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TWO-DIMENSIONAL SHRINKING TARGET PROBLEM IN BETA-DYNAMICAL SYSTEMS

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Abstract

In this paper, we investigate the two-dimensional shrinking target problem in beta-dynamical systems. Let $\beta > 1$ be a real number and define the β-transformation on $[0, 1]$ by $T_\beta : x \to \beta x \bmod 1$. Let Ψ_i $(i = 1, 2)$ be two positive functions on $\mathbb{N}$ such that $\Psi_i \to 0$ when $n \to \infty$. We determine the Lebesgue measure and Hausdorff dimension for the lim sup set

$$W(T_\beta, \Psi_1, \Psi_2) = \{(x, y) \in [0, 1]^2 : |T^n_\beta x - x_0| < \Psi_1(n), |T^n_\beta y - y_0| < \Psi_2(n) \text{ for infinitely many } n \in \mathbb{N}\}$$

for any fixed $x_0, y_0 \in [0, 1]$.

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1. Introduction

This paper deals with the metrical properties of the shrinking target problem in higher dimensional β-dynamical systems. To set the scene, we first briefly review the one-dimensional theory.

Let $\beta > 1$ be a real number and define the transformation $T_\beta : [0, 1) \to [0, 1)$ by $T_\beta(x) = \beta x \bmod 1$ for any $x \in [0, 1)$. This transformation generates the beta-dynamical system $([0, 1], T_\beta)$. It is well known that there exists a unique T_β-invariant ergodic probability measure μ which is equivalent to the Lebesgue measure $\mathcal{L}$ on $[0, 1]$ (see Proposition 2.4 below). The measure μ is often referred to as the Parry measure, after Parry [8].

In 1967, Philipp [9] proved that for any $\beta > 1$, the dynamical Borel–Cantelli lemma holds because the transformation T_β is strongly mixing with respect to the Parry measure μ. Let $\Psi : \mathbb{N} \to \mathbb{R}^+$ be a positive function such that $\Psi(n) \to 0$ as $n \to \infty$. Consider a dynamically defined lim sup set

$$W(T_\beta, \Psi) = \{x \in [0, 1] : |T^n_\beta x - x_0| < \Psi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

Then

$$\mathcal{L}(W(T_\beta, \Psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \Psi(n) < \infty, \\ 1 & \text{if } \sum_{n=1}^{\infty} \Psi(n) = \infty \end{cases}$$

for any fixed $x_0 \in [0, 1]$.

Hausdorff dimension, denoted by $\dim_H$, is an important tool in distinguishing between sets of zero Lebesgue measure. If the approximating function Ψ decreases sufficiently quickly, then the Lebesgue measure is null and gives us no further information about the size of this set. Shen and Wang in [11] determined the Hausdorff dimension of the set $W(T_\beta, \Psi)$:

$$\dim_H W(T_\beta, \Psi) = \frac{1}{1+\alpha} \quad \text{where } \alpha = \liminf_{n\to\infty} -\frac{\log_\beta \Psi(n)}{n}.$$

The set $W(T_\beta, \Psi)$ can be generalised further by letting the fixed parameter x_0 vary in the interval $[0, 1]$, that is,

$$\hat{W}(T_\beta, \Psi) := \{(x, y) \in [0, 1]^2 : |T_\beta^n x - y| < \Psi(n) \text{ for infinitely many } n\}.$$

This set can be viewed as the doubly metric β-dynamical analogue of the classical Diophantine set given by Dodson [4]. By using Fubini's theorem and the slicing property of Hausdorff dimension, Ge and Lü [7] showed that

$$\mathcal{L}(\hat{W}(T_\beta, \Psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \Psi(n) < \infty, \\ 1 & \text{if } \sum_{n=1}^{\infty} \Psi(n) = \infty \end{cases}$$

and that

$$\dim_H \hat{W}(T_\beta, \Psi) = 1 + \frac{1}{1+\alpha} \quad \text{where } \alpha = \liminf_{n\to\infty} -\frac{\log_\beta \Psi(n)}{n}.$$

A complete metrical theory for this set was obtained very recently in [3], where a Jarník-type dichotomy law for the Hausdorff measure was proven.

