Canad. Math. Bull. Vol. 37 (1), 1994 pp. 75–81

HANKEL OPERATORS ASSOCIATED WITH ANALYTIC CROSSED PRODUCTS

YOSHIKI IMINA AND KICHI-SUKE SAITO

ABSTRACT. We introduce the notion of Hankel operators associated with analytic crossed products and consider the Nehari problem in this setting.

1. **Introduction.** The class of Hankel operators has attracted the attention of several mathematicians and plays an important part in operator theory and related fields. Here, we have a special interest in connection with the theory of shift operators, Hankel operators and Hardy spaces of vector valued functions. The notion of analytic crossed products (formally called nonselfadjoint crossed products) has been defined by McAsey, Muhly and the second author in [3, 4]. Roughly speaking, the analytic crossed products we study stand in the same relation to the crossed products as the Hardy space H^∞ over the unit circle stands in relation to L^∞ over the unit circle. Therefore we believe that it is fruitful to study the theory of Toeplitz operators and Hankel operators associated with analytic crossed products in place of the Hardy space $\mathbb{H}^2$ over the unit circle. In [9] and [10], we introduced the notion of Toeplitz operators associated with analytic crossed products. In this note, we shall introduce the notion of Hankel operators associated with analytic crossed products.

Our setting is the following. Suppose that M is a von Neumann algebra acting on the noncommutative L^2-space $L^2(M)$ and α is a $*$-automorphism of M, that is, there exists a unitary operator u in $B\big(L^2(M)\big)$ such that $\alpha(x) = uxu^*$. On the Hilbert space $\mathbb{L}^2$ $(= \ell^2(\mathbb{Z}, L^2(M))$, we define the crossed product $\mathfrak{L}$ defined by $\{L_x\}_{x\in M}$ and L_δ as in §2. Hence the subalgebra which we investigate and call an *analytic crossed product* is the σ-weakly closed subalgebra $\mathfrak{L}_+$ generated by $\{L_x\}_{x\in M}$ and the positive powers of L_δ.

We put $\mathbb{H}^2 = \ell^2\big(\mathbb{Z}_+, L^2(M)\big)$. If P is the orthogonal projection of $\mathbb{L}^2$ onto $\mathbb{H}^2$ and if A is in $\mathfrak{L}$, then the left Hankel operator H_A is defined by

$$H_A f = (I-P)Af, \quad f \in \mathbb{H}^2.$$

Our aim in this note is to study the properties of Hankel operators. In §2, we define the notion of Hankel operators associated with analytic crossed products. In §3, we consider the Nehari problem in this setting. That is, if A is in $\mathfrak{L}$, then $\|H_A\| = d(A, \mathfrak{L}_+)$, where $d(A, \mathfrak{L}_+)$ is the distance of A from $\mathfrak{L}_+$ with respect to the operator norm. Furthermore, there exists an element B in $\mathfrak{L}$ such that $H_A = H_B$ and $\|H_A\| = \|B\|$ (*cf.* Theorem 3.1).

The second author was supported in part by a Grant-in-Aid for Scientific Research from the Japanese Ministry of Education.

Received by the editors March 20, 1992; revised August 11, 1992.

AMS subject classification: Primary: 46L10; secondary: 47D20, 47B35.

2. **Preliminaries and definitions.** Let M be a von Neumann algebra and let α be a $*$-automorphism of M. We regard M as acting on the noncommutative L^2-space $L^2(M)$ in the sense of Haagerup. We refer the reader to [2, 11] for discussions of this space including some of their elementary properties. We denote the operators in the left regular representation of M on $L^2(M)$ by $\ell_x, x \in M$, and those in the right regular representation by $r_x, x \in M$. Put $\ell(M) = \{\ell_x : x \in M\}$ and $r(M) = \{r_x : x \in M\}$, respectively. Let J be the conjugate linear isometric involution $a \longrightarrow a^*$ of $L^2(M)$ and let $L^2(M)_+$ be the positive part of $L^2(M)$. Since $\{\ell(M), L^2(M), J, L^2(M)_+\}$ is a standard form of M in the sense of Haagerup [1], by [1, Theorem 3.2], there exists a unitary operator u in $L^2(M)$ such that $\ell_{\alpha(x)} = u\ell_x u^*$ and $r_{\alpha(x)} = ur_x u^*$. To construct a crossed product, we consider the Hilbert space $\mathbb{L}^2$ defind by

