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Large Time Behavior for the Cubic Nonlinear Schrödinger Equation

Nakao Hayashi and Pavel I. Naumkin

Abstract. We consider the Cauchy problem for the cubic nonlinear Schrödinger equation in one space dimension

(1) $$\begin{cases} iu_t + \frac{1}{2}u_{xx} + \bar{u}^3 = 0, & t \in \mathbf{R}, x \in \mathbf{R}, \\ u(0,x) = u_0(x), & x \in \mathbf{R}. \end{cases}$$

Cubic type nonlinearities in one space dimension heuristically appear to be critical for large time. We study the global existence and large time asymptotic behavior of solutions to the Cauchy problem (1). We prove that if the initial data $u_0 \in \mathbf{H}^{1,0} \cap \mathbf{H}^{0,1}$ are small and such that $\sup_{|\xi| \le 1} |\arg \mathcal{F}u_0(\xi) - \frac{\pi n}{2}| < \frac{\pi}{8}$ for some $n \in \mathbf{Z}$, and $\inf_{|\xi| \le 1} |\mathcal{F}u_0(\xi)| > 0$, then the solution has an additional logarithmic time-decay in the short range region $|x| \le \sqrt{t}$. In the far region $|x| > \sqrt{t}$ the asymptotics have a quasi-linear character.

1 Introduction

The purpose of this paper is to study the global existence and large time asymptotic behavior of solutions to the Cauchy problem for the cubic nonlinear Schrödinger equation in one space dimension

(2) $$\begin{cases} \mathcal{L}u + \bar{u}^3 = 0, & t \in \mathbf{R}, x \in \mathbf{R}, \\ u(0,x) = u_0(x), & x \in \mathbf{R}, \end{cases}$$

where $\mathcal{L} = i\partial_t + \frac{1}{2}\partial_x^2$. Cubic type nonlinearities in one space dimension heuristically appear to be critical for large time. Cubic nonlinear Schrödinger equations have wide physical applications (see [15], [16], [17]).

There are some works (see [2], [3], [6], [8], [12], [14], [19], [20], [23]) concerning the large time asymptotics of solutions to the nonlinear Schrödinger equations with cubic nonlinearities which have the self-conjugate property: $\mathcal{N}(e^{i\theta}u) = e^{i\theta}\mathcal{N}(u)$ for all $\theta \in \mathbf{R}$. Recent developments in this direction can be seen in [11], where we studied the asymptotic behavior in time and scattering problem for the solutions to the Cauchy problem for the derivative cubic nonlinear Schrödinger equation $\mathcal{L}u = \mathcal{N}_1(u)$ with nonlinearity

(3) $$\mathcal{N}_1(u) = \lambda_1|u|^2u + i\lambda_2|u|^2u_x + i\lambda_3u^2\bar{u}_x + \lambda_4|u_x|^2u + \lambda_5\bar{u}u_x^2 + i\lambda_6|u_x|^2u_x,$$

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where the coefficients $\lambda_1, \lambda_6 \in \mathbf{R}$, and $\lambda_2, \lambda_3, \lambda_4, \lambda_5 \in \mathbf{C}$ are such that $\lambda_2 - \lambda_3 \in \mathbf{R}$ and $\lambda_4 - \lambda_5 \in \mathbf{R}$. We proved the global in time existence and large time asymptotics of solutions to the corresponding Cauchy problem. In paper [9] we considered the cubic nonlinear Schrödinger equation $\mathcal{L}u = \mathcal{N}(u)$ without a self-conjugate property, when the nonlinearity can be represented in the form of the full derivative $\mathcal{N}(u) = i\lambda_1(|u|^2u)_x + \lambda_2(u^3)_x + \lambda_3(\bar{u}^2u)_x + \lambda_4(\bar{u}^3)_x$, $\lambda_1 \in \mathbf{R}$, $\lambda_2, \lambda_3, \lambda_4 \in \mathbf{C}$. We proved the nonexistence of the usual scattering states if $\lambda_1 \neq 0$ and if $\lambda_1 = 0$ we proved the existence of the usual scattering states. In [9] we used the techniques developed in our previous work [7], where we introduced an appropriate representation of the solution and instead of the operator $\mathcal{J} = x + it\partial_x$ we used the dilation operator $\mathcal{I}\partial_x^{-1} = x + 2t\partial_t\partial_x^{-1}$, where $\partial_x^{-1} = \int_{-\infty}^{x} dx$. The nonlinear Schrödinger equation with cubic nonlinearities, containing at least one derivative was studied in paper [18], where the large time asymptotics of solutions was found for small initial data $u_0 \in \mathbf{H}^{3,4}$, where $\mathbf{H}^{m,k} = \{\phi \in \mathbf{L}^2 : \|\phi\|_{m,k} \equiv \|\langle x\rangle^k \langle i\partial\rangle^m \phi\|_{\mathbf{L}^2} < \infty\}$ is the weighted Sobolev space, $m, k \in \mathbf{R}$, $\langle x\rangle = \sqrt{1+x^2}$. The special nonlinearities uu_x^2 or $\overline{uu_x^2}$ were considered in [22] and the global existence of small solutions was shown by a different method from [18] and ours (he used the method of the normal forms of Shatah [21]) under different assumptions on the initial data, roughly speaking, $u_0 \in \mathbf{H}^{1,0} \cap \mathbf{L}^1$ for the case uu_x^2 and $(1-\Delta)^3 u_0 \in \mathbf{H}^{1,0} \cap \mathbf{L}^1$ for the case $\overline{uu_x}^2$. Recently in paper [10] we improved the previous result of paper [18] by using much more simple and general approach and estimates in a natural function space. In [10] we studied the Cauchy problem for the cubic nonlinear Schrödinger equation $\mathcal{L}u = \mathcal{N}_1(u) + \mathcal{N}_2(u)$, where the nonlinear term $\mathcal{N}_1$ is given by (3) and the nonlinearity

$$\begin{aligned}\mathcal{N}_2 &= 3a_1u^2u_x + 3a_2uu_x^2 + 3a_3u_x^3 + 3b_1\bar{u}^2\bar{u}_x + 3b_2\bar{u}\bar{u}_x^2 + 3b_3\bar{u}_x^3 \\ &\quad + \mu_1\bar{u}^2u_x + \mu_2|u|^2\bar{u}_x + \mu_3u\bar{u}_x^2 + \mu_4\bar{u}|u_x|^2 + \mu_5|u_x|^2\bar{u}_x\end{aligned}$$

does not satisfy the self-conjugate property, here the coefficients $a_j, b_j, \mu_l \in \mathbf{C}$, $j = 1, 2, 3$, $l = 1, \dots, 5$. Thus the nonlinearity $\mathcal{N}_1 + \mathcal{N}_2$ includes all possible cubic terms with integer powers of u, $\bar{u}$, u_x and $\bar{u}_x$ and contains at least one derivative of the unknown function. We proved that if the initial data $u_0 \in \mathbf{H}^{3,0} \cap \mathbf{H}^{2,1}$ with sufficiently small norm $\|u_0\|_{3,0} + \|u_0\|_{2,1}$, then there exists a unique global solution $u \in \mathbf{C}(\mathbf{R}; \mathbf{H}^{3,0} \cap \mathbf{H}^{2,1})$ of the Cauchy problem for the cubic nonlinear Schrödinger equation $\mathcal{L}u = \mathcal{N}_1(u) + \mathcal{N}_2(u)$, and there exists a unique modified final state $W_+ \in \mathbf{L}^\infty$ such that the asymptotics

$$u(t,x) = M\mathcal{D}W_+e^{i|W_+|^2\Lambda \log t} + O(\varepsilon^3 t^{-\frac{1}{2}-\alpha})$$

is valid for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$, where $M(t) = e^{\frac{ix^2}{2t}}$, $\mathcal{D}(t)$ is the dilation operator $\mathcal{D}(t)\phi = \frac{1}{\sqrt{it}}\phi(\frac{\cdot}{t})$, $\Lambda(\xi) = \lambda_1 - (\lambda_2 - \lambda_3)\xi + (\lambda_4 - \lambda_5)\xi^2 - \lambda_6\xi^3$ and $\alpha \in (0, \frac{1}{4})$.

Thus considering all possible twenty types of cubic nonlinearities: u^3, $|u|^2u$, $u\bar{u}^2$, $\bar{u}^3$, u^2u_x, $|u|^2u_x$, $u^2\bar{u}_x$, $\bar{u}^2u_x$, $|u|^2\bar{u}_x$, $\bar{u}^2\bar{u}_x$, uu_x^2, $\bar{u}u_x^2$, $u|u_x|^2$, $\bar{u}|u_x|^2$, $u\bar{u}_x^2$, $\bar{u}\bar{u}_x^2$, u_x^3, $|u_x|^2u_x$, $|u_x|^2\bar{u}_x$, and $\bar{u}_x^3$, containing integer powers of u, $\bar{u}$, u_x and $\bar{u}_x$, we can see that as far as we know there are no results on the global existence and large time asymptotics

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of solutions for the following three types of nonlinearities u^3, $\bar{u}^3$ and $u\bar{u}^2$ which do not satisfy the self-conjugate property. For the nonlinearity $\lambda|u|^2u$ with complex λ, the large time behavior of solutions is also unknown. In the present paper we make an attempt to cover this gap considering the Cauchy problem (2) with nonlinearity $\bar{u}^3$. We believe that the other two types of nonlinearities u^3 and $u\bar{u}^2$ could be also considered via some modification of our method.

To state our result precisely, we give now some notations. We denote the linear Schrödinger evolution group by

$$\mathcal{U}(t)\phi = \mathcal{F}^{-1}e^{-\frac{i}{2}t\xi^2}\mathcal{F}\phi = \frac{1}{\sqrt{2\pi it}}\int e^{\frac{i}{2t}(x-y)^2}\phi(y)\,dy = M(t)\mathcal{D}(t)\mathcal{F}M(t)\phi,$$

where $M(t) = e^{\frac{ix^2}{2t}}$, $\mathcal{D}(t)$ is the dilation operator $\mathcal{D}(t)\phi = \frac{1}{\sqrt{it}}\phi(\frac{\cdot}{t})$, $\mathcal{F}\phi \equiv \hat{\phi} = \frac{1}{\sqrt{2\pi}}\int e^{-ix\xi}\phi(x)\,dx$ denotes the Fourier transform of the function ϕ. Then the inverse Fourier transformation is $\mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}}\int e^{ix\xi}\phi(\xi)\,d\xi$, $\mathcal{D}(t)^{-1} = i\mathcal{D}(\frac{1}{t})$ and

$$\mathcal{U}(-t)\phi = i\overline{M}(t)\mathcal{F}^{-1}\mathcal{D}\left(\frac{1}{t}\right)\overline{M}(t).$$

We essentially use the estimates involving the operator

$$\mathcal{J} = x + it\partial_x = \mathcal{U}(t)x\mathcal{U}(-t) = itM(t)\partial_x\overline{M}(t)$$

to prove the main result.