It is natural to enquire about the generalisation of the shrinking target problem in beta-dynamical systems to simultaneous settings typical in classical Diophantine approximation. That is, does there exist a metrical theory for the set

$$\begin{aligned} W(T_\beta, \Psi_1, \Psi_2) = \{(x, y) \in [0, 1]^2 : |T_\beta^n x - x_0| < \Psi_1(n), \\ |T_\beta^n y - y_0| < \Psi_2(n) \text{ for infinitely many } n \in \mathbb{N}\}, \end{aligned}$$

where $x_0, y_0 \in [0, 1]$ are two given fixed points? To our surprise, nothing is known as far as the metrical theory is concerned. In this paper, we determine the Lebesgue measure and Hausdorff dimension of this set.

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THEOREM 1.1. *Let Ψ_i $(i = 1, 2)$ be two positive functions defined on $\mathbb{N}$. Then, for any $\beta > 1$,*

$$\mathcal{L}^2(W(T_\beta, \Psi_1, \Psi_2)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \Psi_1(n)\Psi_2(n) < \infty, \\ 1 & \text{if } \sum_{n=1}^{\infty} \Psi_1(n)\Psi_2(n) = \infty. \end{cases}$$

THEOREM 1.2. *Assume that $\Psi_1(n) = \beta^{-n\tau_1}$ and $\Psi_2(n) = \beta^{-n\tau_2}$ with $0 < \tau_1 \le \tau_2$. Then*

$$\dim_H W(T_\beta, \Psi_1, \Psi_2) = \min\Big\{\frac{2}{1+\tau_1}, \frac{2+\tau_2-\tau_1}{1+\tau_1}\Big\}.$$

2. Preliminaries

We begin with a brief account of some basic properties of β-expansions of real numbers and fix some notation. For the definition of Hausdorff dimension, we refer the reader to the excellent book [5].

For a real number $x > 0$, we write $\lfloor x \rfloor$ for the integer part of x. Using the β-transformation T_β, each $x \in [0, 1]$ can be uniquely expressed as a finite or an infinite series, known as the β-expansion of x (see [8]). That is, for each $x \in [0, 1]$ and $n \in \mathbb{N}$,

$$x = \frac{\epsilon_1(x,\beta)}{\beta} + \frac{\epsilon_2(x,\beta)}{\beta^2} + \cdots + \frac{\epsilon_n(x,\beta)}{\beta^n} + \frac{T_\beta^n(x)}{\beta^n}, \tag{2.1}$$

where $\epsilon_i(x,\beta) = \lfloor \beta T_\beta^{i-1} x \rfloor$ for each $i \ge 1$. Call the series (2.1) the β-expansion of x and the elements of the sequence $\{\epsilon_n(x,\beta)\}_{n\ge1}$ the digits of x. We also write (2.1) as

$$x = (\epsilon_1(x,\beta), \ldots, \epsilon_n(x,\beta), \ldots).$$

DEFINITION 2.1. A finite or an infinite sequence $(\epsilon_1, \ldots, \epsilon_n, \ldots)$ is called β-admissible if there exists an $x \in [0, 1]$ such that the β-expansion of x begins with $\epsilon_1, \ldots, \epsilon_n, \ldots$.

Denote by Σ_β^n the set of all β-admissible sequences of length n and by Σ_β the set of all infinite admissible sequences, that is,

$$\Sigma_\beta = \{\epsilon \in \mathcal{A}^{\mathbb{N}} : \epsilon \text{ is the } \beta\text{-expansion of some } x \in [0, 1]\},$$

where

$$\mathcal{A} = \{1, 2, \ldots, \lfloor \beta \rfloor\}.$$

DEFINITION 2.2. For any $(\epsilon_1, \ldots, \epsilon_n) \in \Sigma_\beta^n$, call the set

$$I_n(\epsilon_1, \ldots, \epsilon_n) := \{x \in [0, 1] : \epsilon_j(x,\beta) = \epsilon_j, 1 \le j \le n\}$$

an nth-order cylinder (with respect to the base β).

