad \frac{1}{p_2} < \frac{n-\beta}{n}, \tag{1.4}$$

$$n\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big) \geqslant \beta, \tag{1.5}$$

$$s_2 - \beta + R(\mathbf{p}, \mathbf{q}, \alpha_1, \alpha_2) \leqslant s_1, \tag{1.6}$$

and

$$s_2 - \beta + R(\mathbf{p}, \mathbf{q}, \alpha_1, \alpha_2) + \frac{n(1 - \max(\alpha_1, \alpha_2))}{q_2} \leqslant s_1 + \frac{n(1 - \max(\alpha_1, \alpha_2))}{q_1}, \tag{1.7}$$

with strict inequality in (1.7) when there is strict inequality in (1.6).

Here, we write $\mathbf{p} = (p_1, p_2)$, $\mathbf{q} = (q_1, q_2)$ and

$$R(\mathbf{p}, \mathbf{q}; \alpha_1, \alpha_2) = \begin{cases} n\alpha_1\Big(\dfrac{1}{p_1} - \dfrac{1}{p_2}\Big) + n(\alpha_2 - \alpha_1)\cdot \max\Big\{0, \dfrac{1}{p_2} - \dfrac{1}{q_1}, 1 - \dfrac{1}{p_2} - \dfrac{1}{q_1}\Big\} & \text{if } \alpha_1 \leqslant \alpha_2, \\ n\alpha_2\Big(\dfrac{1}{p_1} - \dfrac{1}{p_2}\Big) + n(\alpha_2 - \alpha_1)\cdot \min\Big\{0, \dfrac{1}{p_1} - \dfrac{1}{q_2}, 1 - \dfrac{1}{p_1} - \dfrac{1}{q_2}\Big\} & \text{if } \alpha_1 > \alpha_2, \end{cases}$$

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By potential lifting (see Lemma 2.4),

$$M^{s,\alpha}_{p,q} = J_{-s}M^{0,\alpha}_{p,q},$$

where $J_s = (1 + |D|^2)^{-s/2}$ is the Bessel potential operator. Thus, α-modulation spaces can be regarded as inhomogeneous potential spaces. As a (homogeneous) potential lifting operator, the behaviour of I_β for high frequency is the same as that of J_β, so we can reduce the boundedness of I_β in the high frequency to the embedding between corresponding function spaces (see Proposition 3.1). On the other hand, Proposition 3.3 characterises the behaviour of I_β for low frequency in the full range $p_1, p_2 \in (0,\infty]$. Theorem 1.1 follows from the equivalent characterisation in Proposition 3.2.

We will only give the proof of Theorem 1.1 for $0 \leqslant \alpha_i < 1$ ($i = 1,2$), since the other cases can be handled similarly. The paper is organised as follows. In Section 2, we introduce various notations and definitions that will be used throughout this paper. We also present several known lemmas that will be used in our proof. Propositions 3.1–3.3 and the proof of Theorem 1.1 follow in Section 3.

2. Preliminaries

We write $\widetilde{p} := \min\{p, 1\}$ for $0 < p \leqslant \infty$ and $[t]$ denotes the integer part of $t \in \mathbb{R}$. Let C be a positive constant that may depend on $n, p_i, q_i, s_i, \alpha_i, \beta$ ($i = 1,2$). The notation $X \lesssim Y$ means $X \leqslant CY$, the notation $X \sim Y$ means $X \lesssim Y \lesssim X$, and the notation $X \simeq Y$ means $X = CY$. For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we denote

$$|x| := \Big(\sum_{i=1}^{n} |x_i|^2\Big)^{1/2} \quad \text{and} \quad \langle x\rangle := (1 + |x|^2)^{1/2}.$$

Let $\mathscr{S} := \mathscr{S}(\mathbb{R}^n)$ be the Schwartz space and $\mathscr{S}' := \mathscr{S}'(\mathbb{R}^n)$ be the space of tempered distributions. The Fourier transform $\mathscr{F}f$ and the inverse Fourier transform $\mathscr{F}^{-1}f$ of $f \in \mathscr{S}(\mathbb{R}^n)$ are defined by

$$\mathscr{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi ix\cdot\xi}\,dx, \quad \mathscr{F}^{-1}f(x) = \hat{f}(-x) = \int_{\mathbb{R}^n} f(\xi)e^{2\pi ix\cdot\xi}\,d\xi.$$

We next recall some definitions of the function spaces treated in this paper. First we give the partition of unity on frequency space associated with $\alpha \in [0,1)$. We take two appropriate constants $c > 0$ and $C > 0$ and choose a Schwartz function sequence $\{\eta^\alpha_k\}_{k\in\mathbb{Z}^n}$ satisfying

$$\begin{cases} |\eta^\alpha_k(\xi)| \gtrsim 1 \quad \text{if } |\xi - \langle k\rangle^{\alpha/(1-\alpha)}k| < c\langle k\rangle^{\alpha/(1-\alpha)}; \\ \operatorname{supp}\eta^\alpha_k \subset \{\xi \in \mathbb{R}^n : |\xi - \langle k\rangle^{\alpha/(1-\alpha)}k| < C\langle k\rangle^{\alpha/(1-\alpha)}\}; \\ \displaystyle\sum_{k\in\mathbb{Z}^n} \eta^\alpha_k(\xi) \equiv 1 \quad \text{for all } \xi \in \mathbb{R}^n; \\ |\partial^\gamma \eta^\alpha_k(\xi)| \leqslant C_{|\alpha|}\langle k\rangle^{-\alpha|\gamma|/(1-\alpha)} \quad \text{for all } \xi \in \mathbb{R}^n, \gamma \in (\mathbb{Z}^+ \cup \{0\})^n. \end{cases}$$

