DECOMPOSITIONS OF THE CONGRUENCE LATTICE OF A SEMIGROUP

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The purpose of this note is to extend the results of Reilly and Scheiblich (**6**) (see also Scheiblich (**7**) and Hall (**2**)) on the θ-class decomposition of the congruence lattice of a regular semigroup and, at the same time, to provide an alternative proof of these results.

Except where otherwise stated, the notation and terminology is that of Howie (**4**).

We begin with some lattice-theoretic definitions. Let $(L, \leqq, \wedge, \vee)$ be a complete lattice. A nonempty subset S of L is called a *complete meet [join] subsemilattice* of L if and only if, for every nonempty family $(a_i)_{i \in I}$ of elements of S,

$$\bigvee_{i \in I} a_i \in S \quad \left[\bigvee_{i \in I} a_i \in S\right].$$

Clearly, every complete meet [join] subsemilattice of L must contain a least [greatest] element under $\leqq$. A nonempty subset of L is called a *complete sublattice* of L if and only if it is both a complete meet subsemilattice and a complete join subsemilattice of L.

An equivalence ϕ on L is called a *complete meet [join] congruence* on L if and only if, for every nonempty family $(a_i, b_i)_{i \in I}$ of elements of ϕ,

$$\left(\bigvee_{i \in I} a_i, \bigvee_{i \in I} b_i\right) \in \phi \quad \left[\left(\bigvee_{i \in I} a_i, \bigvee_{i \in I} b_i\right) \in \phi\right].$$

Further, ϕ is called a *complete lattice congruence* on L if and only if it is both a complete meet congruence and a complete join congruence. It is readily seen that if ϕ is a complete meet [join] congruence on L then each ϕ-class is a complete meet [join] subsemilattice of L.

The set of all congruences on a semigroup S is a complete lattice with respect to inclusion: it will be denoted by $\mathscr{C}(S)$. The meet of any given nonempty family $(\rho_i)_{i \in I}$ of congruences on S is the intersection $\bigcap_{i \in I} \rho_i$ while the join $\bigvee_{i \in I} \rho_i$ of the family is the least equivalence containing every ρ_i (**4**, pp. 27–28). We shall denote these by $\cap \rho_i$ and $\vee \rho_i$ respectively, for typographical simplicity. The identity congruence on S will be denoted by ι_S.

Let $\mathscr{K}$ denote any one of the symbols $\mathscr{L}, \mathscr{R}, \mathscr{H}, \mathscr{D}, \mathscr{J}$ and let S be an arbitrary semigroup. Corresponding to each $\rho \in \mathscr{C}(S)$ we define an equivalence $\varepsilon(\mathscr{K}, \rho)$ on S, in terms of $\mathscr{K}^{S/\rho}$ (Green's equivalence $\mathscr{K}$ on S/ρ), by prescribing that

$$(a, b) \in \varepsilon(\mathscr{K}, \rho) \quad \Leftrightarrow \quad (a\rho, b\rho) \in \mathscr{K}^{S/\rho}$$

and we denote the greatest congruence on S contained in $\varepsilon(\mathcal{K}, \rho)$ by $\mu(\mathcal{K}, \rho)$ (**4**, p. 27). Evidently

$$(\forall \rho \in \mathscr{C}(S)) \quad \rho \subseteq \mu(\mathcal{K}, \rho) \subseteq \varepsilon(\mathcal{K}, \rho).$$

Also, it is easily verified that

$$(\forall \rho, \sigma \in \mathscr{C}(S)) \quad \rho \subseteq \sigma \quad \Rightarrow \quad \varepsilon(\mathcal{K}, \rho) \subseteq \varepsilon(\mathcal{K}, \sigma)$$

and hence that

$$(\forall \rho, \sigma \in \mathscr{C}(S)) \quad \rho \subseteq \sigma \quad \Rightarrow \quad \mu(\mathcal{K}, \rho) \subseteq \mu(\mathcal{K}, \sigma).$$

These properties will be used without further comment below.