$$\Big\{f\colon \mathbb{Z} \longrightarrow L^2(M) \,\Big|\, \sum_{n\in\mathbb{Z}} \|f(n)\|_2^2 < \infty\Big\},$$

where $\|\cdot\|_2$ is the norm of $L^2(M)$. For $x \in M$, we define operators L_x, R_x, L_δ and R_δ on $\mathbb{L}^2$ by the formulae

$$(L_x f)(n) = \ell_x f(n), \quad (R_x f)(n) = r_{\alpha^n(x)} f(n),$$
$$(L_\delta f)(n) = uf(n-1) \text{ and } (R_\delta f)(n) = f(n-1),$$

where $f \in \mathbb{L}^2$ and $n \in \mathbb{Z}$. Put $L(M) = \{L_x : x \in M\}$ and $R(M) = \{r_x : x \in M\}$. We set $\mathfrak{L} = \{L(M), L_\delta\}''$ and $\mathfrak{R} = \{R(M), R_\delta\}''$. We also define the left (resp. right) analytic crossed product $\mathfrak{L}_+$ (resp. $\mathfrak{R}_+$) to be the σ-weakly closed subalgebra of $\mathfrak{L}$ (resp. $\mathfrak{R}$) generated by $L(M)$ (resp. $R(M)$) and L_δ (resp. R_δ).

The automorphism group $\{\beta_t\}_{t\in\mathbb{R}}$ of $\mathfrak{L}$ dual to $\{\alpha^n\}_{n\in\mathbb{Z}}$ is implemented by the unitary representation of $\mathbb{R}$, $\{W_t\}_{t\in\mathbb{R}}$, defined by the formula

$$(W_t f)(n) = e^{2\pi int} f(n), \quad f \in \mathbb{L}^2;$$

that is,

$$\beta_t(T) = W_t T W_t^*, \quad T \in \mathfrak{L},$$

by definition. It is elementary to check that the spectral resolution of $\{W_t\}_{t\in\mathbb{R}}$ is given by the formula

$$W_t = \sum_{n=-\infty}^{\infty} e^{2\pi int} E_n,$$

where E_n is the projection on $\mathbb{L}^2$ defined by the formula

$$(E_n f)(k) = \delta_{nk} f(n).$$

We also define the integral

$$\varepsilon_n(T) = \int_0^1 e^{-2\pi int} \beta_t(T)\, dt, \quad T \in \mathfrak{L}.$$

Furthermore, we define the noncommutative Hardy space $\mathbb{H}^2$ by

$$\mathbb{H}^2 = \{f \in \mathbb{L}^2 : f(n) = 0, n < 0\}$$

and put

$$\mathbb{H}_0^2 = \{f \in \mathbb{H}^2 : f(0) = 0\},$$

respectively. We refer the reader to [3, 4] for discussions of these algebras including some of their elementary properties. At first, we have

PROPOSITION 2.1 (*cf.* [3, 4]). *(1)* $\mathbb{L}^2 = \mathbb{H}^2 \oplus J\mathbb{H}_0^2$.
(2) $\mathbb{H}^2 = \sum_{n=0}^{\infty} \oplus E_n\mathbb{L}^2$ *and* $J\mathbb{H}_0^2 = \sum_{n=-\infty}^{-1} \oplus E_n\mathbb{L}^2$.
(3) $E_n = L_\delta^n E_0 L_\delta^{*n} = R_\delta^n E_0 R_\delta^{*n}$ *for every* $n \in \mathbb{Z}$.