We denote the usual Lebesgue space $\mathbf{L}^p = \{\phi \in \mathbf{S}'; \|\phi\|_p < \infty\}$, where the norm $\|\phi\|_p = (\int_{\mathbf{R}}|\phi(x)|^p\,dx)^{1/p}$ if $1 \le p < \infty$ and $\|\phi\|_\infty = \operatorname{ess.sup}\{|\phi(x)|; x \in \mathbf{R}\}$ if $p = \infty$. For simplicity we write $\|\cdot\| = \|\cdot\|_2 = \|\cdot\|_{\mathbf{L}^2}$. Weighted Sobolev space is $\mathbf{H}^{m,k} = \{\phi \in \mathbf{S}' : \|\phi\|_{m,k} \equiv \|\langle x\rangle^k\langle i\partial\rangle^m\phi\| < \infty\}$, $m, k \in \mathbf{R}$, $\langle x\rangle = \sqrt{1+x^2}$. Different positive constants we denote by the same letter C. We use greek letters for Fourier space variables to make notations more transparent.

The aim of the paper is to prove the following result.

Theorem 1.1 *Let the initial data $u_0 \in \mathbf{H}^{1,0} \cap \mathbf{H}^{0,1}$ have a sufficiently small norm $\|u_0\|_{1,0} + \|u_0\|_{0,1} \equiv \varepsilon > 0$. Also we assume that $\sup_{|\xi|\le 1}|\arg\hat{u}_0(\xi) - \frac{\pi n}{2}| < \frac{\pi}{8} - \varepsilon$ for some $n \in \mathbf{Z}$, and $\inf_{|\xi|\le 1}|\hat{u}_0(\xi)| > C\varepsilon > 0$. Then there exists a unique solution $u \in \mathbf{C}\big([0,\infty), \mathbf{H}^{1,0}\cap\mathbf{H}^{0,1}\big)$ of the Cauchy problem (2). Moreover there exists a unique final state $W_+ \in \mathbf{L}^\infty$ such that the following asymptotics for $t \to \infty$*

$$\begin{aligned} u(t,x) &= e^{\frac{ix^2}{2t}}\frac{1}{\sqrt{it}}\frac{|W_+(\frac{x}{t})|}{\sqrt{1+\frac{2\pi}{\sqrt{3}}|W_+(\frac{x}{t})|^2\log\big(\min(t,\langle\frac{t^2}{x^2}\rangle)\big)}} \\ &\quad + O\left(\frac{1}{\sqrt{t}\log\log(t+1)\sqrt{1+\frac{2\pi}{\sqrt{3}}|W_+(\frac{x}{t})|^2\log\big(\min(t,\langle\frac{t^2}{x^2}\rangle)\big)}}\right) \end{aligned} \tag{4}$$

is valid uniformly with respect to $x \in \mathbf{R}$.

Remark 1.2 Note that the solution u given by the asymptotic formula (4) gains an additional logarithmic time-decay in the short range region $|x| \le \sqrt{t}$. In the far region $|x| > \sqrt{t}$ the asymptotics has a quasi-linear character.

We now give a sketch of the proof. As is well-known it is very important to estimate the norm $\|\mathcal{J}u\|$ to obtain the large time decay estimates of the solution. If we apply operator $\mathcal{J}$ to equation (2) we get

$$\mathcal{L}\mathcal{J}u + \bar{u}^2\overline{\mathcal{J}u} + 4it\partial_x(\bar{u}^3) = 0.$$

The last term in the left-hand side of the above equation has an explicit time growth. To eliminate this term we apply (see Lemma 2.1 below) the method analogous to that of normal forms by Shatah [21]. By making use of the identity given by Lemma 2.1 we get the estimate

$$\|\mathcal{J}u\| \le C\varepsilon + C\varepsilon^3 t^{\frac{1}{4}}(1+\varepsilon^2\log t)^{-\frac{3}{2}}.$$

Then to obtain optimal time decay estimates of the solution in the uniform norm we need to estimate the function $v(t) = \mathcal{F}\mathcal{U}(-t)u(t)$. Here we use the method similar to our paper [5]. Roughly speaking, we get from equation (2)

$$v_t(t,\xi) = -\frac{1}{t}e^{-\frac{2}{3}it\xi^2}\bar{v}^3(t,\xi) + O\big(\varepsilon^5 t^{-1}(1+\varepsilon^2\log t)^{-\frac{5}{2}}\big).$$

In order to eliminate the first divergent term in the right-hand side of the above equation we change the dependent variable $v(t,\xi) = f(t,\xi)e^{-\phi+ig}$ for $|\xi| \le 1$, where ϕ and g are real valued functions such that

$$\phi_t = -\frac{1}{t}e^{-2\phi}\Re\big(e^{-\frac{2}{3}it\xi^2-4ig}\bar{f}^3(t,\xi)f^{-1}(t,\xi)\big)$$

and

$$g_t = -\frac{1}{t}e^{-2\phi}\Im\big(e^{-\frac{2}{3}it\xi^2-4ig}\bar{f}^3(t,\xi)f^{-1}(t,\xi)\big),$$

with initial conditions $\phi(2,\xi) = 0$, $g(2,\xi) = 0$, then we obtain a more rapid time decay for f

$$f_t(t,\xi) = O\big(\varepsilon^5 t^{-1}e^{\phi}(1+\varepsilon^2\log t)^{-\frac{5}{2}}\big),$$

hence f can be easily estimated and as the result we obtain the asymptotic formula (4).

The rest of the paper is organized as follows. In Section 2 we give some preliminary estimates. In Lemma 2.1 we prove an identity which is an analog of the method of normal forms by Shatah. In Lemmas 2.2–2.4 we prepare large time estimates of the solution via the operator $\mathcal{J}$ and estimates of the operator $\mathcal{F}\mathcal{U}(-t)$ acting to the nonlinearity of equation (2). Lemma 2.5 is devoted to the large time asymptotics of the nonlinearity in the $\mathcal{F}\mathcal{U}(-t)$ representation. Section 3 is devoted to the proof of Theorem 1.1.

2 Preliminary Estimates

First we derive an analog of the method of normal forms by Shatah. Define the trilinear operators Q_k by the Fourier transformation

$$Q_k(\varphi, \phi, \psi) = \frac{1}{\pi}\mathcal{F}^{-1}\iint (1+it\Lambda)^{-k}\hat{\varphi}(\xi_1)\hat{\phi}(\xi_2)\hat{\psi}(\xi_3)\,d\eta\,d\zeta,$$

where $\xi_1 = \frac{\xi}{3}+\eta-\zeta$, $\xi_2 = \frac{\xi}{3}-\eta-\zeta$, $\xi_3 = \frac{\xi}{3}+2\zeta$, $\Lambda = \frac{2}{3}\xi^2+\eta^2+3\zeta^2$.

Lemma 2.1 *We have the identity*

$$\begin{aligned} t^n\bar{u}^3 = &-i\mathcal{L}t^{n+1}Q_1(\bar{u},\bar{u},\bar{u}) - t^nQ_2(\bar{u},\bar{u},\bar{u}) - (n-1)t^nQ_1(\bar{u},\bar{u},\bar{u}) \\ &- 3it^{n+1}Q_1(\bar{u},\bar{u},\overline{\mathcal{L}u}), \end{aligned} \tag{5}$$

where $n \geq 0$.

Proof Applying the operator $\mathcal{K} = \mathcal{F}\mathcal{U}(-t) = e^{\frac{i}{2}t\xi^2}\mathcal{F}$, then for the function $v(t) = \mathcal{K}u(t)$ we get

$$\mathcal{K}(\bar{u}^3) = \frac{1}{\pi}\iint e^{it\Lambda}\overline{v(t,-\xi_1)v(t,-\xi_2)v(t,-\xi_3)}\,d\eta\,d\zeta,$$

where $\xi_1 = \frac{\xi}{3}+\eta-\zeta$, $\xi_2 = \frac{\xi}{3}-\eta-\zeta$, $\xi_3 = \frac{\xi}{3}+2\zeta$, $\Lambda = \frac{1}{2}(\xi^2+\sum_{j=1}^3\xi_j^2) = \frac{2}{3}\xi^2+\eta^2+3\zeta^2$. Using the identity

$$e^{it\Lambda} = \partial_t\big(t(1+it\Lambda)^{-1}e^{it\Lambda}\big) - (1+it\Lambda)^{-2}e^{it\Lambda} + (1+it\Lambda)^{-1}e^{it\Lambda},$$

we obtain

$$\begin{aligned} \mathcal{K}(\bar{u}^3) = \partial_t&\left(\frac{t}{\pi}\iint(1+it\Lambda)^{-1}e^{it\Lambda}\overline{v(t,-\xi_1)v(t,-\xi_2)v(t,-\xi_3)}\,d\eta\,d\zeta\right) \\ &-\frac{1}{\pi}\iint(1+it\Lambda)^{-2}e^{it\Lambda}\overline{v(t,-\xi_1)v(t,-\xi_2)v(t,-\xi_3)}\,d\eta\,d\zeta \\ &+\frac{1}{\pi}\iint(1+it\Lambda)^{-1}e^{it\Lambda}\overline{v(t,-\xi_1)v(t,-\xi_2)v(t,-\xi_3)}\,d\eta\,d\zeta \\ &-\frac{3t}{\pi}\iint(1+it\Lambda)^{-1}e^{it\Lambda}\overline{v(t,-\xi_1)v(t,-\xi_2)v_t(t,-\xi_3)}\,d\eta\,d\zeta. \end{aligned}$$

Since $\mathcal{K}^{-1}\partial_t\mathcal{K} = -i\mathcal{L}$, applying the inverse operator $\mathcal{K}^{-1}$, we return to the function $u(t,x)$ to have

$$\bar{u}^3 = -i\mathcal{L}tQ_1(\bar{u},\bar{u},\bar{u}) - Q_2(\bar{u},\bar{u},\bar{u}) + Q_1(\bar{u},\bar{u},\bar{u}) - 3itQ_1(\bar{u},\bar{u},\overline{\mathcal{L}u}),$$

hence via the relation $t^n\mathcal{L} = \mathcal{L}t^n - int^{n-1}$ we get the identity (5). Lemma 2.1 is proved. ■

Denote $\mathcal{K} = \mathcal{F}\mathcal{U}(-t)$, $E = e^{\frac{it}{2}\xi^2}$.