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Note that the unit interval can naturally be partitioned into a disjoint union of cylinders: for any $n \geq 1$,

$$[0,1] = \bigcup_{(\epsilon_1,\dots,\epsilon_n)\in\Sigma_\beta^n} I_n(\epsilon_1,\dots,\epsilon_n). \tag{2.2}$$

Furthermore, the nth-order cylinder $I_n(\epsilon_1,\dots,\epsilon_n)$ is an interval with the left end point

$$\frac{\epsilon_1}{\beta} + \cdots + \frac{\epsilon_n}{\beta^n}$$

and its length satisfies

$$|I_n(\epsilon_1,\dots,\epsilon_n)| \leq \beta^{-n}.$$

If a cylinder $I_n(\epsilon_1,\dots,\epsilon_n)$ of order n is of length β^{-n}, it is called a full cylinder.

In the following proposition, we collect some basic properties of β-expansions.

Proposition 2.3 (Parry [8], Fan and Wang [6], Bugeaud and Wang [1]).

(1) *For any $\beta > 1$, we have $\beta^n \leq \#\Sigma_\beta^n \leq \beta^{n+1}/(\beta-1)$, where # denotes the cardinality of a finite set.*
(2) *An nth-order cylinder $I_n(\epsilon_1,\dots,\epsilon_n)$ is full if and only if, for any admissible sequence $(\epsilon'_1,\epsilon'_2,\dots,\epsilon'_m) \in \Sigma_\beta^m$, the concatenation $(\epsilon_1,\dots,\epsilon_n,\epsilon'_1,\dots,\epsilon'_m) \in \Sigma_\beta^{n+m}$.*
(3) *For $n > 1$, among $n+1$ consecutive cylinders of order n, there exists at least one full cylinder.*

The next proposition relates the Lebesgue measure with the Parry measure.

Proposition 2.4 (Rényi [10], Parry [8]). *The Parry measure μ is equivalent to the Lebesgue measure $\mathcal{L}$ in the sense that if we write $d\mu/d\mathcal{L} = h$, then*

$$1-\beta^{-1} \leq h(x) = \bigg(\int_0^1 \sum_{n:T^n1<x} \frac{1}{\beta^n}\,dx\bigg)^{-1} \sum_{n:T^n1<x} \frac{1}{\beta^n} \leq (1-\beta^{-1})^{-1}.$$

The following strong mixing property due to Philipp is essential in studying the metrical properties of $W(T_\beta,\Psi_1,\Psi_2)$.

Lemma 2.5 (Philipp [9]). *There exists ρ with $0<\rho<1$ such that for any measurable set E and a subinterval F of $[0,1]$,*

$$\mu(T_\beta^{-n}E \cap F) = \mu(E)\mu(F) + O(\rho^n)\mu(F).$$

The next lemma is commonly known as the divergent Borel–Cantelli lemma and often plays an integral part in proving that the measure is positive for a lim sup set in the divergence case.

Lemma 2.6 (Chung [2]). *Let $(X,\mathcal{B},\nu)$ be a probability space and $\{E_n\}_{n=1}^\infty$ be a sequence of measurable sets. Assume that $\sum_{n=1}^\infty \nu(E) = \infty$. Then*

$$\nu\Big(\limsup_{n\to\infty} E_n\Big) \geq \limsup_{n\to\infty} \frac{(\sum_{1\leq i\leq n}\nu(E_n))^2}{\sum_{1\leq i\neq j\leq n}\nu(E_i\cap E_j)}.$$

The proof of Theorem 1.2 crucially relies on the following result, called the mass transference principle for lim sup sets generated by rectangles.

LEMMA 2.7 (Wang *et al.* [12]). *Let* $\{x_n\}_{n\geq 1}$ *be a sequence of points in the unit cube* $[0,1]^d$ *with* $d \geq 1$ *and* $\{r_n\}_{n\geq 1}$ *be a sequence of positive numbers tending to zero. Use* $\mathbf{a}$ *to denote a d-dimensional vector* $(a_1, \dots, a_d)$. *Define*

$$W_1 := \{x \in [0,1]^d : x \in B(x_n, r_n) \text{ for infinitely many } n \in \mathbb{N}\}$$

and, for any $\mathbf{a} = (a_1, \dots, a_d)$ *with* $1 \leq a_1 \leq \cdots \leq a_d$, *define*

$$W_{\mathbf{a}} := \{x \in [0,1]^d : x \in B^{\mathbf{a}}(x_n, r_n) \text{ for infinitely many } n \in \mathbb{N}\},$$

where we use $B^{\mathbf{a}}(x, r)$ *to denote a rectangle with centre x and side lengths* $(r^{a_1}, \dots, r^{a_d})$. *If* $\mathcal{L}^d(W_1) = 1$, *then*

$$\dim_{\mathrm{H}} W_{\mathbf{a}} \geq \min\Big\{\frac{d + ja_j - \sum_{i=1}^{j} a_j}{a_j} : 1 \leq j \leq d\Big\}.$$

3. Proof of Theorem 1.1

The proof of Theorem 1.1 naturally splits into two parts: the convergence case and the divergence case. We deal with them separately below but before that we begin by writing the set $W(T_\beta, \Psi_1, \Psi_2)$ in a way that reflects its lim sup nature.