Let P be the orthogonal projection of $\mathbb{L}^2$ onto $\mathbb{H}^2$. For A in $\mathfrak{L}$ (resp. $\mathfrak{R}$), the left (resp. right) Hankel operator H_A is defined by

$$H_A f = (I - P)Af, \quad \text{for } f \in \mathbb{H}^2.$$

Then H_A is a bounded operator from $\mathbb{H}^2$ into $J\mathbb{H}_0^2$. Throughout this note, all results will be formulated in terms of left Hankel operators. We leave it to the reader to rephrase them to obtain the corresponding results for right Hankel operators.

At first, we easily have the following proposition.

PROPOSITION 2.2. *Let* $A \in \mathfrak{L}$. *Then the matricial representation of* H_A *with respect to* $\mathbb{H}^2 = E_0\mathbb{L}^2 \oplus E_1\mathbb{L}^2 \oplus E_2\mathbb{L}^2 \oplus \cdots$ *and* $J\mathbb{H}_0^2 = E_{-1}\mathbb{L}^2 \oplus E_{-2}\mathbb{L}^2 \oplus E_{-3}\mathbb{L}^2 \oplus \cdots$ *is the Hankel matrix*

$$\begin{bmatrix} \ell_{x_{-1}}u^{-1} & \ell_{x_{-2}}u^{-2} & \ell_{x_{-3}}u^{-3} & \cdots \\ \ell_{x_{-2}}u^{-2} & \ell_{x_{-3}}u^{-3} & \ell_{x_{-4}}u^{-4} & \cdots \\ \ell_{x_{-3}}u^{-3} & \ell_{x_{-4}}u^{-4} & \ell_{x_{-5}}u^{-5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where $x_n \in M$ *for* $n = -1, -2, -3, \ldots$.

3. **The Nehari problem.** Best approximation problems with respect to operator norms which are not Hilbert space norms can hardly ever be solved either easily or explicity. In this section, we consider the Nehari problem. That is, given $A \in \mathfrak{L}$, find an operator B in $\mathfrak{L}_+$ such that $\|A - B\|$ is minimized. That is, we have the following main theorem.

THEOREM 3.1. *Let* A *be in* $\mathfrak{L}$. *Then* $\|H_A\| = d(A, \mathfrak{L}_+)$, *where* $d(A, \mathfrak{L}_+) = \inf\{\|A - B\| : B \in \mathfrak{L}_+\}$. *Furthermore, there exists an element* B *in* $\mathfrak{L}$ *such that* $H_A = H_B$ *and* $\|H_A\| = \|B\|$.

To prove this theorem, we need some lemmas.

LEMMA 3.2. *If* A *is in* $\mathfrak{L}$, *then* $\|H_A\| \leq d(A, \mathfrak{L}_+)$.

PROOF. Since $\mathfrak{L}_+\mathbb{H}^2 \subset \mathbb{H}^2$, we have $(I - P)Cf = 0$ for any $C \in \mathfrak{L}_+$. Thus, for every $f \in \mathbb{H}^2$,

$$H_{A-C}f = (I - P)(A - C)f = (I - P)Af = H_A f.$$

Hence we have $H_{A-C} = H_A$ and so $\|H_A\| \leq \|A - C\|$. Therefore, $\|H_A\| \leq d(A, \mathcal{L}_+)$. This completes the proof.