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Lemma 2.2 *We have the estimates*

$$\|\mathcal{D}(t)\mathcal{F}(M-1)\mathcal{U}(-t)u(t)\|_p \le Ct^{-\frac{3}{4}+\frac{1}{2p}}\|\mathcal{J}u\|,$$

where $2 \le p \le \infty$,

$$\|\partial_\xi E^{\frac{4}{3}}\mathcal{K}\bar{u}^3(t)\| \le C\|u\|_\infty^2\,\|\mathcal{J}u\|$$

and

$$\|\psi^3\mathcal{K}\bar{u}^3(t)\|_\infty \le Ct^{-1}(\|\psi\mathcal{K}u\|_\infty + t^{-\frac{1}{4}}\|\mathcal{J}u\|)^3 + Ct^{-\frac{1}{4}}\|u\|_\infty^2\,\|\mathcal{J}u\|$$

for all $t > 0$, *where* $\psi = \langle t\xi^2\rangle^{-\gamma}$, $\gamma > 0$.

Proof Since $|M-1| \le C\frac{|x|}{\sqrt{t}}$ for all $x \in \mathbf{R}$, $t > 0$ we obtain

$$\begin{aligned}\|\mathcal{D}(t)\mathcal{F}(M-1)\mathcal{U}(-t)u(t)\| &= \|(M-1)\mathcal{U}(-t)u(t)\| \\ &\le Ct^{-\frac{1}{2}}\|x\mathcal{U}(-t)u(t)\| = Ct^{-\frac{1}{2}}\|\mathcal{J}u\|.\end{aligned}$$

Using the estimate $\|\phi\|_\infty \le \sqrt{2}\|\phi\|^{\frac{1}{2}}\,\|\partial_x\phi\|^{\frac{1}{2}}$, we get

$$\begin{aligned}&\|\mathcal{D}(t)\mathcal{F}(M-1)\mathcal{U}(-t)u(t)\|_\infty \\ &\quad= Ct^{-\frac{1}{2}}\|\mathcal{F}(M-1)\mathcal{U}(-t)u(t)\|_\infty \\ &\quad\le Ct^{-\frac{1}{2}}\|\mathcal{F}(M-1)\mathcal{U}(-t)u(t)\|^{\frac{1}{2}}\,\|\partial_\xi\mathcal{F}(M-1)\mathcal{U}(-t)u(t)\|^{\frac{1}{2}} \\ &\quad\le Ct^{-\frac{3}{4}}\|x\mathcal{U}(-t)u(t)\| = Ct^{-\frac{3}{4}}\|\mathcal{J}u\|.\end{aligned}$$

We apply the Hölder inequality $\|\phi\|_p \le C\|\phi\|^{\frac{2}{p}}\,\|\phi\|_\infty^{1-\frac{2}{p}}$ to have the first estimate of the lemma. Via the identity

$$\mathcal{F}\overline{M}\mathcal{F}^{-1}\mathcal{D}\Big(\frac{1}{t}\Big)\overline{M}\phi = \sqrt{i}\mathcal{D}\Big(\frac{\rho}{2}\Big)\overline{E^{\rho(\rho-1)}}\mathcal{F}\overline{M}^{\frac{1}{\rho}}\mathcal{F}^{-1}\mathcal{D}\Big(\frac{1}{t}\Big)\overline{M}^{\rho}\phi,$$

where $E = e^{\frac{it}{2}\xi^2}$, $\rho \ne 0$, we get

$$\mathcal{K}\bar{u}^3 = i\mathcal{F}\overline{M}\mathcal{F}^{-1}\mathcal{D}\Big(\frac{1}{t}\Big)\overline{M}\bar{u}^3 = i^{\frac{3}{2}}\overline{E}^{\frac{4}{3}}\mathcal{D}\Big(-\frac{3}{2}\Big)\mathcal{F}M^{\frac{1}{3}}\mathcal{F}^{-1}\mathcal{D}\Big(\frac{1}{t}\Big)M^3\bar{u}^3.$$

Therefore

$$\begin{aligned}\|\partial_\xi E^{\frac{4}{3}}\mathcal{K}\bar{u}^3\| &= C\Big\|\partial_\xi\mathcal{F}M^{\frac{1}{3}}\mathcal{F}^{-1}\mathcal{D}\Big(\frac{1}{t}\Big)M^3\bar{u}^3\Big\| \\ &\le C\Big\|\partial_x\mathcal{D}\Big(\frac{1}{t}\Big)M^3\bar{u}^3\Big\| \le C\|t\partial_x M^3\bar{u}^3\| \le C\|u\|_\infty^2\,\|\mathcal{J}u\|,\end{aligned}$$

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where we have used the identity $\mathcal{J} = Mit\partial_x\overline{M}$. Thus the second estimate is true. To prove the third estimate, we write via the inequality $\|\phi\|_\infty \le \sqrt{2}\|\phi\|^{\frac{1}{2}}\,\|\partial_x\phi\|^{\frac{1}{2}}$

$$\begin{aligned}
&\Big\|\mathcal{F}(M^{\frac{1}{3}}-1)\mathcal{F}^{-1}\mathcal{D}\Big(\frac{1}{t}\Big)M^3\bar{u}^3\Big\|_\infty \\
&\quad\le C\Big\|\mathcal{F}(M^{\frac{1}{3}}-1)\mathcal{F}^{-1}\mathcal{D}\Big(\frac{1}{t}\Big)M^3\bar{u}^3\Big\|^{\frac{1}{2}}\,\Big\|\partial_\xi\mathcal{F}(M^{\frac{1}{3}}-1)\mathcal{F}^{-1}\mathcal{D}\Big(\frac{1}{t}\Big)M^3\bar{u}^3\Big\|^{\frac{1}{2}} \\
&\quad\le Ct^{-\frac{1}{4}}\Big\|\partial_x\mathcal{D}\Big(\frac{1}{t}\Big)M^3\bar{u}^3\Big\| \le Ct^{-\frac{1}{4}}\Big\|t\partial_x M^3\bar{u}^3\Big\| \le Ct^{-\frac{1}{4}}\,\|u\|_\infty^2\,\|\mathcal{J}u\|.
\end{aligned}$$

Then since

$$\begin{aligned}
&\Big\|\psi\mathcal{D}\Big(\frac{1}{t}\Big)\overline{M}u\Big\|_\infty \\
&\quad\le \|\psi\mathcal{K}u\|_\infty + \Big\|\mathcal{F}(M-1)\mathcal{F}^{-1}\mathcal{D}\Big(\frac{1}{t}\Big)\overline{M}u\Big\|_\infty \\
&\quad\le \|\psi\mathcal{K}u\|_\infty + C\Big\|(M-1)\mathcal{F}^{-1}\mathcal{D}\Big(\frac{1}{t}\Big)\overline{M}u\Big\|^{\frac{1}{2}}\,\Big\|\partial_\xi\mathcal{F}(M-1)\mathcal{F}^{-1}\mathcal{D}\Big(\frac{1}{t}\Big)\overline{M}u\Big\|^{\frac{1}{2}} \\
&\quad\le \|\psi\mathcal{K}u\|_\infty + Ct^{-\frac{1}{4}}\Big\|\partial_x\mathcal{D}\Big(\frac{1}{t}\Big)\overline{M}u\Big\| \le C\|\psi\mathcal{K}u\|_\infty + Ct^{-\frac{1}{4}}\|\mathcal{J}u\|,
\end{aligned}$$

we find that

$$\Big\|\psi^3\mathcal{D}\Big(\frac{1}{t}\Big)M^3\bar{u}^3\Big\|_\infty \le \frac{C}{t}\Big\|\psi\mathcal{D}\Big(\frac{1}{t}\Big)\overline{M}u\Big\|_\infty^3 \le \frac{C}{t}(\|\psi\mathcal{K}u\|_\infty + t^{-\frac{1}{4}}\|\mathcal{J}u\|)^3.$$

Thus we get

$$\begin{aligned}
\|\psi^3\mathcal{K}\bar{u}^3\|_\infty &\le C\Big\|\mathcal{F}(M^{\frac{1}{3}}-1)\mathcal{F}^{-1}\mathcal{D}\Big(\frac{1}{t}\Big)M^3\bar{u}^3\Big\|_\infty + C\Big\|\psi^3\mathcal{D}\Big(\frac{1}{t}\Big)M^3\bar{u}^3\Big\|_\infty \\
&\le Ct^{-1}(\|\psi\mathcal{K}u\|_\infty + t^{-\frac{1}{4}}\|\mathcal{J}u\|)^3 + Ct^{-\frac{1}{4}}\|u\|_\infty^2\,\|\mathcal{J}u\|.
\end{aligned}$$

Therefore the third estimate of the lemma is true. Lemma 2.2 is proved. ■

The next lemma will be used below for estimating nonlinearities of the form $\mathcal{K}Q_k(\bar{u},\bar{u},\bar{u})$, where $\mathcal{K} = \mathcal{F}\mathcal{U}(-t)$. Consider the following integrals

$$\iint e^{itS}A_k\Phi\,d\eta\,d\zeta \quad\text{and}\quad \iint e^{itS}A_k(\Phi-\Phi_0)\,d\eta\,d\zeta,$$

where

$$\boldsymbol{\xi} = (\xi_1,\xi_2,\xi_3),\quad \xi_1 = \lambda\xi+\eta-\zeta,\quad \xi_2 = \lambda\xi-\eta-\zeta,\quad \xi_3 = \frac{\lambda}{\alpha}\xi+2\zeta,$$

$$\boldsymbol{\xi}_0 = \Big(\lambda\xi,\lambda\xi,\frac{\lambda}{\alpha}\xi\Big),\quad \Phi = \Phi(\boldsymbol{\xi}) = \prod_{j=1}^{3}\phi_j(\xi_j),\quad \Phi_0 = \Phi(\boldsymbol{\xi}_0),$$

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$$S = \frac{1}{2}\Big(\xi^2 + \sum_{j=1}^{2} \xi_j^2 + \alpha\xi_3^2\Big) = \omega\xi^2 + \eta^2 + \beta\zeta^2,$$

$$\alpha > 0, \beta = 1 + 2\alpha,\ \omega = \frac{1}{2} + \lambda^2 + \frac{\lambda^2}{2\alpha},$$

the function

$$A_k = A_k(\xi,\eta,\zeta,t) = \frac{1}{\pi}(1 + it\Lambda)^{-k}, \quad k = 1, 2,\ \Lambda = \frac{1}{2}\Big(\xi^2 + \sum_{j=1}^{3} \xi_j^2\Big).$$

Denote $\psi = \langle t\xi^2\rangle^{-\gamma}$, $\gamma > 0$ is small enough.