We now define an equivalence $\theta_{\mathcal{K}}$ on $\mathscr{C}(S)$ by the rule that

$$(\rho, \sigma) \in \theta_{\mathcal{K}} \quad \Leftrightarrow \quad \varepsilon(\mathcal{K}, \rho) = \varepsilon(\mathcal{K}, \sigma).$$

This is the central concept in the present investigation. We note immediately that since $\varepsilon(\mathcal{K}, S \times S) = S \times S$, the $\theta_{\mathcal{K}}$-class $(S \times S)\theta_{\mathcal{K}}$ comprises all congruences on S such that S/ρ is $\mathcal{K}$-simple (that is, consists of a single $\mathcal{K}$-class). In particular, $(S \times S)\theta_{\mathcal{K}}$ is the set of all group congruences on S.

In preparation for Theorem 1, which deals with the four cases $\mathcal{K} = \mathscr{L}, \mathscr{R}, \mathscr{H}$ and $\mathscr{J}$, we have

Lemma 1. *Let S be a semigroup and let $\mathcal{K}$ denote $\mathscr{L}, \mathscr{R}, \mathscr{H}$ or $\mathscr{J}$. Then*
(i) $(\forall \rho, \sigma \in \mathscr{C}(S)) \quad \rho \subseteq \sigma \subseteq \varepsilon(\mathcal{K}, \rho) \quad \Rightarrow \quad (\rho, \sigma) \in \theta_{\mathcal{K}};$
(ii) $(\forall \rho \in \mathscr{C}(S)) \quad \mu(\mathcal{K}, \rho)$ *is the greatest element in $\rho\theta_{\mathcal{K}}$.*

Proof. We establish the result for the case $\mathcal{K} = \mathscr{L}$. The argument is easily adapted to cover each of the remaining cases.

(i) Let $\rho, \sigma \in \mathscr{C}(S)$ and suppose that $\rho \subseteq \sigma \subseteq \varepsilon(\mathscr{L}, \rho)$. Let $(a, b) \in \varepsilon(\mathscr{L}, \sigma)$. Then $(a\sigma, b\sigma) \in \mathscr{L}^{S/\sigma}$ and so there exist $x, y \in S^1$ such that $(xa, b) \in \sigma$ and $(a, yb) \in \sigma$. Thus $(xa, b) \in \varepsilon(\mathscr{L}, \rho)$ and $(a, yb) \in \varepsilon(\mathscr{L}, \rho)$. This means, in particular, that there exist $u, v \in S^1$ such that $(uxa, b) \in \rho$ and $(a, vyb) \in \rho$. Hence $(a\rho, b\rho) \in \mathscr{L}^{S/\rho}$ and so $(a, b) \in \varepsilon(\mathscr{L}, \rho)$. We have thus shown that $\varepsilon(\mathscr{L}, \sigma) \subseteq \varepsilon(\mathscr{L}, \rho)$. But the reverse inclusion holds, since $\rho \subseteq \sigma$. Therefore $\varepsilon(\mathscr{L}, \rho) = \varepsilon(\mathscr{L}, \sigma)$; that is, $(\rho, \sigma) \in \theta_{\mathscr{L}}$.

(ii) Let $\rho \in \mathscr{C}(S)$. Then $\rho \subseteq \mu(\mathscr{L}, \rho) \subseteq \varepsilon(\mathscr{L}, \rho)$ and so, by (i), $(\rho, \mu(\mathscr{L}, \rho)) \in \theta_{\mathscr{L}}$. Since $\mu(\mathscr{L}, \rho)$ contains every congruence contained in $\varepsilon(\mathscr{L}, \rho)$, the result follows.