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The sequence $\{\eta_k^\alpha(\xi)\}_{k\in\mathbb{Z}^n}$ constitutes a smooth decomposition of $\mathbb{R}^n$. The frequency decomposition operators associated with the above function sequence are defined by

$$\Box_k^\alpha := \mathscr{F}^{-1}\eta_k^\alpha\mathscr{F}$$

for $k \in \mathbb{Z}^n$. Let $0 < p, q \leqslant \infty$, $s \in \mathbb{R}$, $\alpha \in [0, 1)$. Then the α-modulation space associated with the above decomposition is defined by

$$M_{p,q}^{s,\alpha}(\mathbb{R}^n) = \Big\{f \in \mathscr{S}'(\mathbb{R}^n) : \|f\|_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} = \Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{sq/(1-\alpha)}\|\Box_k^\alpha f\|_{L^p}^q\Big)^{1/q} < \infty\Big\}$$

with the usual modifications when $q = \infty$. For simplicity, we write $M_{p,q}^s = M_{p,q}^{s,0}$ and $M_{p,q} = M_{p,q}^{0,0}$.

Remark 2.1. The above definition is independent of the exact choice of η_k^α (see [15]). Also, for sufficiently small $\delta > 0$, one can construct a function sequence $\{\eta_k^\alpha\}_{k\in\mathbb{Z}^n}$ such that $\eta_k^\alpha(\xi) = 1$ and $\eta_k^\alpha(\xi)\eta_l^\alpha(\xi) = 0$ if $k \neq l$ and ξ lies in the ball $B(\langle k\rangle^{\alpha/(1-\alpha)}k, \langle k\rangle^{\alpha/(1-\alpha)}\delta)$ (see [12]). We also remark that when $\alpha = 0$, the modulation space was initially defined by a norm (see [10]). For modulation spaces with more general weights, see [20, 22].

To define the Besov spaces, we introduce the dyadic decomposition of $\mathbb{R}^n$. Let φ be a smooth bump function supported in the ball $\{\xi : |\xi| < \frac{3}{2}\}$ and equal to 1 on the ball $\{\xi : |\xi| \leqslant \frac{4}{3}\}$. Define

$$\phi(\xi) = \varphi(\xi) - \varphi(2\xi),$$

and a function sequence

$$\begin{cases} \phi_j(\xi) = \phi(2^{-j}\xi) & \text{for } j \in \mathbb{Z}^+, \\ \phi_0(\xi) = 1 - \sum_{j\in\mathbb{Z}^+}\phi_j(\xi) = \varphi(\xi). \end{cases}$$

For integers $j \in (\mathbb{Z}^+ \cup \{0\})$, we define the Littlewood–Paley operators

$$\Delta_j = \mathscr{F}^{-1}\phi_j\mathscr{F}.$$

Let $0 < p, q \leqslant \infty$ and $s \in \mathbb{R}$. For a tempered distribution f, we define the norm

$$\|f\|_{B_{p,q}^s} = \Big(\sum_{j=0}^{\infty}2^{jsq}\|\Delta_j f\|_{L^p}^q\Big)^{1/q},$$

with the usual modifications when $q = \infty$. The (inhomogeneous) Besov space $B_{p,q}^s$ is the space of all tempered distributions f for which the quantity $\|f\|_{B_{p,q}^s}$ is finite.

We now list some key lemmas which will be used in the proofs. As mentioned before, $M_{p,q}^{s,1}$ denotes the inhomogeneous Besov space $B_{p,q}^s$.

Lemma 2.2 (Young's inequality, see [25]).

(1) *Suppose* $0 < p \leqslant 1$, $R > 0$ *and* $\operatorname{supp}\hat{f}$, $\operatorname{supp}\hat{g} \subseteq B(x, R) \subseteq \mathbb{R}^n$. *Then*

$$\|f * g\|_{L^p} \leqslant CR^{n(1/p-1)}\|f\|_{L^p}\|g\|_{L^p},$$

where C is independent of $x \in \mathbb{R}^n$.

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(2) *Suppose* $1 \leqslant p, q, r \leqslant \infty$ *satisfy* $1 + 1/q = 1/p + 1/r$. *Then*

$$\|f * g\|_{L^q} \lesssim \|f\|_{L^p}\|g\|_{L^r}.$$

The following lemma is useful in dealing with functions having compact support on the time or frequency plane. See [17] and [21] for similar results associated with weighted modulation spaces.

Lemma 2.3 (Local property). *Let* $0 < p, q \leqslant \infty$ *and* $s \in \mathbb{R}$. *For a fixed positive R,*

$$\|f\|_{M^{s,\alpha}_{p,q}} \sim \|f\|_{L^p}$$

for all distributions f satisfying $\operatorname{supp} \hat{f} \subset B(0, R)$.