Lemma 2.3 *We have the estimates*

$$\Big\| \langle t\xi^2\rangle^{\delta} \iint e^{itS} A_k \Phi \, d\eta \, d\zeta \Big\|_{\infty} \le Ct^{-1} \prod_{j=1}^{3} (\|\psi\phi_j\|_{\infty} + t^{-\frac{1}{4}} \|\phi_j'\|)$$

and

$$\Big\| \langle t\xi^2\rangle^{\delta} \iint e^{itS} A_k (\Phi - \Phi_0) \, d\eta \, d\zeta \Big\|_{\infty} \le Ct^{-\frac{5}{4}} \sum_{k=1}^{3} \|\phi_k'\| \prod_{j\ne k} (\|\psi\phi_j\|_{\infty} + t^{-\frac{1}{4}} \|\phi_j'\|)$$

for all $t \ge 1$*, where* $k = 1, 2$*,* $\delta \in [0, 1 - 3\gamma]$*.*

Proof First using the identity $e^{itS} = \mathcal{Y}\frac{\partial}{\partial\eta}(\eta e^{itS})$, with $\mathcal{Y} = (1 + 2it\eta^2)^{-1}$, we integrate by parts with respect to η to get

$$\iint e^{itS} A_k (\Phi - \sigma\Phi_0) \, d\eta \, d\zeta = I_1 + I_2 + I_3,$$

where $\sigma = 0, 1$ and

$$I_1 = C \iint e^{itS} (\Phi - \sigma\Phi_0) \eta \partial_\eta (A_k \mathcal{Y}) \, d\eta \, d\zeta,$$

$$I_2 = -\iint e^{itS} A_k \eta \mathcal{Y} \partial_{\xi_1} \Phi \, d\eta \, d\zeta \quad \text{and} \quad I_3 = \iint e^{itS} A_k \eta \mathcal{Y} \partial_{\xi_2} \Phi \, d\eta \, d\zeta.$$

Using the identity

$$e^{itS} = \mathcal{Z}\frac{\partial}{\partial\zeta}(\zeta e^{itS}),$$

where $\mathcal{Z} = (1 + 2it\beta\zeta^2)^{-1}$, we integrate by parts with respect to ζ in the integral I_1 to find that

$$I_1 = C \iint e^{itS} (\Phi - \sigma\Phi_0) \eta\zeta \partial_\eta \partial_\zeta (A_k \mathcal{Y}\mathcal{Z}) \, d\eta \, d\zeta + C \iint e^{itS} \eta \partial_\eta (A_k \mathcal{Y}) \zeta \mathcal{Z} \partial_\zeta \Phi \, d\eta \, d\zeta.$$

Via the estimates

$$\langle t\Lambda\rangle \geq C\langle t\boldsymbol{\xi}^2\rangle + C\langle t\xi^2\rangle, \quad |\eta\partial_\eta(A_k\mathcal{Y})| \leq C\langle t\eta^2\rangle^{-1}\langle t\Lambda\rangle^{-1}, \quad |\mathcal{Z}| \leq C\langle t\zeta^2\rangle^{-1}$$

and

$$|\eta\zeta\partial_\eta\partial_\zeta(A_k\mathcal{Y}\mathcal{Z})| \leq C\langle t\eta^2\rangle^{-1}\langle t\zeta^2\rangle^{-1}\langle t\Lambda\rangle^{-1},$$

we obtain

$$\begin{aligned}\|\langle t\xi^2\rangle^\delta I_1\|_\infty \leq C&\left\| \iint \frac{\langle t\xi^2\rangle^\delta |\Phi - \sigma\Phi_0|\, d\eta\, d\zeta}{\langle t\eta^2\rangle\langle t\zeta^2\rangle\langle t\Lambda\rangle}\right\|_\infty \\ &+ C\left\| \iint \frac{\langle t\xi^2\rangle^\delta |\zeta|\, |\partial_\zeta\Phi|\, d\eta\, d\zeta}{\langle t\eta^2\rangle\langle t\zeta^2\rangle\langle t\Lambda\rangle}\right\|_\infty,\end{aligned}$$

hence in the case $\sigma = 0$

$$\begin{aligned}\|\langle t\xi^2\rangle^\delta I_1\|_\infty &\leq C\prod_{k=1}^3 \|\psi\phi_k\|_\infty \iint \frac{d\eta d\zeta}{\langle t\eta^2\rangle\langle t\zeta^2\rangle} \\ &\quad + Ct^{-\frac{1}{2}}\sum_{j=1}^3 \|\phi_j'\| \prod_{k\neq j} \|\psi\phi_k\|_\infty \int \frac{d\eta}{\langle t\eta^2\rangle}\sqrt{\int \frac{d\zeta}{\langle t\zeta^2\rangle}} \\ &\leq Ct^{-1}\prod_{k=1}^3 \|\psi\phi_k\|_\infty + Ct^{-\frac{5}{4}}\sum_{j=1}^3 \|\phi_j'\| \prod_{k\neq j}\|\psi\phi_k\|_\infty\end{aligned}$$

and in the case $\sigma = 1$

$$\begin{aligned}&\|\langle t\xi^2\rangle^\delta I_1\|_\infty \\ &\quad \leq C\sum_{j=1}^3 \|\phi_j'\| \prod_{k\neq j}\|\psi\phi_k\|_\infty \left(\iint \frac{\sqrt{|\eta| + |\zeta|}\, d\eta\, d\zeta}{\langle t\eta^2\rangle\langle t\zeta^2\rangle} + t^{-\frac{1}{2}}\int \frac{d\eta}{\langle t\eta^2\rangle}\sqrt{\int \frac{d\zeta}{\langle t\zeta^2\rangle}}\right) \\ &\quad \leq Ct^{-\frac{5}{4}}\sum_{j=1}^3 \|\phi_j'\| \prod_{k\neq j}\|\psi\phi_k\|_\infty.\end{aligned}$$

Now we estimate the second integral I_2. We make a change of variables of integration $\eta = \beta\tilde{\zeta} - \tilde{\eta}$ and $\zeta = -\tilde{\zeta} - \tilde{\eta}$ to get

$$I_2 = C\iint e^{it\tilde{\Omega}} A_k\eta\mathcal{Y}\partial_{\xi_1}\Phi\, d\tilde{\zeta}\, d\tilde{\eta},$$

where

$$\tilde{\Omega} = \omega\xi^2 + \alpha\beta(1+\beta)\tilde{\zeta}^2 + (1+\beta)\tilde{\eta}^2, \quad \xi_1 = \lambda\xi + (1+\beta)\tilde{\zeta},$$

$$\xi_2 = \lambda\xi + (1-\beta)\tilde{\zeta} + 2\tilde{\eta}, \quad \xi_3 = \frac{\lambda}{\alpha}\xi - \lambda\tilde{\zeta} - \lambda\tilde{\eta}.$$

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We integrate by parts with respect to $\tilde{\eta}$ via the identity $e^{it\tilde{\Omega}} = H\frac{\partial}{\partial\tilde{\eta}}(\tilde{\eta}e^{it\tilde{\Omega}})$, where $H = \left(1 + 2it(1+\beta)\tilde{\eta}^2\right)^{-1}$ to obtain

$$I_2 = C\iint e^{it\tilde{\Omega}}\tilde{\eta}\partial_{\tilde{\eta}}(A_k\eta\mathfrak{Y}H)\partial_{\xi_1}\Phi\, d\tilde{\eta}\, d\tilde{\zeta} + C\iint e^{it\tilde{\Omega}}A_k\eta\mathfrak{Y}\tilde{\eta}H\partial_{\xi_1}\partial_{\tilde{\eta}}\Phi\, d\tilde{\eta}\, d\tilde{\zeta}.$$

Then via estimates

$$|\tilde{\eta}\partial_{\tilde{\eta}}(A_k\eta\mathfrak{Y}H)| \le C(|\tilde{\eta}| + |\eta|)\langle t\eta^2\rangle^{-1}\langle t\tilde{\eta}^2\rangle^{-1}\langle t\tilde{\Omega}\rangle^{-1}$$

and $|H| \le C\langle t\tilde{\eta}^2\rangle^{-1}$ we obtain

$$\begin{aligned}
&\|\langle t\xi^2\rangle^{\delta} I_2\|_\infty \\
&\quad\le C\left\|\iint \frac{\langle t\xi^2\rangle^{\delta}(|\tilde{\eta}| + |\eta|)|\partial_{\xi_1}\Phi|\, d\eta\, d\tilde{\eta}}{\langle t\tilde{\eta}^2\rangle\langle t\eta^2\rangle\langle t\tilde{\Omega}\rangle}\right\|_\infty \\
&\qquad + C\sum_{j=2}^{3}\left\|\iint \frac{\langle t\xi^2\rangle^{\delta}|\tilde{\eta}|\,|\eta|\,|\partial_{\xi_1}\partial_{\xi_j}\Phi|\, d\eta\, d\tilde{\eta}}{\langle t\tilde{\eta}^2\rangle\langle t\eta^2\rangle\langle t\tilde{\Omega}\rangle}\right\|_\infty \\
&\quad\le Ct^{-\frac{1}{2}}\|\phi_1'\|\left(\prod_{k=2}^{3}\|\psi\phi_k\|_\infty\left(\int\frac{d\eta}{\langle t\eta^2\rangle}\sqrt{\int\frac{d\tilde{\eta}}{\langle t\tilde{\eta}^2\rangle}} + \int\frac{d\tilde{\eta}}{\langle t\tilde{\eta}^2\rangle}\sqrt{\int\frac{d\eta}{\langle t\eta^2\rangle}}\right)\right. \\
&\qquad\qquad\left. + t^{-\frac{1}{2}}\sum_{j=2}^{3}\|\phi_j'\|\,\|\psi\phi_{5-j}\|_\infty\sqrt{\int\frac{d\eta}{\langle t\eta^2\rangle}}\sqrt{\int\frac{d\tilde{\eta}}{\langle t\tilde{\eta}^2\rangle}}\right) \\
&\quad\le Ct^{-\frac{5}{4}}\|\phi_1'\|\sum_{j=2}^{3}\|\phi_j'\|\,\|\psi\phi_{5-j}\|_\infty.
\end{aligned}$$

Integral I_3 is considered in the same manner as the integral I_2. Lemma 2.3 is proved. ■

Denote $\mathcal{K} = \mathcal{F}\mathcal{U}(-t)$, $\psi(t,\xi) = \langle t\xi^2\rangle^{-\gamma}$, $\gamma > 0$ is small.