The proof of (i) above closely resembles that of Theorem 13 of (**3**). In addition, the example cited by Hall (**3**, p. 174) to show that this theorem does not hold for $\mathcal{K} = \mathscr{D}$ also shows that (i) does not hold for $\mathcal{K} = \mathscr{D}$, as we shall demonstrate. Let S be a simple idempotent-free semigroup containing at least two minimal right ideals. (The existence of such semigroups is established in (**1**, §8.2).) Then it can be shown that $\mathscr{D}(= \mathscr{R})$ is a congruence on S and that $S/\mathscr{D}$ is a nontrivial left zero semigroup (**1**, Exercise 1 for §8.2). Hence we have that $\iota_S \subseteq \mathscr{D} = \varepsilon(\mathscr{D}, \iota_S)$, while $(\iota_S, \mathscr{D}) \notin \theta_{\mathscr{D}}$ (for $\varepsilon(\mathscr{D}, \mathscr{D}) = S \times S \neq \mathscr{D}$).

We now come to the first main result.

Theorem 1. *Let S be a semigroup and let $\mathcal{K}$ denote $\mathscr{L}, \mathscr{R}, \mathscr{H}$ or $\mathscr{J}$. Then*
(i) *$\theta_{\mathcal{K}}$ is a complete join congruence on $\mathscr{C}(S)$;*
(ii) *each $\theta_{\mathcal{K}}$-class is a convex complete join subsemilattice of $\mathscr{C}(S)$.*

Proof.   For brevity, we shall denote $\varepsilon(\mathcal{K}, \rho)$ and $\mu(\mathcal{K}, \rho)$ by $\varepsilon(\rho)$ and $\mu(\rho)$ respectively.

(i) We first show that, for any nonempty family $(\tau_i)_{i \in I}$ of congruences on S,

$$(\vee \tau_i, \ \vee \mu(\tau_i)) \in \theta_{\mathcal{K}}. \tag{α}$$

Let $(\tau_i)_{i \in I}$ be such a family and let τ denote $\vee \tau_i$. Then $\mu(\tau_i) \subseteq \mu(\tau)$ for all $i \in I$ and so $\vee \mu(\tau_i) \subseteq \mu(\tau)$. Also, $\tau \subseteq \vee \mu(\tau_i)$; hence $\varepsilon(\tau) \subseteq \varepsilon(\vee \mu(\tau_i))$ and therefore $\mu(\tau) \subseteq \varepsilon(\vee \mu(\tau_i))$. Combining these facts, we have that

$$\vee \mu(\tau_i) \subseteq \mu(\tau) \subseteq \varepsilon(\vee \mu(\tau_i)).$$

Thus, by Lemma 1 (i), $(\vee \mu(\tau_i), \mu(\tau)) \in \theta_{\mathcal{K}}$. But, by Lemma 1 (ii), $(\tau, \mu(\tau)) \in \theta_{\mathcal{K}}$. Hence (α) holds.

Now let $(\rho_i, \sigma_i)_{i \in I}$ be a nonempty family of elements of $\theta_{\mathcal{K}}$ and let $\rho = \vee \rho_i$, $\sigma = \vee \sigma_i$. Since $\varepsilon(\rho_i) = \varepsilon(\sigma_i)$ for all $i \in I$ it follows that $\mu(\rho_i) = \mu(\sigma_i)$ for all $i \in I$ and so $\vee \mu(\rho_i) = \vee \mu(\sigma_i)$. But, from (α), $(\rho, \vee \mu(\rho_i)) \in \theta_{\mathcal{K}}$ and $(\sigma, \vee \mu(\sigma_i)) \in \theta_{\mathcal{K}}$. Hence $(\rho, \sigma) \in \theta_{\mathcal{K}}$. Thus $\theta_{\mathcal{K}}$ is a complete join congruence on $\mathcal{C}(S)$.

(ii) It follows from (i) that each $\theta_{\mathcal{K}}$-class is a complete join subsemilattice of $\mathcal{C}(S)$. To complete the proof we show that each $\theta_{\mathcal{K}}$-class is also convex. Let $\rho, \xi, \sigma \in \mathcal{C}(S)$ be such that $\rho \subseteq \xi \subseteq \sigma$ and $(\rho, \sigma) \in \theta_{\mathcal{K}}$. Then $(\rho \vee \xi, \sigma \vee \xi) \in \theta_{\mathcal{K}}$; that is, $(\xi, \sigma) \in \theta_{\mathcal{K}}$, as required.