For every $k \in \mathbb{Z}$, we define the closed subspace $\mathbb{L}_k^2$ of $\mathbb{L}^2$ by

$$\mathbb{L}_k^2 = \{f \in \mathbb{L}^2 : f(n) = 0 \text{ for every } n < k\}.$$

Then it is clear the $\mathbb{L}_k^2 = E_k\mathbb{L}^2 \oplus E_{k+1}\mathbb{L}^2 \oplus E_{k+2}\mathbb{L}^2 \oplus \cdots$. Then we have

LEMMA 3.3. *For any $k \in \mathbb{N}$, we set*

$$S_k = \begin{bmatrix} \ell_{x_k}u^k & \ell_{x_{k-1}}u^{k-1} & \ell_{x_{k-2}}u^{k-2} & \cdots \\ \ell_{x_{k-1}}u^{k-1} & \ell_{x_{k-2}}u^{k-2} & \ell_{x_{k-3}}u^{k-3} & \cdots \\ \ell_{x_{k-2}}u^{k-2} & \ell_{x_{k-3}}u^{k-3} & \ell_{x_{k-4}}u^{k-4} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where $x_i \in M$ for every $i \leq k$. If S_k is a contraction on $\mathbb{L}_k^2$, then there is an element $x_{k+1} \in M$ such that

$$S_{k+1} = \begin{bmatrix} \ell_{x_{k+1}}u^{k+1} & \ell_{x_k}u^k & \ell_{x_{k-1}}u^{k-1} & \cdots \\ \ell_{x_k}u^k & \ell_{x_{k-1}}u^{k-1} & \ell_{x_{k-2}}u^{k-2} & \cdots \\ \ell_{x_{k-1}}u^{k-1} & \ell_{x_{k-2}}u^{k-2} & \ell_{x_{k-3}}u^{k-3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is a contraction on $\mathbb{L}_{k+1}^2$.

PROOF. Put

$$A = \begin{bmatrix} \ell_{x_{k-1}}u^{k-1} & \ell_{x_{k-2}}u^{k-2} & \ell_{x_{k-3}}u^{k-3} & \cdots \\ \ell_{x_{k-2}}u^{k-2} & \ell_{x_{k-3}}u^{k-3} & \ell_{x_{k-4}}u^{k-4} & \cdots \\ \ell_{x_{k-3}}u^{k-3} & \ell_{x_{k-4}}u^{k-4} & \ell_{x_{k-5}}u^{k-5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad B = \begin{bmatrix} \ell_{x_k}u^k \\ \ell_{x_{k-1}}u^{k-1} \\ \ell_{x_{k-2}}u^{k-2} \\ \vdots \end{bmatrix},$$

and $C = [\,\ell_{x_k}u^k \quad \ell_{x_{k-1}}u^{k-1} \quad \cdots\,]$, respectively. For some $x \in B\big(L^2(M)\big)$, we put $M_X = \begin{bmatrix} X & C \\ A & B \end{bmatrix}$. Since $[\,B \quad A\,] = \begin{bmatrix} C \\ A \end{bmatrix} = S_k$, by [5, Theorem 1], there exist sequences of real numbers $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ such that the weak limit D of $c_nC(I - d_nA^*A)^{-1}A^*B$ exists and satisfies $\|M_D\| = \max\left\{\left\|\begin{bmatrix} C \\ A \end{bmatrix}\right\|, \|\,B \quad A\,\|\right\} \leq 1$. Therefore, we have to prove that there exists an element $x_{k+1} \in M$ such that $D = \ell_{x_{k+1}}u^{k+1}$. To do this, it is sufficient to prove that $C(I - d_nA^*A)^{-1}A^*Bu^{-k+1} \in r(M)' = \ell(M)$. We now put, for each $x \in M$,

$$V(x) = \begin{bmatrix} r_{\alpha^{-1}(x)} & 0 & 0 & \cdots \\ 0 & r_{\alpha^{-2}(x)} & 0 & \cdots \\ 0 & 0 & r_{\alpha^{-3}(x)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$W(x) = \begin{bmatrix} r_{\alpha^{-k}(x)} & 0 & 0 & \cdots \\ 0 & r_{\alpha^{-k+1}(x)} & 0 & \cdots \\ 0 & 0 & r_{\alpha^{-k+2}(x)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Since $r_{\alpha(x)} = ur_x u^*$, it is clear that the following relations hold: for each $x \in M$,

(1) $r_x C = CW(x)$;

(2) $V(x)A = AW(x)$; and

(3) $V(x)B = Br_{\alpha^{-k-1}(x)}$.