Proof. We only give the proof for $p, q < \infty$, since the other cases can be handled similarly. By the assumption, f can be represented by

$$f = \sum_{|k|\leqslant N} \square^\alpha_k f.$$

Using the Minkowski inequality, we deduce that

$$\|f\|_{L^p} = \Big\| \sum_{|k|\leqslant N} \square^\alpha_k f \Big\|_{L^p} \lesssim \Big(\sum_{|k|\leqslant N} \|\square^\alpha_k f\|^{\tilde{p}}_{L^p} \Big)^{1/\tilde{p}} \sim \Big(\sum_{|k|\leqslant N} \langle k\rangle^{sq} \|\square^\alpha_k f\|^q_{L^p} \Big)^{1/q} \sim \|f\|_{M^{s,\alpha}_{p,q}}.$$

On the other hand, by Lemma 2.2,

$$\|\square^\alpha_k f\|_{L^p} \lesssim \langle k\rangle^{\alpha n(1/\tilde{p}-1)/(1-\alpha)} \|\mathscr{F}^{-1}\eta^\alpha_k\|_{L^{\tilde{p}}} \cdot \|f\|_{L^p} \lesssim \|f\|_{L^p}.$$

Thus,

$$\|f\|_{M^{s,\alpha}_{p,q}} = \Big(\sum_{|k|\leqslant N} \langle k\rangle^{sq} \|\square^\alpha_k f\|^q_{L^p} \Big)^{1/q} \lesssim \Big(\sum_{|k|\leqslant N} \langle k\rangle^{sq} \|f\|^q_{L^p} \Big)^{1/q} \lesssim \|f\|_{L^p}. \qquad \square$$

Lemma 2.4 (Potential lifting, see [15]). *Let* $0 < p, q \leqslant \infty$. *For any* $s \in \mathbb{R}$, *the mapping* $J_\beta : M^{s,\alpha}_{p,q} \to M^{s+\beta,\alpha}_{p,q}$ *is isomorphic.*

Lemma 2.5 ([25], Section 1.4.1, Remark 4). *Let* $0 < p_1 \leqslant p_2 \leqslant \infty$ *and* $R > 0$. *If* $\operatorname{supp} \hat{f} \subseteq B(x, R)$, *then*

$$\|f\|_{L^{p_2}} \leqslant C R^{n(1/p_1 - 1/p_2)} \|f\|_{L^{p_1}},$$

where C is independent of f and $x \in \mathbb{R}^n$.

Lemma 2.6 (Bernstein's multiplier theorem [30]). *Let* $0 < p \leqslant 2$. *If* $\partial^\gamma f \in L^2$ *for all multi-indices* γ *with* $|\gamma| \leqslant [n(1/p - 1/2)] + 1$, *then*

$$\|\mathscr{F}^{-1} f\|_{L^p} \lesssim \sum_{|\gamma|\leqslant[n(1/p-1/2)]+1} \|\partial^\gamma f\|_{L^2}.$$

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Proof. Write $N = [n(1/p - 1/2)] + 1$ and $1/r = 1/p - 1/2$, so $r \geq 0$ and $-Nr + n < 0$. By Hölder's inequality and Plancherel's equality,

$$\begin{aligned}\|\mathscr{F}^{-1}f\|_{L^p} &= \|\langle x\rangle^{-N}\langle x\rangle^{N}\mathscr{F}^{-1}f(x)\|_{L^p} \leqslant \|\langle x\rangle^{-N}\|_{L^r}\|\langle x\rangle^{N}\mathscr{F}^{-1}f(x)\|_{L^2}\\ &\lesssim \|\langle x\rangle^{N}\mathscr{F}^{-1}f(x)\|_{L^2} \lesssim \Big\|\sum_{|\gamma|\leqslant N}|x^{\gamma}\mathscr{F}^{-1}f(x)\Big\|_{L^2}\\ &\lesssim \sum_{|\gamma|\leqslant N}\|x^{\gamma}\mathscr{F}^{-1}f(x)\|_{L^2} \sim \sum_{|\gamma|\leqslant N}\|\partial^{\gamma}f\|_{L^2}.\end{aligned}$$ □

Lemma 2.7 (Sharpness of embedding [14]). *Let $0 < p_i, q_i \leqslant \infty$, $s_i \in \mathbb{R}$ and $\alpha_i \in [0,1]$ for $i = 1, 2$. Then*

$$M^{s_1,\alpha_1}_{p_1,q_1} \subseteq M^{s_2,\alpha_2}_{p_2,q_2}$$

if and only if

$$\frac{1}{p_2} \leqslant \frac{1}{p_1}, \quad \frac{1}{q_2} \leqslant \frac{1}{q_1}, \quad s_2 + R(\mathbf{p},\mathbf{q},\alpha_1,\alpha_2) \leqslant s_1,$$

or

$$\begin{cases}\dfrac{1}{p_2} \leqslant \dfrac{1}{p_1}, \quad \dfrac{1}{q_2} > \dfrac{1}{q_1},\\ s_2 + R(\mathbf{p},\mathbf{q},\alpha_1,\alpha_2) + \dfrac{n(1-\max\{\alpha_1,\alpha_2\})}{q_2} < s_1 + \dfrac{n(1-\max\{\alpha_1,\alpha_2\})}{q_1}.\end{cases}$$

3. Boundedness of the fractional integral operators

In this section, we give the proof of Theorem 1.1. To do this, we first show that the boundedness of fractional integral operators at high frequency can be reduced to the corresponding embedding between two α-modulation spaces (see Proposition 3.1). Then we establish a mild characterisation for the boundedness of fractional integral operators (see Proposition 3.2). Finally, we establish the sharp conditions for the boundedness of fractional integral operators at low frequency (see Proposition 3.3).