Lemma 2.4 *We have the estimates*

$$\|\langle t\xi^2\rangle^{\delta}\mathcal{K}Q_k(\bar{u},\bar{u},\bar{u})\|_\infty + t^{\frac{3}{4}}\|\partial_x Q_k(\bar{u},\bar{u},\bar{u})\| \le Ct^{-1}(\|\psi\mathcal{K}u\|_\infty + t^{-\frac{1}{4}}\|\mathcal{J}u\|)^3$$

and

$$\begin{aligned}
&\|\langle t\xi^2\rangle^{\delta}\mathcal{K}Q_1(\bar{u},\bar{u},u^3)\|_\infty + t^{\frac{3}{4}}\|\partial_x Q_1(\bar{u},\bar{u},u^3)\| \\
&\quad\le Ct^{-2}(\|\psi\mathcal{K}u\|_\infty + t^{-\frac{1}{4}}\|\mathcal{J}u\|)^5 + Ct^{-\frac{5}{4}}\|u\|_\infty^2\,\|\mathcal{J}u\|(\|\psi\mathcal{K}u\|_\infty + t^{-\frac{1}{4}}\|\mathcal{J}u\|)^2
\end{aligned}$$

for all $t > 0$, where $k = 1, 2$, $\delta \in [0, 1-3\gamma]$.

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Proof Since

$$\mathcal{K}Q_k(\bar{u},\bar{u},\bar{u}) = \frac{1}{\pi}\iint \frac{e^{itS}}{(1+it\Lambda)^k}\prod_{j=1}^{3}\overline{\mathcal{K}u(t,-\xi_j)}\,d\eta\,d\zeta = \iint e^{itS}A_k\Phi\,d\eta\,d\zeta,$$

where

$$\xi_1 = \frac{\xi}{3}+\eta-\zeta, \quad \xi_2 = \frac{\xi}{3}-\eta-\zeta, \quad \xi_3 = \frac{\xi}{3}+2\zeta,$$

$$\Lambda = \frac{1}{2}\Big(\xi^2+\sum_{j=1}^{3}\xi_j^2\Big) = \frac{2}{3}\xi^2+\eta^2+3\zeta^2,$$

the functions

$$A_k = \frac{1}{\pi}(1+it\Lambda)^{-k}, \quad k=1,2,\ \Phi = \Phi(\boldsymbol{\xi}) = \prod_{j=1}^{3}\overline{\mathcal{K}u(t,-\xi_j)},$$

where

$$\boldsymbol{\xi} = (\xi_1,\xi_2,\xi_3), \quad \boldsymbol{\xi}_0 = \Big(\frac{1}{3}\xi,\frac{1}{3}\xi,\frac{1}{3}\xi\Big).$$

Then by the first estimate of Lemma 2.3 with $\alpha = 1$, $\lambda = \frac{1}{3}$, $\omega = \frac{2}{3}$, $\beta = 3$, we get

$$\begin{aligned}\|\langle t\xi^2\rangle^{\delta}\mathcal{K}Q_k(\bar{u},\bar{u},\bar{u})\|_\infty &= \Big\|\langle t\xi^2\rangle^{\delta}\iint e^{itS}A_k\Phi\,d\eta\,d\zeta\Big\|_\infty\\ &\le Ct^{-1}\Big(\|\psi\mathcal{K}u\|_\infty + t^{-\frac{1}{4}}\|\partial_\xi(\mathcal{K}u)\|\Big)^3\\ &\le Ct^{-1}(\|\psi\mathcal{K}u\|_\infty + t^{-\frac{1}{4}}\|\mathcal{J}u\|)\end{aligned}$$

since $\partial_\xi\mathcal{K} = \mathcal{K}\mathcal{J}$ and $\|\mathcal{K}\phi\| = \|\phi\|$. Similarly

$$\begin{aligned}\|\partial_x Q_k(\bar{u},\bar{u},\bar{u})\| &= \|\mathcal{K}\partial_x Q_k(\bar{u},\bar{u},\bar{u})\|\\ &= \|\xi\mathcal{K}Q_k(\bar{u},\bar{u},\bar{u})\| = t^{-\frac{1}{2}}\Big\|\xi\sqrt{t}\iint e^{itS}A_k\Phi\,d\eta\,d\zeta\Big\|\\ &\le Ct^{-\frac{1}{2}}\|\langle t\xi^2\rangle^{-\frac{1}{4}-\gamma}\|\Big\|\langle t\xi^2\rangle^{\frac{3}{4}+\gamma}\iint e^{itS}A_k\Phi\,d\eta\,d\zeta\Big\|_\infty\\ &\le Ct^{-\frac{3}{4}}\Big\|\langle t\xi^2\rangle^{\delta}\iint e^{itS}A_k\Phi\,d\eta\,d\zeta\Big\|_\infty\\ &\le Ct^{-\frac{7}{4}}(\|\psi\mathcal{K}u\|_\infty + t^{-\frac{1}{4}}\|\mathcal{J}u\|)^3.\end{aligned}$$

Therefore the first estimate of the lemma follows. To prove the second estimate we write

$$\mathcal{K}Q_1(\bar{u},\bar{u},u^3) = \iint e^{itS}A_1\Psi\,d\eta\,d\zeta,$$

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where

$$\xi_1 = \xi + \eta - \zeta, \quad \xi_2 = \xi - \eta - \zeta, \quad \xi_3 = 3\xi + 2\zeta,$$
$$S = \frac{1}{2}\Big(\xi^2 + \sum_{j=1}^{2}\xi_j^2 + \frac{1}{3}\xi_3^2\Big) = \frac{5}{2}\xi^2 + \eta^2 + \frac{5}{3}\zeta^2,$$

the functions

$$A_1 = A_1(\xi,\eta,\zeta,t) = \frac{1}{\pi}(1 + it\Lambda)^{-1}, \quad \Lambda = \frac{1}{2}\Big(\xi^2 + \sum_{j=1}^{3}\xi_j^2\Big),$$
$$\Psi = \Psi(\boldsymbol{\xi}) = \prod_{j=1}^{2}\overline{\mathcal{K}u(t,-\xi_j)}E^{\frac{4}{3}}(\mathcal{K}u^3)(t,\xi_3),$$

where $E = e^{\frac{it}{2}\xi_3^2}$, $\boldsymbol{\xi} = (\xi_1,\xi_2,\xi_3)$. Then by Lemma 2.3 with $\alpha = \frac{1}{3}$, $\lambda = 1$, $\omega = \frac{5}{2}$, $\beta = \frac{5}{3}$, and by Lemma 2.2 we get

$$\begin{aligned}
&\|\langle t\xi^2\rangle^{\delta}\mathcal{K}Q_1(\bar{u},\bar{u},u^3)\|_\infty \\
&\quad = \Big\|\langle t\xi^2\rangle^{\delta}\iint e^{itS}A_1\Psi\,d\eta\,d\zeta\Big\|_\infty \\
&\quad \le Ct^{-1}\Big(\|\psi\mathcal{K}u\|_\infty + t^{-\frac{1}{4}}\|\partial_\xi(\mathcal{K}u)\|\Big)^2\Big(\|\psi^3(\mathcal{K}u^3)\|_\infty + t^{-\frac{1}{4}}\Big\|\partial_\xi\Big(E^{\frac{4}{3}}(\mathcal{K}u^3)\Big)\Big\|\Big) \\
&\quad \le Ct^{-2}(\|\psi\mathcal{K}u\|_\infty + t^{-\frac{1}{4}}\|\mathcal{J}u\|)^5 + Ct^{-\frac{5}{4}}\|u\|_\infty^2\|\mathcal{J}u\|(\|\psi\mathcal{K}u\|_\infty + t^{-\frac{1}{4}}\|\mathcal{J}u\|)^2.
\end{aligned}$$

In the same manner

$$\begin{aligned}
&\|\partial_x Q_1(\bar{u},\bar{u},u^3)\| \\
&\quad = \|\mathcal{K}\partial_x Q_1(\bar{u},\bar{u},u^3)\| \\
&\quad = \|\xi\mathcal{K}Q_1(\bar{u},\bar{u},u^3)\| = t^{-\frac{1}{2}}\Big\|\xi\sqrt{t}\iint e^{itS}A_1\Psi\,d\eta\,d\zeta\Big\| \\
&\quad \le Ct^{-\frac{1}{2}}\|\langle t\xi^2\rangle^{-\frac{1}{4}-\gamma}\|\Big\|\langle t\xi^2\rangle^{\frac{3}{4}+\gamma}\iint e^{itS}A_1\Psi\,d\eta\,d\zeta\Big\|_\infty \\
&\quad \le Ct^{-\frac{3}{4}}\Big\|\langle t\xi^2\rangle^{\delta}\iint e^{itS}A_1\Psi\,d\eta\,d\zeta\Big\|_\infty \\
&\quad \le Ct^{-\frac{7}{4}}\Big(\|\psi\mathcal{K}u\|_\infty + t^{-\frac{1}{4}}\|\partial_\xi(\mathcal{K}u)\|\Big)^2\Big(\|\psi^3(\mathcal{K}u^3)\|_\infty + t^{-\frac{1}{4}}\Big\|\partial_\xi\Big(E^{\frac{4}{3}}(\mathcal{K}u^3)\Big)\Big\|\Big) \\
&\quad \le Ct^{-\frac{11}{4}}(\|\psi\mathcal{K}u\|_\infty + t^{-\frac{1}{4}}\|\mathcal{J}u\|)^5 + Ct^{-2}\|u\|_\infty^2\|\mathcal{J}u\|(\|\psi\mathcal{K}u\|_\infty + t^{-\frac{1}{4}}\|\mathcal{J}u\|)^2.
\end{aligned}$$

Lemma 2.4 is proved. ∎

Denote as above $\mathcal{K} = \mathcal{F}\mathcal{U}(-t)$, $\psi(t,\xi) = \langle t\xi^2\rangle^{-\gamma}$. Define

$$Q_3 = Q_2 - Q_1, \quad \Omega(\eta) = \frac{\pi}{\sqrt{3}} \int_0^\infty \frac{(\eta + r)e^{ir} dr}{\big(1 - i(\eta + r)\big)^2}.$$

Note that

$$|\Omega(\eta)| \le C\langle\eta\rangle^{-1}$$

and

$$\Omega(0) = \frac{\pi}{\sqrt{3}} \int_0^\infty e^{-ir} \frac{r dr}{(1 - ir)^2} = \frac{\pi}{\sqrt{3}}.$$

Lemma 2.5 *We have the asymptotics*

$$\mathcal{K}Q_3(\bar{u}, \bar{u}, \bar{u}) = \frac{i}{t} e^{-\frac{2}{3}it\xi^2} \Omega(t\xi^2)(\overline{\mathcal{K}u})^3\Big(t, -\frac{\xi}{3}\Big) \\ + O\big(\langle t\xi^2\rangle^{-\delta} t^{-\frac{5}{4}} \|\mathcal{J}u\|(\|\psi\mathcal{K}u\|_\infty + t^{-\frac{1}{4}}\|\mathcal{J}u\|)^2\big)$$

for all $t > 0$.