From (3), we have $W(x)A^* = A^*W(x)$. Thus $W(x)A^*A = A^*AW(x)$. This implies that $W(x)(I - d_nA^*A)^{-1} = (I - d_nA^*A)^{-1}W(x)$. From (1)–(3), we have, for every $x \in M$,

$$\begin{aligned} r_xC(I - d_nA^*A)^{-1}A^*Bu^{-k-1} &= CW(x)(I - d_nA^*A)^{-1}A^*Bu^{-k-1} \\ &= C(I - d_nA^*A)^{-1}W(x)A^*Bu^{-k-1} \\ &= C(I - d_nA^*A)^{-1}A^*V(x)Bu^{-k-1} \\ &= C(I - d_nA^*A)^{-1}A^*Br_{\alpha^{-(k+1)}(x)}u^{-k-1} \\ &= C(I - d_nA^*A)^{-1}A^*Bu^{-k-1}r_x. \end{aligned}$$

This implies that $D \in \ell(M)u^{k+1}$. This completes the proof.

LEMMA 3.4. *Let T be the infinite matrix $T = [t_{ij}]_{i,j=-\infty}^{\infty}$, where $t_{ij} \in B\big(L^2(M)\big)$ for every $i, j \in \mathbb{Z}$. For every $k, m \in \mathbb{Z}$, let T_{km} be the submatrix of T, $T_{km} = [t_{ij}]_{i=k,j=m}^{\infty}$. If T_{km} defines a contraction on $\mathbb{L}_m^2$ for every k, $m \in \mathbb{Z}$, then T defines a contraction on $\mathbb{L}^2$.*

PROOF. We now put $\mathbb{L}_0^2 = \{f \in \mathbb{L}^2 : f(n) = 0$ for all but finitely many $n\}$. For every $f \in \mathbb{L}_0^2$, we set $g(k) = \sum_{j=-\infty}^{\infty} t_{kj}f(j)$ for every $k \in \mathbb{Z}$. Then we may write $Tf = g$. Suppose that $\|f\|_2 = 1$. Further, we define $f_m(i) = 0$ if $i < m$ and $= f(i)$ if $i \geq m$. Since $\|T_{km}f_m\|_2 \leq 1$, we have

$$\sum_{i=k}^{\infty} \Big\| \sum_{j=m}^{\infty} t_{ij}f_m(j) \Big\|_2^2 \leq 1.$$

Letting $m \longrightarrow \infty$, we obtain $\sum_{i=k}^{\infty} \|g(i)\|_2^2 \leq 1$. As $k \longrightarrow \infty$, we find that $\sum_{i=-\infty}^{\infty} \|g(i)\|_2^2 \leq 1$. Thus we have $g \in \mathbb{L}^2$ and so T is a contraction on L_0^2. Since $\mathbb{L}_0^2$ is dense in $\mathbb{L}^2$, T defines a contraction on $\mathbb{L}^2$. This completes the proof.

PROOF OF THEOREM 3.1. By Lemma 3.2, we have $\|H_A\| \leq d(A, \mathfrak{L}_+)$. To prove the opposite inequality, we may suppose that $\|H_A\| = 1$, replacing A by $(1/\|A\|)A$. By Proposition 2.2, H_A has the Hankel matrix

$$\begin{bmatrix} \ell_{x_{-1}}u^{-1} & \ell_{x_{-2}}u^{-2} & \ell_{x_{-3}}u^{-3} & \cdots \\ \ell_{x_{-2}}u^{-2} & \ell_{x_{-3}}u^{-3} & \ell_{x_{-4}}u^{-4} & \cdots \\ \ell_{x_{-3}}u^{-3} & \ell_{x_{-4}}u^{-4} & \ell_{x_{-5}}u^{-5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