Let S and $\mathcal{K}$ be as in the statement of the theorem. Evidently $\varepsilon(\mathcal{K}, \iota_S) = \mathcal{K}(= \mathcal{K}^S)$; hence if $\rho \in \iota_S \theta_{\mathcal{K}}$ then $\rho \subseteq \mathcal{K}$. Conversely, if $\rho \in \mathcal{C}(S)$ is such that $\rho \subseteq \mathcal{K}$ then $\iota_S \subseteq \rho \subseteq \varepsilon(\mathcal{K}, \iota_S)$ and so, by Lemma 1 (i), $\rho \in \iota_S \theta_{\mathcal{K}}$. Thus $\iota_S \theta_{\mathcal{K}}$ consists of all congruences contained in $\mathcal{K}$. It follows that $\iota_S \theta_{\mathcal{K}}$ is a complete sublattice of $\mathcal{C}(S)$, with least element ι_S and greatest element $\mu(\mathcal{K}, \iota_S)$.

In general, however, a $\theta_{\mathcal{K}}$-class need not be a complete sublattice of $\mathcal{C}(S)$. (To see this we need only consider $(S \times S) \theta_{\mathcal{K}}$, where S is any semigroup that does not possess a least group congruence.) It may even happen that the meet of two elements of a $\theta_{\mathcal{K}}$-class does not itself lie in that $\theta_{\mathcal{K}}$-class — a possibility that is illustrated for $\mathcal{K} = \mathcal{J}$ by the example that follows Theorem 2 below.

We next supplement the results of Lemma 1 and Theorem 1 for the case $\mathcal{K} = \mathcal{H}$ with an extension of a theorem due to Lallement (5).

Lemma 2.   *For all congruences ρ on a semigroup S*

$$\{\xi \in \mathcal{C}(S): \rho \subseteq \xi \subseteq \varepsilon(\mathcal{H}, \rho)\}$$

is a convex modular complete sublattice of $\mathcal{C}(S)$ contained in $\rho \theta_{\mathcal{H}}$.

Proof.   Let $\rho \in \mathcal{C}(S)$ and let $\mathcal{S}(\rho)$ denote the set of all $\xi \in \mathcal{C}(S)$ such that $\rho \subseteq \xi \subseteq \varepsilon(\mathcal{H}, \rho)$. Evidently $\mathcal{S}(\rho)$ is a convex complete sublattice of $\mathcal{C}(S)$, with least element ρ and greatest element $\mu(\mathcal{H}, \rho)$: also, by Lemma 1 (i), $\mathcal{S}(\rho) \subseteq \rho \theta_{\mathcal{H}}$.

It remains to prove that $\mathcal{S}(\rho)$ is modular. We see easily that $\mathcal{S}(\rho)$ is lattice-isomorphic to the complete sublattice $\mathcal{T}(\rho)$ of $\mathcal{C}(S/\rho)$ consisting of all congruences contained in $\mathcal{H}^{S/\rho}$. But $\mathcal{T}(\rho)$ is modular (5, Théorème 3.2); hence so also is $\mathcal{S}(\rho)$.

In particular, Lemma 2 shows that if a $\theta_{\mathcal{H}}$-class P has a least element then P is itself a convex complete modular sublattice of $\mathcal{C}(S)$.

For the rest of the discussion we restrict ourselves to the case of regular semigroups.