Proof As in the proof of Lemma 2.4 we get

$$\mathcal{K}Q_3(\bar{u}, \bar{u}, \bar{u}) = \Phi_0 \iint e^{it\Lambda} A_3 \, d\eta \, d\zeta + \iint e^{it\Lambda} A_3(\Phi - \Phi_0) \, d\eta \, d\zeta,$$

where

$$\xi_1 = \frac{\xi}{3} + \eta - \zeta, \quad \xi_2 = \frac{\xi}{3} - \eta - \zeta, \quad \xi_3 = \frac{\xi}{3} + 2\zeta,$$

$$\Lambda = \frac{1}{2}\Big(\xi^2 + \sum_{j=1}^{3} \xi_j^2\Big) = \frac{2}{3}\xi^2 + \eta^2 + 3\zeta^2,$$

the functions

$$A_3 = A_3(\xi, \eta, \zeta, t) = \frac{it}{\pi}\Lambda(1 + it\Lambda)^{-2} = A_2 - A_1,$$

$$\Phi = \Phi(\boldsymbol{\xi}) = \prod_{j=1}^{3} \overline{\mathcal{K}u(t, -\xi_j)},$$

where

$$\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3), \quad \boldsymbol{\xi}_0 = \Big(\frac{1}{3}\xi, \frac{1}{3}\xi, \frac{1}{3}\xi\Big).$$

Then by Lemma 2.3 with $\alpha = 1$, $\lambda = \frac{1}{3}$, $\omega = \frac{2}{3}$, $\beta = 3$, we get

$$\Big\| \langle t\xi^2\rangle^\delta \iint e^{it\Lambda} A_3(\Phi - \Phi_0) \, d\eta \, d\zeta \Big\|_\infty \le Ct^{-\frac{5}{4}} \|\mathcal{J}u\|(\|\psi\mathcal{K}u\|_\infty + t^{-\frac{1}{4}}\|\mathcal{J}u\|)^2.$$

Therefore we obtain the result of the lemma, since

$$\begin{aligned}\Omega(t\xi^2) &= -ite^{\frac{2}{3}it\xi^2}\iint e^{it\Lambda}A_3\,d\eta\,d\zeta\\ &= \frac{1}{\sqrt{3}}\iint e^{-i(\eta^2+\zeta^2)}\frac{(\xi^2t+\eta^2+\zeta^2)}{\left(1-i(\xi^2t+\eta^2+\zeta^2)\right)^2}\,d\eta\,d\zeta\\ &= \frac{\pi}{\sqrt{3}}\int_0^\infty e^{-ir}\frac{(\xi^2t+r)dr}{\left(1-i(\xi^2t+r)\right)^2}.\end{aligned}$$

Lemma 2.5 is proved. ■

3 Proof of Theorem 1.1

By virtue of the method of papers [1], [4], [13], [19] we obtain the existence of local solutions in the functional space $\mathbf{H}^{1,0}\cap\mathbf{H}^{0,1}$. Denote

$$\begin{aligned}\|\phi\|_{\mathbf{X}} &= \|\mathcal{K}\phi\|_\infty + \sqrt{1+\varepsilon^{2+3\gamma}\log(t+1)}\|\psi\mathcal{K}\phi\|_\infty\\ &\quad+\left(1+\varepsilon^{2-3\gamma}(t+1)^{\frac{1}{4}}\left(1+\varepsilon^{2+3\gamma}\log(t+1)\right)^{-\frac{3}{2}}\right)^{-1}\|\mathcal{J}\phi\|,\end{aligned}$$

where $\mathcal{K}=\mathcal{F}\mathcal{U}(-t)$, $\psi(t,\xi)=\langle t\xi^2\rangle^{-\gamma}$, $\gamma\in(0,\frac{1}{100})$, $\varepsilon>0$ is small.

Theorem 3.1 *Let the initial data $u_0\in\mathbf{H}^{1,0}\cap\mathbf{H}^{0,1}$. Then for some time $T>0$ there exists a unique solution $u\in\mathbf{C}([0,T],\mathbf{H}^{1,0}\cap\mathbf{H}^{0,1})$ of the Cauchy problem (2). If in addition we assume that the initial data $u_0\in\mathbf{H}^{1,0}\cap\mathbf{H}^{0,1}$ have a sufficiently small norm $\|u_0\|_{1,0}+\|u_0\|_{0,1}\equiv\varepsilon>0$, then there exists a unique solution $u\in\mathbf{C}([0,T],\mathbf{H}^{1,0}\cap\mathbf{H}^{0,1})$ of the Cauchy problem (2) on a finite time interval $[0,T]$ with $T>2$, such that*

$$\|u\|_{\mathbf{X}}\le C\varepsilon$$

for all $t\in[0,T]$.

In the next lemma we obtain the *a priori* estimate of solutions in the norm $\mathbf{X}$.

Lemma 3.2 *Let the initial data $u_0\in\mathbf{H}^{1,0}\cap\mathbf{H}^{0,1}$ have a sufficiently small norm $\|u_0\|_{1,0}+\|u_0\|_{0,1}\equiv\varepsilon>0$. Also we assume that $\sup_{|\xi|\le1}|\arg\hat{u}_0(\xi)-\frac{\pi n}{2}|<\frac{\pi}{8}-\varepsilon$ for some $n\in\mathbf{Z}$, and $\inf_{|\xi|\le1}|\hat{u}_0(\xi)|>C\varepsilon$. Then there exists a unique global solution of the Cauchy problem (2) such that*

$$\|u\|_{\mathbf{X}}<\varepsilon^{1-\gamma} \tag{6}$$

for all $t\ge0$.

Proof Applying the result of Theorem 3.1 and using a standard continuation argument we can find a maximal time $T > 2$ such that

$$\|u\|_{\mathrm{X}} \le \varepsilon^{1-\gamma} \tag{7}$$

is true for all $t \in [0, T]$. If we prove the estimates (6) for all $t \in [0, T]$, then by the contradiction argument we obtain the desired result of the lemma. In view of the local existence Theorem 3.1 it is sufficient to consider the estimates of the solution for time interval $t \ge 2$ only.

We start with the estimate of the norm $\|\mathcal{J}u\|$. Applying the operator $\mathcal{J}$ to equation (2) we get

$$\mathcal{L}\mathcal{J}u + \bar{u}^2\overline{\mathcal{J}u} + 4it\partial_x(\bar{u}^3) = 0.$$

By virtue of Lemma 2.1 with $n = 1$, we obtain

$$\mathcal{L}\big(\mathcal{J}u + 4t^2\partial_x Q_1(\bar{u}, \bar{u}, \bar{u})\big) = -\bar{u}^2\overline{\mathcal{J}u} + 4it\partial_x Q_2(\bar{u}, \bar{u}, \bar{u}) + 12t^2\partial_x Q_1(\bar{u}, \bar{u}, u^3). \tag{8}$$

Since

$$u(t) = \mathcal{U}(t)\mathcal{U}(-t)u(t) = M\mathcal{D}\mathcal{K}u + M\mathcal{D}\mathcal{F}(M-1)\mathcal{U}(-t)u,$$

by Lemma 2.2 and (7) we have the estimate

$$\|u\|_\infty \le Ct^{-\frac{1}{2}}\|\mathcal{K}u\|_\infty + Ct^{-\frac{3}{4}}\|\mathcal{J}u\| \le C\varepsilon^{1-\gamma}t^{-\frac{1}{2}},$$

hence by Theorem 3.1

$$\|u^2\mathcal{J}u\| \le C\|u\|_\infty^2\,\|\mathcal{J}u\| \le C\varepsilon^{3-3\gamma}t^{-\frac{3}{4}}\big(1 + \varepsilon^{2+3\gamma}\log(t+1)\big)^{-\frac{3}{2}}. \tag{9}$$

Since by Theorem 3.1,

$$\|\mathcal{K}u\|_\infty \le \varepsilon^{1-\gamma}, \quad \|\psi\mathcal{K}u\|_\infty \le \varepsilon^{1-\gamma}\big(1 + \varepsilon^{2+3\gamma}\log(t+1)\big)^{-\frac{1}{2}}$$

and

$$\|\mathcal{J}u\| \le \varepsilon^{1-\gamma}t^{\frac{1}{4}}\big(1 + \varepsilon^{2+3\gamma}\log(t+1)\big)^{-\frac{3}{2}}$$

we have by Lemma 2.4

$$\begin{aligned}
&t\|\partial_x Q_k(\bar{u}, \bar{u}, \bar{u})\| + t^2\|\partial_x Q_1(\bar{u}, \bar{u}, u^3)\| \\
&\quad\le Ct^{-\frac{3}{4}}(\|\psi\mathcal{K}u\|_\infty + t^{-\frac{1}{4}}\|\mathcal{J}u\|)^3 + Ct^{-\frac{3}{4}}(\|\psi\mathcal{K}u\|_\infty + t^{-\frac{1}{4}}\|\mathcal{J}u\|)^5 \\
&\qquad + C\|u\|_\infty^2\,\|\mathcal{J}u\|(\|\psi\mathcal{K}u\|_\infty + t^{-\frac{1}{4}}\|\mathcal{J}u\|)^2 \\
&\quad\le C\varepsilon^{3-3\gamma}t^{-\frac{3}{4}}\big(1 + \varepsilon^{2+3\gamma}\log(t+1)\big)^{-\frac{3}{2}}.
\end{aligned} \tag{10}$$