Proposition 3.1. *Let $0 < \beta < n$, $s_i \in \mathbb{R}$ for $i = 1, 2$, and let $\varrho \in \mathscr{S}$ be a nonzero smooth function such that $\varrho(\xi) = 1$ in $B(0, r)$, where $r > 0$. Set $P_0 f = \mathscr{F}^{-1}(\varrho\hat{f})$ and $P_\infty f = \mathscr{F}^{-1}((1-\varrho)\hat{f})$. Then the mapping $I_\beta \circ P_\infty : M^{s_1,\alpha_1}_{p_1,q_1} \to M^{s_2,\alpha_2}_{p_2,q_2}$ is bounded if and only if $J_\beta : M^{s_1,\alpha_1}_{p_1,q_1} \to M^{s_2,\alpha_2}_{p_2,q_2}$ is bounded.*

Proof. We only state the proof for $q_1, q_2 < \infty$, since the other cases can be handled similarly.

We first give the proof of the 'IF' part. Let $\zeta \in \mathscr{S}$ be a nonzero smooth function with compact support such that $\zeta^{\alpha_2}_k \eta^{\alpha_2}_k = \eta^{\alpha_2}_k$ for all $k \in \mathbb{Z}^n$, where

$$\zeta^{\alpha_2}_k(\xi) := \zeta\Big(\frac{\xi - \langle k\rangle^{\alpha_2/(1-\alpha_2)}k}{\langle k\rangle^{\alpha_2/(1-\alpha_2)}}\Big).$$

By Lemma 2.2,

$$\begin{aligned}
\|I_\beta \circ P_\infty f\|_{M^{s_2,\alpha_2}_{p_2,q_2}} &= \Big(\sum_{k\in\mathbb{Z}^n} \|\square_k^{\alpha_2} I_\beta \circ P_\infty f\|_{L^{p_2}}^{q_2} \langle k\rangle^{s_2q_2/(1-\alpha_2)}\Big)^{1/q_2} \\
&= \Big(\sum_{k\in\mathbb{Z}^n} \|(\mathscr{F}^{-1}\zeta_k^{\alpha_2}) * (\square_k^{\alpha_2} I_\beta \circ P_\infty f)\|_{L^{p_2}}^{q_2} \langle k\rangle^{s_2q_2/(1-\alpha_2)}\Big)^{1/q_2} \\
&= \Big(\sum_{k\in\mathbb{Z}^n} \Big\|\mathscr{F}^{-1}\Big(\zeta_k^{\alpha_2}(\xi)\frac{|\xi|^\beta}{(1+|\xi|^2)^{\beta/2}}(1-\varrho(\xi))\Big) * (\square_k^{\alpha_2} J_\beta f)\Big\|_{L^{p_2}}^{q_2} \langle k\rangle^{s_2q_2/(1-\alpha_2)}\Big)^{1/q_2} \\
&\lesssim \Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{(n\alpha_2/1-\alpha_2)(1/\widetilde{p_2}-1)q_2}\Big\|\mathscr{F}^{-1}\Big(\zeta_k^{\alpha_2}(\xi)\frac{|\xi|^\beta}{(1+|\xi|^2)^{\beta/2}}(1-\varrho(\xi))\Big)\Big\|_{L^{\widetilde{p_2}}}^{q_2} \\
&\quad \times \|\square_k^{\alpha_2} J_\beta f\|_{L^{p_2}}^{q_2} \langle k\rangle^{s_2q_2/1-\alpha_2}\Big)^{1/q_2}
\end{aligned}$$

and this is $\lesssim \|J_\beta f\|_{M^{s_2,\alpha_2}_{p_2,q_2}} \lesssim \|f\|_{M^{s_1,\alpha_1}_{p_1,q_1}}$ if we prove that, for all $k \in \mathbb{Z}^n$,

$$\langle k\rangle^{n\alpha_2(1/\widetilde{p_2}-1)/(1-\alpha_2)}\Big\|\mathscr{F}^{-1}\Big(\zeta_k^{\alpha_2}(\xi)\frac{|\xi|^\beta}{(1+|\xi|^2)^{\beta/2}}(1-\varrho(\xi))\Big)\Big\|_{L^{\widetilde{p_2}}} \lesssim 1. \tag{3.1}$$

To verify (3.1), we write $g_k(\xi) = \zeta_k^{\alpha_2}(\xi)(|\xi|^\beta/(1+|\xi|^2)^{\beta/2})(1-\varrho(\xi))$. Observing that g_k is a Schwartz function for every fixed k, we only need to verify (3.1) for large k such that $\zeta_k^{\alpha_2}(1-\varrho(\xi)) = \zeta_k^{\alpha_2}$ and $\operatorname{supp}\zeta_k^{\alpha_2} \cap B(0,100) = \emptyset$. Using the Bernstein multiplier theorem and the scaling property of L^p, we see that

$$\begin{aligned}
&\langle k\rangle^{n\alpha_2(1/\widetilde{p_2}-1)/(1-\alpha_2)}\|\mathscr{F}^{-1}g_k\|_{L^{\widetilde{p_2}}} = \|\mathscr{F}^{-1}(g_k(\langle k\rangle^{\alpha_2/(1-\alpha_2)}\xi))\|_{L^{\widetilde{p_2}}} \\
&\quad\lesssim \sum_{|\gamma|\leqslant[n(1/\widetilde{p_2}-1/2)]+1} \langle k\rangle^{\alpha_2(|\gamma|-n/2)/(1-\alpha_2)}\|\partial^\gamma g_k\|_{L^2} \\
&\quad\lesssim \sum_{|\gamma|\leqslant[n(1/\widetilde{p_2}-1/2)]+1} \langle k\rangle^{\alpha_2(|\gamma|-n/2)/(1-\alpha_2)} \sum_{\gamma_1+\gamma_2+\gamma_3=\gamma} \|\partial^{\gamma_1}|\xi|^\beta \cdot \partial^{\gamma_2}(1+|\xi|^2)^{-\beta/2} \cdot \partial^{\gamma_3}\zeta_k^{\alpha_2}\|_{L^2}.
\end{aligned}$$