The following argument, based on a result in (3), shows that the conclusions of Lemma 1 and Theorem 1 hold when S is a regular semigroup and $\mathscr{K} = \mathscr{D}$. It suffices to establish assertion (i) of Lemma 1 for such S and $\mathscr{K}$. Accordingly, suppose that S is a regular semigroup and that $\rho, \sigma \in \mathscr{C}(S)$ are such that $\rho \subseteq \sigma \subseteq \varepsilon(\mathscr{D}, \rho)$. Let $(a, b) \in \varepsilon(\mathscr{D}, \sigma)$. Then $(a\sigma, b\sigma) \in \mathscr{D}^{S/\sigma}$. Hence, by (3, Theorem 10), there exist $c, d \in S$ such that $(c, d) \in \mathscr{D}^{S}$, $(a, c) \in \sigma$ and $(b, d) \in \sigma$. Consequently, $(c\rho, d\rho) \in \mathscr{D}^{S/\rho}$, $(a\rho, c\rho) \in \mathscr{D}^{S/\rho}$ and $(b\rho, d\rho) \in \mathscr{D}^{S/\rho}$, from which it follows that $(a\rho, b\rho) \in \mathscr{D}^{S/\rho}$. Thus $(a, b) \in \varepsilon(\mathscr{D}, \rho)$. This shows that $\varepsilon(\mathscr{D}, \sigma) \subseteq \varepsilon(\mathscr{D}, \rho)$. But the reverse inclusion holds since $\rho \subseteq \sigma$ and so $(\rho, \sigma) \in \theta_{\mathscr{D}}$.

A substantially stronger result than Theorem 1 holds for regular semigroups in the three cases $\mathscr{K} = \mathscr{L}, \mathscr{R}, \mathscr{H}$.

Theorem 2.   *Let S be a regular semigroup and let $\mathscr{K}$ denote $\mathscr{L}, \mathscr{R}$ or $\mathscr{H}$. Then*
(i) *$\theta_{\mathscr{K}}$ is a complete lattice congruence on $\mathscr{C}(S)$;*
(ii) *each $\theta_{\mathscr{K}}$-class is a convex complete sublattice of $\mathscr{C}(S)$ and is modular if $\mathscr{K} = \mathscr{H}$.*

Proof.   Again, for brevity, we denote $\varepsilon(\mathscr{K}, \rho)$ and $\mu(\mathscr{K}, \rho)$ by $\varepsilon(\rho)$ and $\mu(\rho)$ respectively.

We begin by showing that, for an arbitrary nonempty family $(\tau_i)_{i \in I}$ of congruences on S,

$$\mu(\cap \tau_i) = \cap \mu(\tau_i). \tag{β}$$

Let $(\tau_i)_{i \in I}$ be such a family and let τ denote $\cap \tau_i$. First, $\mu(\tau) \subseteq \mu(\tau_i)$ for all $i \in I$ and so $\mu(\tau) \subseteq \cap \mu(\tau_i)$. We now establish the reverse inclusion.

Consider the case $\mathscr{K} = \mathscr{L}$. Let $(a, b) \in \cap \mu(\tau_i)$ and let a', b' denote inverses in S of a, b respectively. Now $(a, b) \in \cap \varepsilon(\tau_i)$ and so $(a\tau_i, b\tau_i) \in \mathscr{L}^{S/\tau_i}$ for all $i \in I$. Hence $(ab'b, a) \in \tau_i$ and $(ba'a, b) \in \tau_i$ for all $i \in I$. It follows that $(ab'b, a) \in \tau$ and $(ba'a, b) \in \tau$. Consequently, $(a\tau, b\tau) \in \mathscr{L}^{S/\tau}$; that is, $(a, b) \in \varepsilon(\tau)$. We have thus shown that $\cap \mu(\tau_i) \subseteq \varepsilon(\tau)$. Hence $\cap \mu(\tau_i) \subseteq \mu(\tau)$. A similar argument shows that this result also holds for the cases $\mathscr{K} = \mathscr{R}$ and $\mathscr{K} = \mathscr{H}$. Thus (β) holds.