Application of the energy method to equation (8) via estimates (9) and (10) yields

$$\begin{aligned}
\|\mathcal{J}u + 4t^2\partial_x Q_1(\bar{u}, \bar{u}, \bar{u})\| &\le C\varepsilon + C\varepsilon^{3-3\gamma}\int_2^t \big(1 + \varepsilon^{2+3\gamma}\log(\tau+1)\big)^{-\frac{3}{2}}\frac{d\tau}{\tau^{\frac{3}{4}}} \\
&\le C\varepsilon + C\varepsilon^{3-3\gamma}t^{\frac{1}{4}}\big(1 + \varepsilon^{2+3\gamma}\log(t+1)\big)^{-\frac{3}{2}}.
\end{aligned}$$

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Hence in view of (10) we get the estimate

$$\|\mathcal{J}u\| < C\varepsilon + C\varepsilon^{3-3\gamma} t^{\frac{1}{4}} \left(1 + \varepsilon^{2+3\gamma} \log(t+1)\right)^{-\frac{3}{2}}. \tag{11}$$

Now we prove the estimate $\|\mathcal{K}u\|_\infty + \sqrt{1 + \varepsilon^{2+3\gamma} \log(t+1)}\|\psi\mathcal{K}u\|_\infty < C\varepsilon$. By virtue of Lemma 2.1 with $n = 0$, we have

$$\mathcal{L}w + Q_3(\bar{u}, \bar{u}, \bar{u}) + 3itQ_1(\bar{u}, \bar{u}, u^3) = 0,$$

here $Q_3 = Q_1 - Q_2$ and $w = u + itQ_1(\bar{u}, \bar{u}, \bar{u})$. Then via Lemma 2.5 we get for the function $v \equiv \mathcal{K}w$

$$\begin{aligned} v_t(t, \xi) = {} & -\frac{1}{t} e^{-\frac{2}{3}it\xi^2} \Omega(t\xi^2)(\overline{\mathcal{K}u})^3\left(t, -\frac{\xi}{3}\right) + O(\varepsilon^{3-3\gamma} t^{-\frac{5}{4}}) \\ & + O\left(\varepsilon^{5-5\gamma} t^{-1} \psi\left(1 + \varepsilon^{2+3\gamma} \log(t+1)\right)^{-\frac{5}{2}}\right). \end{aligned}$$

Since

$$\begin{aligned} \left|\mathcal{K}u\left(t, -\frac{\xi}{3}\right) - \mathcal{K}u(t,\xi)\right| & \le C\sqrt{|\xi|}\|\partial_\xi \mathcal{K}u\| = C\sqrt{|\xi|}\|\mathcal{J}u\| \\ & \le C\varepsilon t^{-\frac{1}{4}} \sqrt[4]{\xi^2 t} + C\varepsilon^{3-3\gamma} \sqrt[4]{\xi^2 t}\left(1 + \varepsilon^{2+3\gamma} \log(t+1)\right)^{-\frac{3}{2}}, \end{aligned}$$

therefore in view of estimates (7) and $|\Omega(t\xi^2)| \le C\langle t\xi^2\rangle^{-1}$ we obtain

$$\begin{aligned} v_t(t, \xi) = {} & -\frac{1}{t} e^{-\frac{2}{3}it\xi^2} \Omega(t\xi^2)(\overline{\mathcal{K}u})^3(t, \xi) + O(\varepsilon^{3-3\gamma} t^{-\frac{5}{4}}) \\ & + O\left(\varepsilon^{5-5\gamma} t^{-1} \psi\left(1 + \varepsilon^{2+3\gamma} \log(t+1)\right)^{-\frac{5}{2}}\right). \end{aligned}$$

Then applying Lemma 2.4 to the relation $v = \mathcal{K}u + it\mathcal{K}Q_1(\bar{u}, \bar{u}, \bar{u})$, we obtain

$$\mathcal{K}u = v + O\left(\varepsilon^{3-3\gamma}\left(1 + \varepsilon^{2+3\gamma} \log(t+1)\right)^{-\frac{3}{2}}\right). \tag{12}$$

Thus, we get

$$\begin{aligned} v_t(t, \xi) = {} & -\frac{1}{t} e^{-\frac{2}{3}it\xi^2} \Omega(t\xi^2)\bar{v}^3(t, \xi) + O(\varepsilon^{3-3\gamma} t^{-\frac{5}{4}}) \\ & + O\left(\varepsilon^{5-5\gamma} t^{-1} \psi\left(1 + \varepsilon^{2+3\gamma} \log(t+1)\right)^{-\frac{5}{2}}\right). \end{aligned}$$

We change the dependent variable $v(t, \xi) = f(t, \xi)e^{-\phi+ig}$ for $|\xi| \le 1$, then

$$\begin{aligned} f_t e^{-\phi+ig} - \phi_t v + ig_t v & = -\frac{1}{t} e^{-\frac{2}{3}it\xi^2} \Omega(t\xi^2)\bar{f}^3 e^{-3\phi-3ig} + O(\varepsilon^{3-3\gamma} t^{-\frac{5}{4}}) \\ & \quad + O\left(\varepsilon^{5-5\gamma} t^{-1} \psi\left(1 + \varepsilon^{2+3\gamma} \log(t+1)\right)^{-\frac{5}{2}}\right) \\ & = -\frac{1}{t} e^{-\frac{2}{3}it\xi^2} \Omega(t\xi^2)\bar{f}^3 f^{-1} v e^{-2\phi-4ig} + O(\varepsilon^{3-3\gamma} t^{-\frac{5}{4}}) \\ & \quad + O\left(\varepsilon^{5-5\gamma} t^{-1} \psi\left(1 + \varepsilon^{2+3\gamma} \log(t+1)\right)^{-\frac{5}{2}}\right). \end{aligned}$$

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We let ϕ and g be real valued functions such that

$$\phi_t = \frac{1}{t} e^{-2\phi} \Re\Big(e^{-\frac{2}{3}it\xi^2 - 4ig} \bar{f}^3(t,\xi) f^{-1}(t,\xi) \Omega(t\xi^2)\Big)$$

and

$$g_t = -\frac{1}{t} e^{-2\phi} \Im\Big(e^{-\frac{2}{3}it\xi^2 - 4ig} \bar{f}^3(t,\xi) f^{-1}(t,\xi) \Omega(t\xi^2)\Big),$$

then

$$f_t(t,\xi) = O(\varepsilon^{3-3\gamma} e^{\phi} t^{-\frac{5}{4}}) + O\Big(\varepsilon^{5-5\gamma} t^{-1} \psi e^{\phi} \big(1 + \varepsilon^{2+3\gamma} \log(t+1)\big)^{-\frac{5}{2}}\Big).$$

Thus we get a system

$$(13)\quad \begin{cases} \phi_t = \frac{1}{t} e^{-2\phi} \Re\big(e^{-\frac{2}{3}it\xi^2 - 4ig} \bar{f}^3(t,\xi) f^{-1}(t,\xi) \Omega(t\xi^2)\big), \\ g_t = -\frac{1}{t} e^{-2\phi} \Im\big(e^{-\frac{2}{3}it\xi^2 - 4ig} \bar{f}^3(t,\xi) f^{-1}(t,\xi) \Omega(t\xi^2)\big), \\ f_t(t,\xi) = O(\varepsilon^{3-3\gamma} e^{\phi} t^{-\frac{5}{4}}) + O\Big(\varepsilon^{5-5\gamma} t^{-1} \psi e^{\phi} \big(1 + \varepsilon^{2+3\gamma} \log(t+1)\big)^{-\frac{5}{2}}\Big), \\ \phi(2,\xi) = 0,\ g(2,\xi) = 0,\ f(2,\xi) = v(2,\xi). \end{cases}$$

Let us prove the following estimate

$$(14)\qquad \sup_{|\xi|\le 1} |f| < \varepsilon^{1-\gamma}$$

for all $t \in [2, T]$. We prove estimate (14) by contradiction. Suppose that there exists a maximal time $T_1 \in (2, T]$, such that

$$(15)\qquad \sup_{|\xi|\le 1} |f| \le \varepsilon^{1-\gamma}$$

for all $t \in [2, T_1]$. Then from the first equation of system (13) we get

$$\partial_t e^{2\phi} = O(\varepsilon^{2-2\gamma} t^{-1}),$$

which gives us the estimate

$$(16)\qquad e^{2\phi} \le 1 + C\varepsilon^{2-2\gamma} \log(t+1)$$

for all $t \in [2, T_1]$. Now from the third equation of system (13) we see that

$$\partial_t f = O(\varepsilon^{3-3\gamma} t^{\gamma-\frac{5}{4}}) + O\Big(\varepsilon^{5-9\gamma} t^{-1} \big(1 + \varepsilon^{2+3\gamma} \log(t+1)\big)^{-2}\Big).$$