From the defining property of $\zeta_k^{\alpha_2}$ and $\operatorname{supp}\zeta_k^{\alpha_2} \cap B(0,100) = \emptyset$, the above quantity is

$$\begin{aligned}
\lesssim &\sum_{|\gamma|\leqslant[n(1/\widetilde{p_2}-1/2)]+1} \langle k\rangle^{\alpha_2(|\gamma|-n/2)/(1-\alpha_2)} \\
&\sum_{\gamma_1+\gamma_2+\gamma_3=\gamma} \langle k\rangle^{-|\gamma_1|-|\gamma_2|/(1-\alpha_2)}\langle k\rangle^{-\alpha_2(|\gamma_3|)/(1-\alpha_2)}\langle k\rangle^{n\alpha_2/(2(1-\alpha_2))} \lesssim 1.
\end{aligned}$$

Now, we turn to the proof of the 'ONLY IF' part. We deal with the high frequency part first. Let τ be a smooth function with compact support in $B(0,r)$ with $\tau(\xi) = 1$ in $B(0,r/2)$. Obviously, $(1-\varrho)(1-\tau) = 1-\varrho$. In addition, as in the proof of the 'IF' part,

$$\langle k\rangle^{n\alpha_2(1/\widetilde{p_2}-1)/(1-\alpha_2)}\Big\|\mathscr{F}^{-1}\Big(\zeta_k^{\alpha_2}(\xi)\frac{(1+|\xi|^2)^{\beta/2}}{|\xi|^\beta}(1-\tau(\xi))\Big)\Big\|_{L^{\widetilde{p_2}}} \lesssim 1$$

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for all $k \in \mathbb{Z}^n$. By Lemma 2.2,

$$\begin{aligned}\|\square_k^{\alpha_2} J_\beta \circ P_\infty f\|_{p_2} &= \Big\|\mathscr{F}^{-1}\Big(\zeta_k^{\alpha_2}(\xi)\frac{(1+|\xi|^2)^{\beta/2}}{|\xi|^\beta}(1-\tau(\xi))\Big) * \square_k^{\alpha_2} I_\beta \circ P_\infty f\Big\|_{p_2} \\ &\lesssim \langle k\rangle^{n\alpha_2(1/\widetilde{p_2}-1)/(1-\alpha_2)}\Big\|\mathscr{F}^{-1}\Big(\zeta_k^{\alpha_2}(\xi)\frac{(1+|\xi|^2)^{\beta/2}}{|\xi|^\beta}(1-\tau(\xi))\Big)\Big\|_{L^{\widetilde{p_2}}}\Big\|\square_k^{\alpha_2} I_\beta \circ P_\infty f\Big\|_{p_2} \\ &\lesssim \Big\|\square_k^{\alpha_2} I_\beta \circ P_\infty f\Big\|_{p_2}.\end{aligned}$$

Thus,

$$\begin{aligned}\|J_\beta \circ P_\infty f\|_{M^{s_2,\alpha_2}_{p_2,q_2}} &= \Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{s_2q_2/(1-\alpha_2)}\|\square_k^{\alpha_2} J_\beta \circ P_\infty f\|_{p_2}^{q_2}\Big)^{1/q_2} \\ &\lesssim \Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{s_2q_2/(1-\alpha_2)}\|\square_k^{\alpha_2} I_\beta f\|_{p_2}^{q_2}\Big)^{1/q_2} = \|I_\beta \circ P_\infty f\|_{M^{s_2,\alpha_2}_{p_2,q_2}} \lesssim \|f\|_{M^{s_1,\alpha_1}_{p_1,q_1}}.\end{aligned} \tag{3.2}$$

Before dealing with the low frequency part, we verify that $1/p_2 \leqslant 1/p_1$. Indeed, we can take a large positive number ξ_0 and a nonzero smooth function v with Fourier support in $B(0,1)$, such that

$$J_\beta v_{\xi_0,\lambda} = J_\beta \circ P_\infty v_{\xi_0,\lambda}$$

for all $\lambda \leqslant 1$, where $\widehat{v_{\xi_0,\lambda}}(\xi) := \widehat{v}((\xi-\xi_0)/\lambda)$. From (3.2) and Lemmas 2.3 and 2.4,

$$\begin{aligned}\|v_{\xi_0,\lambda}\|_{L^{p_2}} \sim \|v_{\xi_0,\lambda}\|_{M^{s_2-\beta,\alpha_2}_{p_2,q_2}} &\sim \|J_\beta v_{\xi_0,\lambda}\|_{M^{s_2,\alpha_2}_{p_2,q_2}} \sim \|J_\beta \circ P_\infty v_{\xi_0,\lambda}\|_{M^{s_2,\alpha_2}_{p_2,q_2}} \\ &\lesssim \|v_{\xi_0,\lambda}\|_{M^{s_1,\alpha_1}_{p_1,q_1}} \sim \|v_{\xi_0,\lambda}\|_{L^{p_1}},\end{aligned}$$

which implies

$$\lambda^{1-1/p_2} \sim \|v_{\xi_0,\lambda}\|_{L^{p_2}} \lesssim \|v_{\xi_0,\lambda}\|_{L^{p_1}} \sim \lambda^{1-1/p_1}.$$

Letting $\lambda \to 0$, we deduce that $1/p_2 \leqslant 1/p_1$.