From Theorem 1, $\theta_{\mathscr{K}}$ is a complete join congruence on $\mathscr{C}(S)$ and each $\theta_{\mathscr{K}}$-class is a convex complete join subsemilattice of $\mathscr{C}(S)$. Now let $(\rho_i, \sigma_i)_{i \in I}$ be a nonempty family of elements of $\theta_{\mathscr{K}}$. For all $i \in I$, $\varepsilon(\rho_i) = \varepsilon(\sigma_i)$ and so $\mu(\rho_i) = \mu(\sigma_i)$. Hence $\cap \mu(\rho_i) = \cap \mu(\sigma_i)$. Thus, from (β), $\mu(\rho) = \mu(\sigma)$, where $\rho = \cap \rho_i$ and $\sigma = \cap \sigma_i$. But $(\rho, \mu(\rho)) \in \theta_{\mathscr{K}}$ and $(\sigma, \mu(\sigma)) \in \theta_{\mathscr{K}}$, by Lemma 1(ii). Hence $(\rho, \sigma) \in \theta_{\mathscr{K}}$. This shows that $\theta_{\mathscr{K}}$ is also a complete meet congruence on $\mathscr{C}(S)$. It follows that $\theta_{\mathscr{K}}$ is a complete lattice congruence on $\mathscr{C}(S)$ and that each $\theta_{\mathscr{K}}$-class is a convex complete sublattice of $\mathscr{C}(S)$. Moreover, from Lemma 2, if $\mathscr{K} = \mathscr{H}$ then each $\theta_{\mathscr{K}}$-class is modular. This completes the proof.

The following example, again taken from (3), shows that we cannot extend the hypotheses of Theorem 2 to include the cases $\mathscr{K} = \mathscr{D}$ or $\mathscr{K} = \mathscr{J}$. Let B denote the bicyclic semigroup: specifically, take $B = N \times N$, where N is the set of all non-negative integers and multiplication is given by

$$(m, n)(p, q) = (m - n + t, q - p + t), \quad t = \max(n, p).$$

Now let S denote the subset

$$\{((m, n), (p, q)): m + p = n + q\}$$

of the direct product $B \times B$ (an inverse semigroup). Then S is a (full) inverse subsemigroup of $B \times B$, $\mathscr{D} = \mathscr{J}$ on S and S has infinitely many $\mathscr{D}$-classes (3, §6). (In fact, $S \cong \mathscr{F}\mathscr{I}_1^1$, where $\mathscr{F}\mathscr{I}_1$ is the free inverse semigroup of rank 1.) Next, let ψ_1 and ψ_2 denote the homomorphisms of S onto B defined by the rules

$$((m, n), (p, q))\psi_1 = (m, n), \quad ((m, n), (p, q))\psi_2 = (p, q)$$

and let $\rho_i = \psi_i \circ \psi_i^{-1} (i = 1, 2)$. Clearly $S/\rho_1 \cong S/\rho_2 \cong B$ and $\rho_1 \cap \rho_2 = \iota_S$. Hence, since B is bisimple [simple] while S is not, it follows that $(S \times S)\theta_{\mathscr{D}} \; [= (S \times S)\theta_{\mathscr{J}}]$ is not closed under intersection.

To conclude, we restrict our attention to the case $\mathscr{K} = \mathscr{H}$. Let S be a regular semigroup and let E denote its set of idempotents. As in (6), we define a relation θ on $\mathscr{C}(S)$ as follows:

$$(\rho, \sigma) \in \theta \quad \Leftrightarrow \quad \rho \cap (E \times E) = \sigma \cap (E \times E).$$

Then we have

Lemma 3. *Let S be a regular semigroup. Then $\theta_{\mathscr{H}} = \theta$ on $\mathscr{C}(S)$.*

Proof. Let E denote the set of all idempotents of S. For all $\tau \in \mathscr{C}(S)$, denote $\tau \cap (E \times E)$ by $\tau|_E$ and $\varepsilon(\mathscr{H}, \tau)$ by $\varepsilon(\tau)$.