Integrating with respect to $t \in [2, T_1]$ we have

$$|f(t,\xi)| \le C\varepsilon + C\varepsilon^{3-3\gamma} \int_2^t \tau^{\gamma-\frac{5}{4}}\, d\tau + C \int_2^t \frac{\varepsilon^{5-9\gamma}\, d\tau}{\tau\big(1 + \varepsilon^{2+3\gamma} \log(\tau+1)\big)^2} \le C\varepsilon$$

for all $t \in [2, T_1]$. By the contradiction estimate (14) is true for all $t \in [2, T]$. Now taking estimate (16) for all $t \in [2, T]$ into account, we find from the third equation of system (13)

$$|f(T,\xi) - f(t,\xi)| \le C\varepsilon^{3-3\gamma} \int_t^T \tau^{\gamma-\frac{5}{4}}\, d\tau + C \int_t^T \frac{\varepsilon^{5-7\gamma}\, d\tau}{\tau\big(1+\varepsilon^{2+3\gamma}\log(\tau+1)\big)^2} \le C\varepsilon^{3-10\gamma}\big(1+\varepsilon^{2+3\gamma}\log(t+1)\big)^{-1}$$

for all $t \in [2, T]$. And also we have

$$f(t,\xi) = f(2,\xi) + O(\varepsilon^{3-9\gamma}),$$

hence

$$\inf_{|\xi|\le 1} |f(t,\xi)| \ge \inf_{|\xi|\le 1} |f(2,\xi)| + O(\varepsilon^{3-9\gamma}) > \varepsilon^{1+\gamma},$$

and

$$\sup_{|\xi|\le 1}\Big|\arg f(t,\xi) - \frac{\pi n}{2}\Big| \le \sup_{|\xi|\le 1}\Big|\arg f(2,\xi) - \frac{\pi n}{2}\Big| + O(\varepsilon^{2-10\gamma}) < \frac{\pi}{8} - \varepsilon^{1+\gamma}$$

for all $t \in [2, T]$ since by the local existence Theorem 3.1 we have for the initial data $f(2,\xi) = v(2,\xi) : |v(2,\xi)| \ge |\mathcal{K}u| - 2|\mathcal{K}Q_1(\bar{u},\bar{u},\bar{u})| \ge C\varepsilon$. Then we obtain from system (13)

$$\begin{cases} \phi_t = \frac{1}{t} e^{-2\phi} \Re\Big(e^{-\frac{2}{3}it\xi^2 - 4ig}\overline{f(T,\xi)}^3 f^{-1}(T,\xi)\Omega(t\xi^2)\Big) \\ \qquad\qquad + O\Big(\varepsilon^{4-10\gamma}\psi t^{-1} e^{-2\phi}\big(1+\varepsilon^{2+3\gamma}\log(t+1)\big)^{-1}\Big), \\ g_t = -\frac{1}{t} e^{-2\phi} \Im\Big(e^{-\frac{2}{3}it\xi^2 - 4ig}\overline{f(T,\xi)}^3 f^{-1}(T,\xi)\Omega(t\xi^2)\Big) \\ \qquad\qquad + O\Big(\varepsilon^{4-10\gamma}\psi t^{-1} e^{-2\phi}\big(1+\varepsilon^{2+3\gamma}\log(t+1)\big)^{-1}\Big). \end{cases}$$

Denote $\mu_T = \arg f(T,\xi)$ and since $\arg\Omega(0) = 0$, we have

$$e^{-\frac{2}{3}it\xi^2}\Omega(t\xi^2) = |\Omega(t\xi^2)| + O\bigg(\frac{(t\xi^2)^\gamma}{\langle t\xi^2\rangle}\bigg),$$

therefore denoting $h = 4g + 4\mu_T$ we get a system

$$\begin{cases} \phi_t = \frac{1}{t} e^{-2\phi} |\overline{f(T,\xi)}^2 \Omega(t\xi^2)| \cos h + O\Big(\varepsilon^{4-10\gamma} t^{-1} e^{-2\phi} \frac{(t\xi^2)^\gamma}{\langle t\xi^2\rangle}\Big) \\ \qquad\qquad + O\Big(\varepsilon^{4-10\gamma} t^{-1} e^{-2\phi}\big(1+\varepsilon^{2+3\gamma}\log(t+1)\big)^{-1}\Big), \\ h_t = -\frac{4}{t} e^{-2\phi} |\overline{f(T,\xi)}^2 \Omega(t\xi^2)| \sin h + O\Big(\varepsilon^{4-10\gamma} t^{-1} e^{-2\phi} \frac{(t\xi^2)^\gamma}{\langle t\xi^2\rangle}\Big) \\ \qquad\qquad + O\Big(\varepsilon^{4-10\gamma} t^{-1} e^{-2\phi}\big(1+\varepsilon^{2+3\gamma}\log(t+1)\big)^{-1}\Big). \end{cases} \tag{17}$$

Let us prove the following estimates

$$e^{2\phi} > 1 + \varepsilon^{2+3\gamma}\log\big(\min(t,\xi^{-2})\big) \quad \text{and} \quad |h| < \varepsilon \tag{18}$$

for all $|\xi| \le 1$, $t \in [2, T]$. We prove estimate (18) by contradiction. Suppose that there exists a maximal time $T_1 \in (2, T]$, such that

$$e^{2\phi} \ge 1 + \varepsilon^{2+3\gamma} \log\big(\min(t, \xi^{-2})\big) \quad \text{and} \quad |h| \le \varepsilon \tag{19}$$

for all $|\xi| \le 1$ and $t \in [2, T_1]$. Since $\phi(2, \xi) = 0$ and

$$\begin{aligned}\int_{\xi^2}^{t\xi^2} |\Omega(\eta)| \frac{d\eta}{\eta} &= \int_{\xi^2}^{\min(1,t\xi^2)} |\Omega(\eta)| \frac{d\eta}{\eta} + O(1)\\ &= |\Omega(0)| \int_{\xi^2}^{\min(1,t\xi^2)} \frac{d\eta}{\eta} + O(1)\\ &= |\Omega(0)| \log\big(\min(t, \xi^{-2})\big) + O(1),\end{aligned}$$

multiplying by $e^{2\phi}$ the first equation of system (17)

$$\begin{aligned}\partial_t e^{2\phi} = \frac{2}{t} |\overline{f(T, \xi)}^2 \Omega(t\xi^2)| \cos h + O\Big(\varepsilon^{4-10\gamma} t^{-1} \frac{(t\xi^2)^\gamma}{\langle t\xi^2 \rangle} \Big)\\ + O\Big(\varepsilon^{4-10\gamma} t^{-1} \big(1 + \varepsilon^{2+3\gamma} \log(t+1)\big)^{-1} \Big),\end{aligned}$$

hence integrating with respect to $t \ge 2$ we get

$$\begin{aligned}e^{2\phi} &\ge 1 + 2|f(T, \xi)|^2 \int_2^t \cos h |\Omega(t\xi^2)| \frac{d\tau}{\tau}\\ &\quad + O\Big(\varepsilon^{2-13\gamma} \log\big(1 + \varepsilon^{2+3\gamma} \log(t+1)\big) \Big)\\ &\ge 1 + 2|f(T, \xi)|^2 \cos \varepsilon \int_{2\xi^2}^{t\xi^2} |\Omega(\eta)| \frac{d\eta}{\eta}\\ &\quad + O\Big(\varepsilon^{2-13\gamma} \log\big(1 + \varepsilon^{2+3\gamma} \log(t+1)\big) \Big)\\ &> 1 + \varepsilon^{2+3\gamma} \log\big(\min(t, \xi^{-2})\big)\end{aligned} \tag{20}$$

for all $|\xi| \le 1$ and $t \in [2, T_1]$.

By virtue of the second equation of system (17) via (19) we see that

$$\partial_t h^2 \le C\varepsilon^{5-10\gamma} t^{-1} \big(1 + \varepsilon^{2+3\gamma} \log(t+1)\big)^{-2} + C\varepsilon^{5-10\gamma} t^{-1} e^{-2\phi} \frac{(t\xi^2)^\gamma}{\langle t\xi^2 \rangle},$$

hence integrating with respect to time t, we obtain

$$|h| \le C\varepsilon^{2-13\gamma} < \varepsilon \tag{21}$$

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for all $|\xi| \le 1$ and $t \in [2, T_1]$. By (20) and (21) we have (18) for all $|\xi| \le 1$ and $t \in [2, T]$. From (21) it follows that

$$\sup_{|\xi|\le 1}\Big|\arg v(t,\xi) - \frac{\pi n}{2}\Big| \le \sup_{|\xi|\le 1}\Big|\arg f(t,\xi) - \frac{\pi n}{2}\Big| + |g| < \frac{\pi}{8} - \varepsilon^{1+\gamma}$$

for all $|\xi| \le 1$ and $t \in [0, T]$. By (12) we get

$$\begin{aligned}|\mathcal{K}u| &\le |f|e^{-\phi} + O\Big(\varepsilon^{3-3\gamma}\big(1+\varepsilon^{2+3\gamma}\log(t+1)\big)^{-\frac{3}{2}}\Big)\\ &< C\varepsilon\Big(1+\varepsilon^{2+3\gamma}\log\big(\min(t,\xi^{-2})\big)\Big)^{-\frac{1}{2}}\end{aligned}$$

for all $|\xi| \le 1$, and if $|\xi| > 1$ we have $|\Omega(t\xi^2)| \le Ct^{-1}$, therefore

$$\partial_t v(t,\xi) = O(\varepsilon^{3-3\gamma}t^{-\frac{5}{4}}) + O(\varepsilon^{5-5\gamma}t^{-1-\gamma}).$$

Integration with respect to time $t \in [0, T]$ yields $|\mathcal{K}u| \le |v| \le C\varepsilon$. Hence

$$\|\mathcal{K}u\|_\infty + \sqrt{1+\varepsilon^{2+3\gamma}\log(t+1)}\|\psi\mathcal{K}u\|_\infty < C\varepsilon$$

for all $t \in [0, T]$. Thus in view of (11) we get (6) for all $t \in [0, T]$. The contradiction obtained proves (6) for all $t > 0$. Lemma 3.2 is proved. ■

Proof of Theorem 1.1 The global existence of solutions follows from Lemma 3.2. We need only to prove asymptotic formula (4). By Lemmas 2.2, 3.2 we have

$$u(t) = M\mathcal{D}\mathcal{K}u + M\mathcal{D}\mathcal{F}(M-1)\mathcal{U}(-t)u = M\mathcal{D}fe^{-\phi+ig} + O\Big(t^{-\frac{1}{2}}\big(\log(t+1)\big)^{-\frac{3}{2}}\Big).$$

By the third equation of system (13) we see that there exists a unique final state $W_+(\xi) = \lim_{t\to\infty} f(t,\xi) \in \mathbf{L}^\infty(\mathbf{R})$ such that

$$|f(t) - W_+| \le C(\log t)^{-1}.$$

Denote $\mu_\infty = \arg W_+(\xi)$. Since $\Omega(0) = \frac{\pi}{\sqrt{3}}$ in the same manner as in the proof of Lemma 3.2 we obtain

$$e^{2\phi} = 1 + \frac{2\pi}{\sqrt{3}}|W_+|^2\log\big(\min(t,\langle\xi^{-2}\rangle)\big) + O(\log\log t)$$

and $h = 4g + 4\mu_\infty = O\big((\log\log t)^{-1}\big)$ as $t \to \infty$, therefore $g = -\mu_\infty + O\big((\log\log t)^{-1}\big)$ as $t \to \infty$. Hence we get the asymptotics (4). Theorem 1.1 is proved.

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Department of Mathematics
Graduate School of Science
Osaka University
Osaka
Toyonaka 560-0043
Japan
email: nhayashi@math.wani.osaka-u.ac.jp

Instituto de Física y Matemáticas
Universidad Michoacana, AP 2-82
Morelia, CP 58040
Michoacán
Mexico
email: pavelni@ccu.umich.mx