Now, we turn to the estimates for the low frequency part. By Lemma 2.3, Lemma 2.4 and the embedding of L^p with Fourier support (see Lemma 2.5),

$$\begin{aligned}\|J_\beta \circ P_0 f\|_{M^{s_2,\alpha_2}_{p_2,q_2}} &\sim \|P_0 f\|_{M^{s_2-\beta,\alpha_2}_{p_2,q_2}} \sim \|P_0 f\|_{L^{p_2}} \\ &\lesssim \|P_0 f\|_{L^{p_1}} \sim \|P_0 f\|_{M^{s_1,\alpha_1}_{p_1,q_1}} \lesssim \|f\|_{M^{s_1,\alpha_1}_{p_1,q_1}}.\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3),

$$\|J_\beta f\|_{M^{s_2,\alpha_2}_{p_2,q_2}} \lesssim \|J_\beta \circ P_0 f\|_{M^{s_2,\alpha_2}_{p_2,q_2}} + \|J_\beta \circ P_\infty f\|_{M^{s_2,\alpha_2}_{p_2,q_2}} \lesssim \|f\|_{M^{s_1,\alpha_1}_{p_1,q_1}},$$

which is the desired conclusion. □

As an application of Proposition 3.1, we can give an equivalent characterisation to that of Theorem 1.1.

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PROPOSITION 3.2. *Let* $0 \leqslant \alpha \leqslant 1$, $0 < \beta < n$, $0 < p_i, q_i \leqslant \infty$ *and* $s_i \in \mathbb{R}$, $i = 1, 2$. *Let* $\varrho \in \mathscr{S}$ *be a nonzero smooth function such that* $\varrho(\xi) = 1$ *in* $B(0, r)$ *for some* $r > 0$. *Set* $P_0 f = \mathscr{F}^{-1}(\varrho \hat{f})$ *and* $P_\infty f = \mathscr{F}^{-1}((1-\varrho)\hat{f})$. *Then* $I_\beta : M^{s_1,\alpha_1}_{p_1,q_1} \to M^{s_2,\alpha_2}_{p_2,q_2}$ *is bounded, if and only if*

(1) $\|I_\beta \circ P_0 f\|_{L^{p_2}} \lesssim \|P_0 f\|_{L^{p_1}}$ *for all* $f \in \mathscr{S}'$;

(2) *the embedding relation* $M^{s_1,\alpha_1}_{p_1,q_1} \subset M^{s_2-\beta,\alpha_2}_{p_2,q_2}$ *holds.*

PROOF. We divide the proof into two parts.

'IF' part. By Lemma 2.4, the embedding relation $M^{s_1,\alpha_1}_{p_1,q_1} \subset M^{s_2-\beta,\alpha_2}_{p_2,q_2}$ implies the boundedness of $J_\beta : M^{s_1,\alpha_1}_{p_1,q_1} \to M^{s_2,\alpha_2}_{p_2,q_2}$, and we use Proposition 3.1 to deduce the boundedness of $I_\beta \circ P_\infty : M^{s_1,\alpha_1}_{p_1,q_1} \to M^{s_2,\alpha_2}_{p_2,q_2}$. Now $\|I_\beta \circ P_0 f\|_{L^{p_2}} \lesssim \|P_0 f\|_{L^{p_1}}$ and, by Lemma 2.3, $\|I_\beta \circ P_0 f\|_{L^{p_2}} \sim \|I_\beta \circ P_0 f\|_{M^{s_2,\alpha_2}_{p_2,q_2}}$ and $\|P_0 f\|_{L^{p_1}} \sim \|P_0 f\|_{M^{s_1,\alpha_1}_{p_1,q_1}}$, which implies $\|I_\beta \circ P_0 f\|_{M^{s_2,\alpha_2}_{p_2,q_2}} \lesssim \|P_0 f\|_{M^{s_1,\alpha_1}_{p_1,q_1}}$. Thus,

$$\begin{aligned}\|I_\beta f\|_{M^{s_2,\alpha_2}_{p_2,q_2}} &\lesssim \|I_\beta \circ P_0 f\|_{M^{s_2,\alpha_2}_{p_2,q_2}} + \|I_\beta \circ P_\infty f\|_{M^{s_2,\alpha_2}_{p_2,q_2}} \\ &\lesssim \|P_0 f\|_{M^{s_1,\alpha_1}_{p_1,q_1}} + \|f\|_{M^{s_1,\alpha_1}_{p_1,q_1}} \lesssim \|f\|_{M^{s_1,\alpha_1}_{p_1,q_1}}.\end{aligned}$$