First, suppose that $(\rho, \sigma) \in \theta_{\mathscr{H}}$. Let $(e, f) \in \rho|_E$. Then $(e, f) \in \varepsilon(\rho)$ and so $(e, f) \in \varepsilon(\sigma)$. Thus $(e\sigma, f\sigma) \in \mathscr{H}^{S/\sigma}$. Hence, since $e\sigma$ and $f\sigma$ are idempotents in S/σ, $e\sigma = f\sigma$ and so $(e, f) \in \sigma|_E$. Thus $\rho|_E \subseteq \sigma|_E$. Similarly, $\sigma|_E \subseteq \rho|_E$. Hence $\rho|_E = \sigma|_E$; that is, $(\rho, \sigma) \in \theta$.

Conversely, suppose that $(\rho, \sigma) \in \theta$. Let $(a, b) \in \varepsilon(\rho)$ and let a', b' denote inverses in S of a, b respectively. Since $(a\rho, b\rho) \in \mathscr{H}^{S/\rho}$ it follows, in particular, that $(ab'b, a) \in \rho$. Hence,

$$(1) \quad (a'ab'b, a'a) \in \rho.$$

Now, by Lallement's lemma (5; 4, Lemma II 4.6) there exist $e \in E$ and $x \in S$ such that

$$(2) \quad (a'ab'b, e) \in \rho, \quad (3) \quad e = xa'ab'b.$$

By (1) and (2), $(e, a'a) \in \rho|_E$ and so, by hypothesis, $(e, a'a) \in \sigma|_E$. Hence $(ae, a) \in \sigma$ and therefore, by (3), $(axa'ab'b, a) \in \sigma$. Thus $L_{a\sigma} \leq L_{b\sigma}$ in S/σ. Interchanging a and b, we see that $L_{b\sigma} \leq L_{a\sigma}$. It follows that $(a\sigma, b\sigma) \in \mathscr{L}^{S/\sigma}$. A dual argument shows that $(a\sigma, b\sigma) \in \mathscr{R}^{S/\sigma}$. Hence $(a\sigma, b\sigma) \in \mathscr{H}^{S/\sigma}$; that is, $(a, b) \in \varepsilon(\sigma)$. We have thus proved that $\varepsilon(\rho) \subseteq \varepsilon(\sigma)$. Similarly, $\varepsilon(\sigma) \subseteq \varepsilon(\rho)$. Hence $\varepsilon(\rho) = \varepsilon(\sigma)$ and so $(\rho, \sigma) \in \theta_{\mathscr{H}}$.

Combining Lemma 3 with Theorem 2 for the case $\mathscr{K} = \mathscr{H}$, we obtain the following well-known theorem on θ-classes established by Reilly and Scheiblich (6), Scheiblich (7) and Hall (2).

Theorem 3. *Let S be a regular semigroup. Then*
  (i) *θ is a complete lattice congruence on $\mathscr{C}(S)$;*
  (ii) *each θ-class is a convex modular complete sublattice of $\mathscr{C}(S)$.*

Note added in proof. Suppose that S and $\mathscr{K}$ are as in Theorem 1. Then, for all $\rho \in \mathscr{C}(S)$, $\mu(\mathscr{K}, \rho)$ is the greatest element in $\rho\theta_{\mathscr{K}}$. Thus the mapping $\rho \mapsto \mu(\mathscr{K}, \rho)$ is a closure operation on $\mathscr{C}(S)$ and so the set of greatest elements of the $\theta_{\mathscr{K}}$-classes is a complete meet

subsemilattice of $\mathscr{C}(S)$. The same conclusions hold if S is a regular semigroup and $\mathscr{H} = \mathscr{D}$.

For S and $\mathscr{H}$ as in Theorem 2, $\theta_{\mathscr{H}}$ is a complete lattice congruence on $\mathscr{C}(S)$ and it follows that the set of least elements of the $\theta_{\mathscr{H}}$-classes is a complete join subsemilattice of $\mathscr{C}(S)$. I am grateful to Dr. T. E. Hall for pointing out the corresponding result for θ-classes.

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