For any fixed real numbers x_0, y_0, denote

$$W_n(T_\beta, \Psi_1, \Psi_2) = \{(x, y) \in [0,1]^2 : |T_\beta^n x - x_0| < \Psi_1(n), |T_\beta^n y - y_0| < \Psi_2(n)\}.$$

Then

$$W(T_\beta, \Psi_1, \Psi_2) = \limsup_{n\to\infty} W_n(T_\beta, \Psi_1, \Psi_2) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} W_n(T_\beta, \Psi_1, \Psi_2). \tag{3.1}$$

In view of Proposition 2.4, we know that the Lebesgue measure $\mathcal{L}$ and the Parry measure μ are equivalent. Therefore, we interchange between these terms appropriately. For convenience, we fix $C = 1 - \beta^{-1}$.

3.1. The convergent case. For any N, it follows from (3.1) that

$$W(T_\beta, \Psi_1, \Psi_2) \subseteq \bigcup_{n=N}^{\infty} W_n(T_\beta, \Psi_1, \Psi_2).$$

By using the convergence part of the Borel–Cantelli lemma, it can readily be seen that the set $W(T_\beta, \Psi_1, \Psi_2)$ is of Lebesgue measure zero if

$$\sum_{n\geq 1} \mu \times \mu(W_n(T_\beta, \Psi_1, \Psi_2)) < \infty.$$

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Denote by $B(a,r)$ a ball of radius r centred at a and by χ_A the characteristic function of the set A. Then

$$\begin{aligned}\mu\times\mu(W_n(T_\beta,\Psi_1,\Psi_2)) &= \int_0^1\int_0^1 \chi_{B(x_0,\Psi_1(n))}(T^n x)\chi_{B(y_0,\Psi_2(n))}(T^n y)\,d\mu(x)\,d\mu(y)\\ &= \int_0^1 \chi_{B(x_0,\Psi_1(n))}(T^n x)\,d\mu(x)\int_0^1 \chi_{B(y_0,\Psi_2(n))}(T^n y)\,d\mu(y).\end{aligned}$$

By the invariance property of μ under T,

$$\begin{aligned}\mu\times\mu(W_n(T_\beta,\Psi_1,\Psi_2)) &= \int_0^1 \chi_{B(x_0,\Psi_1(n))}(x)\,d\mu(x)\int_0^1 \chi_{B(y_0,\Psi_2(n))}(y)\,d\mu(y)\\ &= \mu(B(x_0,\Psi_1(n)))\mu(B(y_0,\Psi_2(n))).\end{aligned}$$

Combining this estimate with Proposition 2.4,

$$C^{-2}\Psi_1(n)\Psi_2(n)\le \mu\times\mu(W_n(T_\beta,\Psi_1,\Psi_2))\le 4C^2\Psi_1(n)\Psi_2(n). \tag{3.2}$$

Thus, under the condition that $\sum_{n=1}^\infty \Psi_1(n)\Psi_2(n)<\infty$, we conclude that

$$\mu\times\mu(W(T_\beta,\Psi_1,\Psi_2))=0,$$

which is equivalent to $\mathcal{L}^2(W(T_\beta,\Psi_1,\Psi_2))=0$.