'ONLY IF' part. By the boundedness of I_β, we have $\|I_\beta \circ P_0 f\|_{M^{s_2,\alpha_2}_{p_2,q_2}} \lesssim \|P_0 f\|_{M^{s_1,\alpha_1}_{p_1,q_1}}$. Recalling $\|I_\beta \circ P_0 f\|_{L^{p_2}} \sim \|I_\beta \circ P_0 f\|_{M^{s_2,\alpha_2}_{p_2,q_2}}$ and $\|P_0 f\|_{L^{p_1}} \sim \|P_0 f\|_{M^{s_1,\alpha_1}_{p_1,q_1}}$, we deduce that $\|I_\beta \circ P_0 f\|_{L^{p_2}} \lesssim \|P_0 f\|_{L^{p_1}}$. By the boundedness of I_β, we also conclude that

$$\|I_\beta \circ P_\infty f\|_{M^{s_2,\alpha_2}_{p_2,q_2}} \lesssim \|P_\infty f\|_{M^{s_1,\alpha_1}_{p_1,q_1}} \lesssim \|f\|_{M^{s_1,\alpha_1}_{p_1,q_1}}.$$

Using Proposition 3.1, we deduce that the map $J_\beta : M^{s_1,\alpha_1}_{p_1,q_1} \to M^{s_2,\alpha_2}_{p_2,q_2}$ is bounded, which implies the embedding relation $M^{s_1,\alpha_1}_{p_1,q_1} \subset M^{s_2-\beta,\alpha_2}_{p_2,q_2}$. □

The following proposition gives the sharp conditions for boundedness of I_β at low frequency.

PROPOSITION 3.3. *Let* $0 < \beta < n$ *and* $0 < p_i \leqslant \infty$ *for* $i = 1, 2$. *Suppose* ρ *is a smooth function supported in* $B(0, R)$ *satisfying* $\rho(\xi) = 1$ *for* $\xi \in B(0, r)$, *where* $0 < r < R < \infty$. *Set* $Pf := \rho(D)f$. *Then*

$$\|I_\beta \circ Pf\|_{L^{p_2}} \leqslant C\|Pf\|_{L^{p_1}},$$

for all $f \in \mathscr{S}'$, *if and only if*

$$\frac{1}{p_2} + \frac{\beta}{n} \leqslant \min\Big\{\frac{1}{p_1}, 1\Big\}, \quad p_1 \neq \frac{n}{\beta}, \quad p_2 \neq \frac{n}{n-\beta}, \tag{3.4}$$

where C *depends only on* n, p_1, p_2 *and* R *and is independent of* f.

PROOF. The conditions (3.4) are equivalent to

$$\frac{1}{p_2} + \frac{\beta}{n} \leqslant \frac{1}{p_1}, \quad p_1 \neq \frac{n}{\beta} \quad \text{and} \quad p_2 > \frac{n}{n-\beta}.$$

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We prove the sufficiency by considering the following three cases.

Case 1: $1/p_2 + \beta/n \leqslant 1/p_1$ and $0 < 1/p_2 < (n-\beta)/n$.

In this case, there exists τ_1 such that $1/\tau_1 = 1/p_2 + \beta/n \leqslant 1/p_1$ and $1 < \tau_1, p_2 < \infty$. Using the Hardy–Littlewood–Sobolev embedding theorem and Lemma 2.5,

$$\|I_\beta \circ Pf\|_{L^{p_2}} = \|I_\beta(Pf)\|_{L^{p_2}} \lesssim \|Pf\|_{L^{\tau_1}} \lesssim \|Pf\|_{L^{p_1}}.$$

Case 2: $\beta/n < 1/p_1 < 1$ and $p_2 = \infty$.

In this case, there exists τ_2 such that $(1/p_2 =)\ 0 < 1/\tau_2 = 1/p_1 - \beta/n$ and $1 < p_1, \tau_2 < \infty$. By Lemma 2.5 and the Hardy–Littlewood–Sobolev embedding theorem,

$$\|I_\beta \circ Pf\|_{L^{p_2}} \lesssim \|I_\beta \circ Pf\|_{L^{\tau_2}} = \|I_\beta(Pf)\|_{L^{\tau_2}} \lesssim \|Pf\|_{L^{p_1}}.$$

Case 3: $1/p_1 \geqslant 1$ and $p_2 = \infty$.

In this case, there exist τ_1, τ_2, such that $1/p_2 + \beta/n < 1/\tau_2 + \beta/n = 1/\tau_1 < 1/p_1$ and $1 < \tau_1, \tau_2 < \infty$. For instance, we may choose $1/\tau_1 = (n+\beta)/2n$ and $1/\tau_2 = (n-\beta)/2n$. By Lemma 2.5 and the Hardy–Littlewood–Sobolev embedding theorem,

$$\|I_\beta \circ Pf\|_{L^{p_2}} \lesssim \|I_\beta \circ Pf\|_{L^{\tau_2}} = \|I_\beta(Pf)\|_{L^{\tau_2}} \lesssim \|Pf\|_{L^{\tau_1}} \lesssim \|Pf\|_{L^{p_1}}.$$

Next we turn to the proof of necessity. We first verify that $1/p_2 + \beta/n \leqslant 1/p_1$. Let $f \in \mathscr{S}(\mathbb{R}^n)$ be a nonzero function whose Fourier transform has support in $B(0,1)$. Write $f_\lambda(x) := \lambda^n f(\lambda x)$ for $\lambda > 0$ so that $f_\lambda \in L^{p_1}$. By checking the Fourier transform,

$$\|I_\beta f_\lambda\|_{L^{p_2}} = \|I_\beta \circ P f_\lambda\|_{L^{p_2}} \leqslant C\|Pf_\lambda\|_{L^{p_1}} = C\|f_\lambda\|_{L^{p_1}}$$

for sufficiently small λ. Then a direct calculation yields

$$\lambda^{-\beta}\lambda^{n(1-1/p_2)}\|I_\beta f\|_{L^{p_2}} \leqslant C\lambda^{n(1-1/p_1)}\|f\|_{L^{p_1}}.$$