https://doi.org/10.4153/CMB-1994-011-6 Published online by Cambridge University Press

where $x_n \in M$ for $n = -1, -2, -3, \ldots$. We shall construct a sequence $\{x_n\}_{n=0}^{\infty}$ of M inductively as follows. For $k \in \mathbb{N}$, let H_k be an operator

$$H_k = \begin{bmatrix} \ell_{x_{k-1}} u^{k-1} & \ell_{x_{k-2}} u^{k-2} & \ell_{x_{k-3}} u^{k-3} & \cdots \\ \ell_{x_{k-2}} u^{k-2} & \ell_{x_{k-3}} u^{k-3} & \ell_{x_{k-4}} u^{k-4} & \cdots \\ \ell_{x_{k-3}} u^{k-3} & \ell_{x_{k-4}} u^{k-4} & \ell_{x_{k-5}} u^{k-5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note that H_0 defines a contraction $H_0 = H_A$. We suppose that H_k is a contraction on $\mathbb{L}^2_{k-1}$. The Hankel matrix pattern fixes all the new first column except it first entry. Let

$$H_k(b) = \begin{bmatrix} b & \ell_{x_{k-1}} u^{k-1} & \ell_{x_{k-2}} u^{k-2} & \cdots \\ \ell_{x_{k-1}} u^{k-1} & \ell_{x_{k-2}} u^{k-2} & \ell_{x_{k-3}} u^{k-3} & \cdots \\ \ell_{x_{k-2}} u^{k-2} & \ell_{x_{k-3}} u^{k-3} & \ell_{x_{k-4}} u^{k-4} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$
$$= \begin{bmatrix} b & Q \\ R & S \end{bmatrix}$$

where Q, R and S are suitable matrices of type $1 \times \infty$, $\infty \times 1$ and $\infty \times \infty$, respectively. Then we have $[\,R \quad S\,] = H_k = \begin{bmatrix} Q \\ S \end{bmatrix}$ and H_k is a contraction on $\mathbb{L}^2_{k-1}$, by the inductive hypothesis. By Lemma 3.3, there exists an operator x_k in M such that $H_k(\ell_{x_k} u^k)$ defines a contraction on $\mathbb{L}^2_k$. By putting $H_{k+1} = H_k(\ell_{x_k} u^k)$, H_{k+1} is a contraction on $\mathbb{L}^2_k$. By induction, the sequence $\{x_k\}_{k=0}^{\infty}$ in M has the property that $\|H_k\| \leq 1$ for all $k \geq 0$.

Next, we consider the infinite matrix T defined by

$$T = [\ell_{x_{-i-j+1}} u^{-i-j+1}]_{i,j=-\infty}^{\infty}.$$

In the notation of Lemma 3.4, for every $k, m \in \mathbb{Z}$, let

$$T_{km} = \begin{bmatrix} \ell_{x_{-k-m+1}} u^{-k-m+1} & \ell_{x_{-k-m}} u^{-k-m} & \ell_{x_{-k-m-1}} u^{-k-m-1} & \cdots \\ \ell_{x_{-k-m}} u^{-k-m} & \ell_{x_{-k-m-1}} u^{-k-m-1} & \ell_{x_{-k-m-2}} u^{-k-m-2} & \cdots \\ \ell_{x_{-k-m-1}} u^{-k-m-1} & \ell_{x_{-k-m-2}} u^{-k-m-2} & \ell_{x_{-k-m-3}} u^{-k-m-3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

be the submatrix of T. In either case, T_{km} is a contraction on $\mathbb{L}^2_m$ for every $k, m \in \mathbb{Z}$. By Lemma 3.4, T defines a contraction on $\mathbb{L}^2$. Therefore, it is clear that $T \in \mathfrak{L}$. Then we put $B = A - T$. Since A and T have the same negative Fourier coefficients, we have $B \in \mathfrak{L}_+$. Hence we have

$$d(A, \mathfrak{L}_+) \leq \|A - B\| = \|T\| \leq \|H_A\| \leq d(A, \mathfrak{L}_+).$$

Then we have $\|H_A\| = \|T\| = d(A, \mathfrak{L}_+)$. Since $B \in \mathfrak{L}_+$, we deduce that $H_T = H_{A-B} = H_A$. This completes the proof.