3.2. The divergent case. It is clear that $\mu\times\mu(W(T_\beta,\Psi_1,\Psi_2))\le 1$. To prove that $\mu\times\mu(W(T_\beta,\Psi_1,\Psi_2))\ge 1$, first notice that, in view of (3.2) and the given divergent sum condition,

$$\sum_{n=1}^\infty \mu\times\mu(W_n(T_\beta,\Psi_1,\Psi_2))\ge C^{-2}\sum_{n=1}^\infty \Psi_1(n)\Psi_2(n)=\infty.$$

Next we show that the measurable sets $W_n(\Psi_1,\Psi_2)$ are quasi-pairwise independent, so that Lemma 2.6 gives the desired conclusion. So, we estimate

$$\mu\times\mu(W_n(T_\beta,\Psi_1,\Psi_2)\cap W_m(T_\beta,\Psi_1,\Psi_2)).$$

For convenience, write $W_n(T_\beta,\Psi_1,\Psi_2):=W_n$. Without loss of generality, we assume that $n>m$. Then

$$\begin{aligned}\mu\times\mu(W_n\cap W_m) &= \int_0^1\int_0^1 \chi_{B(x_0,\Psi_1(n))}(T^n x)\chi_{B(y_0,\Psi_2(n))}(T^n y)\\ &\qquad\times\chi_{B(x_0,\Psi_1(m))}(T^m x)\chi_{B(y_0,\Psi_2(m))}(T^m y)\,d\mu(x)\,d\mu(y)\\ &= \mu(T_\beta^{-n}B(x_0,\Psi_1(n))\times T_\beta^{-m}B(x_0,\Psi_2(m)))\\ &\qquad\times\mu(T_\beta^{-n}B(y_0,\Psi_1(n))\times T_\beta^{-m}B(y_0,\Psi_2(m))).\end{aligned}$$

By the invariance of μ under T and Lemma 2.5,

$$\begin{aligned}\mu\times\mu(W_n\cap W_m) &= [\mu(B(x_0,\Psi_1(n)))\mu(B(x_0,\Psi_1(m)))+O(\rho^{n-m})\mu(B(x_0,\Psi_1(n)))]\\ &\qquad\times[\mu(B(y_0,\Psi_2(n)))\mu(B(y_0,\Psi_2(m)))+O(\rho^{n-m})\mu(B(y_0,\Psi_2(n)))]\\ &= \mu\times\mu(W_n)\cdot\mu\times\mu(W_m)+O(\rho^{n-m})\mu\times\mu(W_n).\end{aligned}$$

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Thus, by Lemma 2.6,

$$
\begin{aligned}
\mu \times \mu(W(T_\beta, \Psi_1, \Psi_2)) &= \mu \times \mu\Big(\limsup_{n\to\infty} W_n\Big) \\
&\geq \limsup_{N\to\infty} \frac{(\sum_{n=1}^{N} \mu \times \mu(W_n))^2}{\sum_{1\leq n\neq m\leq N} \mu \times \mu(W_n \cap W_m)} \\
&= \limsup_{N\to\infty} \frac{(\sum_{n=1}^{N} \mu \times \mu(W_n))^2}{2\sum_{1\leq m<n\leq N}(\mu \times \mu(W_n)\cdot \mu \times \mu(W_m) + O(\rho^{n-m})\mu(W_n))} \\
&\geq 1.
\end{aligned}
$$

Since $\mu \times \mu$ is the Parry measure, by Proposition 2.4, it is equivalent to the two-dimensional Lebesgue measure $\mathcal{L}^2$. Hence, $\mathcal{L}^2(W(T_\beta, \Psi_1, \Psi_2)) = 1$, as required.

4. Proof of Theorem 1.2

4.1. The upper bound. As is common in obtaining upper bounds for the Hausdorff dimension, we first construct a natural cover for the lim sup set $W(T_\beta, \Psi_1, \Psi_2)$ and then show that the s-dimensional Hausdorff measure of this set is zero whenever $s > \dim_{\mathrm{H}} W(T_\beta, \Psi_1, \Psi_2)$.

Let $\Psi_1(n) = \beta^{-n\tau_1}$ and $\Psi_2(n) = \beta^{-n\tau_2}$ with $\tau_1 \leq \tau_2$. From (2.2), for any $n \in \mathbb{N}$,

$$
[0, 1] \times [0, 1] = \bigcup_{w,v\in\Sigma_\beta^n} I_n(w) \times I_n(v).
$$

For any $w = (w_1, \ldots, w_n), v = (v_1, \ldots, v_n) \in \Sigma_\beta^n$, write

$$
x_n(w) = \frac{w_1}{\beta} + \frac{w_2}{\beta^2} + \cdots + \frac{w_n + x_0}{\beta^n}, \qquad y_n(v) = \frac{v_1}{\beta} + \frac{v_2}{\beta^2} + \cdots + \frac{v_n + y_0}{\beta^n}. \tag{4.1}
$$