For a fixed $f \neq 0$, letting $\lambda \to 0$ on both sides of the above inequality, yields

$$\frac{1}{p_2} + \frac{\beta}{n} \leqslant \frac{1}{p_1}.$$

Next, we verify the conditions $p_1 \neq n/\beta$ and $p_2 > n/(n-\beta)$. Suppose first, for a contradiction, that $p_1 = n/\beta$. Since $1/p_2 + \beta/n \leqslant 1/p_1$, we get $p_2 = \infty$. Let $f = h_\varepsilon * v$, where $0 < \beta/n < \varepsilon \leqslant 1$ and h_ε is defined by

$$h_\varepsilon = \begin{cases} |x|^{-\beta}(\log|x|)^{-\varepsilon}, & |x| \geqslant e, \\ 0 & \text{otherwise,} \end{cases}$$

and $v \in \mathscr{S}(\mathbb{R}^n)$ is a positive function whose Fourier transform has compact support near the origin such that $\rho\hat{v} = \hat{v}$ and $v(0) = 1$. We remark that such a function v exists. For instance, we can choose $v = \omega\bar{\omega} = |\omega|^2$ for $\omega \in \mathscr{S}(\mathbb{R}^n)$ satisfying $\omega(0) = 1$ and its Fourier transform having compact support. Thus, f is also a positive function and its Fourier transform has compact support near the origin. Since $h_\varepsilon \in L^{n/\beta}$ and $v \in \mathscr{S}(\mathbb{R}^n)$ with compact Fourier support, we obtain $f \in L^{n/\beta}$ by Young's inequality. By the choice

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of the function v, there exists δ with $0 < \delta < \frac{1}{2}e$ such that $v(x) \geqslant \frac{1}{2}$ for all $|x| \leqslant \delta$. Take $R > e$. Then $|y|/|x| \sim 1$ for all $|x| \geqslant R$, $|x - y| \leqslant \delta$. Consequently

$$\begin{aligned}\|I_\beta \circ Pf\|_{L^\infty} &= \|I_\beta f\|_{L^\infty} \geqslant I_\beta f(0) \sim \int_{\mathbb{R}^n} |x|^{\beta-n} f(x)\,dx \\ &\geqslant \int_{B(0,R)^c} |x|^{\beta-n} f(x)\,dx = \int_{B(0,R)^c} |x|^{\beta-n} (h_\varepsilon * v)(x)\,dx \\ &\geqslant \frac{1}{2}\int_{B(0,R)^c} |x|^{\beta-n} \int_{B(x,\delta)} h_\varepsilon(y)\,dydx \gtrsim \frac{1}{2}\delta^n \int_{B(0,R)^c} |x|^{\beta-n} h_\varepsilon(x)\,dx = \infty.\end{aligned}$$

But, since $\rho \in \mathscr{S}$, Young's inequality (Lemma 2.2) gives $\|Pf\|_{L^{n/\beta}} \lesssim \|f\|_{L^{n/\beta}} < \infty$, contradicting the hypothesis that $\|I_\beta \circ Pf\|_{L^\infty} \leqslant C\|Pf\|_{L^{n/\beta}}$. Therefore, $p_1 \neq n/\beta$.

Next, suppose for a contradiction that $p_2 \leqslant n/(n - \beta)$. Since $1/p_2 + \beta/n \leqslant 1/p_1$, we get $p_1 \leqslant 1$. As above, let $v \in \mathscr{S}(\mathbb{R}^n)$ be a positive function whose Fourier transform has compact support near the origin such that $\rho\hat{v} = \hat{v}$ and $v(0) = 1$. There is a $\delta > 0$ such that $v(x) \geqslant \frac{1}{2}$ for all $|x| \leqslant \delta$. Take $R > 2\delta$. Then

$$\begin{aligned}\|I_\beta v\|_{L^{p_2}} &= \Big\| \int_{\mathbb{R}^n} |y|^{\beta-n} v(x-y)\,dy \Big\|_{L^{p_2}} \geqslant \Big\| \int_{B(x,\delta)} |y|^{\beta-n} v(x-y)\,dy \Big\|_{L^{p_2}(B(0,R)^c)} \\ &\gtrsim \Big\| \int_{B(x,\delta)} |y|^{\beta-n}\,dy \Big\|_{L^{p_2}(B(0,R)^c)} \gtrsim \|\delta^n |x|^{\beta-n}\|_{L^{p_2}(B(0,R)^c)} = \infty.\end{aligned}$$

But $\|I_\beta \circ Pv\|_{L^{p_2}} \leqslant C\|Pv\|_{L^{p_1}} = C\|v\|_{L^{p_1}} < \infty$ by hypothesis, which is a contradiction. So $p_2 > n/(n - \beta)$. □

Proof of Theorem 1.1. Using Proposition 3.2, we obtain conditions (1.4) and (1.5) by Proposition 3.3, and we obtain conditions (1.6) and (1.7) by Proposition 3.1 and Lemma 2.7. □

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