By Proposition 2.2, any left Hankel matrix operator has the left Hankel matrix. Next, we consider the converse.

THEOREM 3.5 [NEHARI]. *If T is a bounded linear operator from $\mathbb{H}^2$ into $J\mathbb{H}_0^2$. Then the following conditions are equivalent.*

(1) T has the matricial representation of the form $[\ell_{x_{-i-j+1}} u^{-i-j+1}]_{i,j=1}^{\infty}$, where $x_{-n} \in M$ for $n \in \mathbb{N}$.

(2) There exists an element A in $\mathfrak{L}$ such that $H_A = T$.

Moreover, under the above condition, then

$$||T|| = \inf\{||A|| : A \in \mathfrak{L} \text{ such that } H_A = T\}.$$

PROOF. Without loss of generality, we may assume that $||T|| = 1$.

(1) $\Rightarrow$ (2). By the proof of Theorem 3.1, there exists a contraction A in $\mathfrak{L}$ which satisfies $A = [\ell_{x_{-i-j+1}} u^{-i-j+1}]_{i,j=-\infty}^{\infty}$, where $x_n \in M$ for $n \geq 0$. Then we have $H_A = T$.

(2) $\Rightarrow$ (1). By Proposition 2.2, we have (1).

Next let B be an operator in $\mathfrak{L}$ such that $H_B = T$. Then $||T|| = ||H_A|| \leq ||B||$ and so

$$||T|| \leq \inf\{||B|| : B \in \mathfrak{L}, T = H_B\}.$$

Since we can take a contraction A in $\mathfrak{L}$ such that $H_A = T$, we have the converse inequality. This completes the proof.

REFERENCES

1. U. Haagerup, *The standard form of von Neumann algebras*, Math. Scand. **37**(1975), 271–283.

2. ———, *L^p-spaces associated with an arbitrary von Neumann algebras*, Colloques internationaux du CNRS **274**(1979), 175–184.

3. M. McAsey, P. S. Mulhy and K.-S. Saito, *Nonselfadjoint crossed products (Invariant subspaces and maximality)*, Trans. Amer. Math. Soc. **248**(1979), 381–409.

4. ———, *Nonselfadjoint crossed products III (Infinite algebras)*, J. Operator Theory **12**(1984), 3–22.

5. S. Parrott, *On a quotient norm and the Sz.-Nagy-Foias lifting theorem*, J. Funct. Anal. **30**(1978), 311–328.

6. S. C. Power, *Hankel operators on Hilbert spaces*, Bull London Math. Soc. **12**(1980), 422–442.

7. ———, *Hankel operators on Hilbert spaces*, Research Notes in Math. **64**(1982).

8. M. Rosenblum and J. Rovnyak, *Hardy spaces and operator theory*, Oxford Math. Monographs, Oxford University Press, 1985.

9. K.-S. Saito, *Toeplitz operators associated with analytic crossed products*, Math. Proc. Camb. Philo. Soc. **108**(1990), 539–549.

10. ———, *Toeplitz operators associated with analytic crossed products II (Invariant subspaces and factorization)*, Integral Equations and Operator Theory **14**(1991), 251–275.

11. M. Terp, *L^p-spaces associated with von Neumann algebras*, Rapport, 1981.

12. N. Young, *An introduction to Hilbert space*, Cambridge Math., Textbooks, 1988.

Department of Mathematics
Faculty of Science
Niigata University
Niigata, 950-21
Japan