Then, for any $x \in I_n(w)$,

$$
|x - x_n(w)| = \frac{1}{\beta^n}|T_\beta^n x - x_0|.
$$

So,

$$
\begin{aligned}
W(T_\beta, \Psi_1, \Psi_2) &= \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} W_n(T_\beta, \Psi_1, \Psi_2) \\
&= \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{w,v\in\Sigma_\beta^n} \{(x, y) : x \in I_n(w), y \in I_n(v), \\
&\qquad |T_\beta^n x - x_0| < \Psi_1(n), |T_\beta^n y - y_0| < \Psi_2(n)\} \\
&\subset \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{w,v\in\Sigma_\beta^n} B\Big(x_n(w), \frac{\Psi_1(n)}{\beta^n}\Big) \times B\Big(y_n(v), \frac{\Psi_2(n)}{\beta^n}\Big) \\
&= \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{w,v\in\Sigma_\beta^n} B(x_n(w), \beta^{-n(1+\tau_1)}) \times B(y_n(v), \beta^{-n(1+\tau_2)}).
\end{aligned}
$$

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4.1.1. Case I. Recall that $\tau_1 \le \tau_2$. The rectangle

$$B(x_n(w), \beta^{-n(1+\tau_1)}) \times B(y_n(v), \beta^{-n(1+\tau_2)})$$

can be covered by a ball of radius $\beta^{-n(1+\tau_1)}$. Thus, for any $s > 0$, the s-dimensional Hausdorff measure $\mathcal{H}^s$ can be estimated as

$$\begin{aligned}\mathcal{H}^s(W(T_\beta, \Psi_1, \Psi_2)) &\le \liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{w,v\in\Sigma_\beta^n} (\beta^{-n(1+\tau_1)})^s \\ &\le \Big(\frac{\beta}{\beta-1}\Big)^2 \liminf_{N\to\infty} \sum_{n=N}^{\infty} \beta^{2n}\beta^{-n(1+\tau_1)s}.\end{aligned}$$

If $s > 2/(1+\tau_1)$, it follows that $\mathcal{H}^s(W(T_\beta, \Psi_1, \Psi_2)) = 0$, which shows that

$$\dim_{\mathrm{H}} W(T_\beta, \Psi_1, \Psi_2) \le \frac{2}{1+\tau_1}.$$

4.1.2. Case II. We cover the rectangle

$$B(x_n(w), \beta^{-n(1+\tau_1)}) \times B(y_n(v), \beta^{-n(1+\tau_2)})$$

by balls of radius $\beta^{-n(1+\tau_2)}$. Then each rectangle can be covered by at most

$$\frac{\beta^{-n(1+\tau_1)}}{\beta^{-n(1+\tau_2)}} + 1 \le 2\beta^{n(\tau_2-\tau_1)}$$

balls of radius $\beta^{-n(1+\tau_2)}$. Thus, for any $s > 0$,

$$\begin{aligned}\mathcal{H}^s(W(T_\beta, \Psi_1, \Psi_2)) &\le 2\liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{w,v\in\Sigma_\beta^n} \beta^{n(\tau_2-\tau_1)}(\beta^{-n(1+\tau_2)})^s \\ &\le 2\Big(\frac{\beta}{\beta-1}\Big)^2 \liminf_{N\to\infty} \sum_{n=N}^{\infty} \beta^{2n}\beta^{n(\tau_2-\tau_1)}\beta^{-n(1+\tau_2)s}.\end{aligned}$$

Therefore, for any $s > (2+\tau_2-\tau_1)/(1+\tau_2)$, we have $\mathcal{H}^s(W(T_\beta, \Psi_1, \Psi_2)) = 0$, which shows that

$$\dim_{\mathrm{H}} W(T_\beta, \Psi_1, \Psi_2) \le \frac{2+\tau_2-\tau_1}{1+\tau_2}.$$

4.2. The lower bound. Fix $\delta > 0$. Let n_0 be an integer such that, for all $n \ge n_0$,

$$(n+1)\beta^{-n} \le \beta^{-n(1-\delta)} \quad \text{and} \quad \tfrac{1}{2} \ge \beta^{-n\delta}.$$

For any $w = (w_1, \ldots, w_n) \in \Sigma_\beta^n$, define

$$J_n(\Psi_1, w, x_0) = \{x \in I_n(w) : |T_\beta^n(x) - x_0| < \Psi_1(n)\} = I_n(w) \cap B\Big(x_n(w), \frac{\Psi_1(n)}{\beta^n}\Big),$$

where $x_n(w)$ is given in (4.1). It is possible that the cylinder $I_n(w)$ may be very small, so that $J_n(\Psi_1, w, x_0)$ is an empty set. So, one should focus on cylinders whose length is large. In view of Proposition 2.3, we know that the *full* cylinders are in abundance and therefore should play a major role in determining the Hausdorff dimension of $W(\Psi_1, \Psi_2)$.

Fix $w = (w_1, \dots, w_n) \in \Sigma_\beta^n$ for which $I_n(w)$ is full. Then

$$I_n(w) = \Big[\frac{w_1}{\beta} + \cdots + \frac{w_n}{\beta^n}, \frac{w_1}{\beta} + \cdots + \frac{w_n + 1}{\beta^n}\Big)$$

and $x_n(w) \in I_n(w)$. It follows that $J_n(\Psi_1, w, x_0)$ contains an interval of length at least $\Psi_1(n)\beta^{-n} = \beta^{-n(1+\tau_1)}$. We denote this interval by

$$B(x_n'(w), \tfrac{1}{2}\beta^{-n(1+\tau_1)}) \quad \text{for some } x_n'(w) \in I_n(w).$$

As a result,

$$\begin{aligned}
W(T_\beta, \Psi_1, \Psi_2) &= \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} W_n(T_\beta, \Psi_1, \Psi_2) \\
&= \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{w,v\in\Sigma_\beta^n} J_n(\Psi_1, w, x_0) \times J_n(\Psi_2, v, y_0) \\
&\supset \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{w,v\in\Sigma_\beta^n, I_{n,\beta}(w), I_{n,\beta}(v) \text{ full}} B\Big(x_n'(w), \frac{1}{2}\beta^{-n(1+\tau_1)}\Big) \times B\Big(y_n'(v), \frac{1}{2}\beta^{-n(1+\tau_2)}\Big) \\
&\supset \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{w,v\in\Sigma_\beta^n, I_{n,\beta}(w), I_{n,\beta}(v) \text{ full}} B(x_n'(w), \beta^{-n(1+\tau_1+\delta)}) \times B(y_n'(v), \beta^{-n(1+\tau_2+\delta)}) \\
&:= W_{\mathbf{a}} \quad \text{with } \mathbf{a} = \Big(\frac{1+\tau_1+\delta}{1-\delta}, \frac{1+\tau_2+\delta}{1-\delta}\Big).
\end{aligned}$$

On the other hand, since the interval $[0, 1]$ can be partitioned disjointly by (2.2), from Proposition 2.3, for any $x \in [0, 1]$, among $n + 1$ consecutive cylinders of order n, there is at least one full cylinder of order n around x. So, there exists $w \in \Sigma_\beta^n$ for which $I_n(w)$ is full and such that

$$|x - x'| \le (n + 1)\beta^{-n} \le \beta^{-n(1-\delta)}$$

for any $x' \in I_n(w)$. Thus,

$$[0, 1] = \bigcup_{w\in\Sigma_\beta^n, I_n(w) \text{ full}} B(x_n'(w), \beta^{-n(1-\delta)}).$$

Clearly, the set

$$W_{\mathbf{1}} := \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{w,v\in\Sigma_\beta^n, I_n(w), I_n(v) \text{ full}} B(x_n'(w), \beta^{-n(1-\delta)}) \times B(y_n'(v), \beta^{-n(1-\delta)})$$

equals $[0, 1]^2$, so it is of full Lebesgue measure.

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Finally, we use the mass transference principle generated by rectangles (Lemma 2.7) to conclude that

$$\dim_{\mathrm{H}} W(T_\beta, \Psi_1, \Psi_2) \geq \min\left\{\frac{2}{1+\tau_1+\delta}, \frac{2+\tau_2-\tau_1}{1+\tau_2+\delta}\right\}.$$

By letting $\delta \to 0$, we reach the desired lower bound.

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