Can. J. Math., Vol. XXIX, No. 5, 1977, pp. 947–956

THE DICHROMATE AND ORIENTATIONS OF A GRAPH

GERALD BERMAN

Internal and external activities are defined for any orientation of a graph $\mathscr{G}$ relative to a fixed labelling of its edges. It is shown that the number of such orientations of $\mathscr{G}$ having internal activity r and external activity s is $2^{r+s}\chi_{rs}$ where χ_{rs} is the coefficient of $x^r y^s$ in the dichromate $\chi(\mathscr{G}; x, y)$. It follows that the number of orientations of $\mathscr{G}$ in which the resulting digraph $\mathscr{D}$ is acyclic is given by $|P(\mathscr{G}; -1)|$, where $P(\mathscr{G}; \lambda)$ is the chromatic polynomial associated with $\mathscr{G}$. This result was obtained by Stanley [**5**] using enumeration techniques. In case $\mathscr{G}$ is planar the number of orientations of $\mathscr{G}$ in which $\mathscr{D}$ is strongly connected is equal to $|P(\mathscr{G}', -1)|$ where $\mathscr{G}'$ is the planar dual of $\mathscr{G}$.

1. Introduction. Let $\mathscr{G}$ be a connected finite graph (possibly with loops and multiple edges). We shall use the following notation: E the set of edges, $m = |E|$ the cardinality of E, V the set of vertices, $n = |V|$, ρ the *cycle rank* (or *cyclomatic number*), ρ' the *cocycle rank* (or *coboundary rank*), T a *spanning tree*, $T' = E - T$ the corresponding *spanning cotree*, $R_T e$ the unique *circuit* determined by $e \in T'$, $R_T' e$ the unique *cocircuit* (or *bond*) determined by $e \in T$. If $e \in T$, $R_T e = \emptyset$ and if $e \in T'$, $R_T' e = \emptyset$.

Let $e_1, e_2, \ldots, e_m$, $m = |E|$ be any labelling of the edges of $\mathscr{G}$. An edge $e \in T'$ is *externally active* with respect to T if e is the first edge of $R_T e$ in the ordering determined by the labelling. An edge $e \in T$ is *internally active* if e is the first edge of $R_T' e$. We adopt the convention used in [**2**; **9**] rather than that used by Tutte [**6**] in which the last edges are used to define the activities.

Let χ_{rs} denote the number of trees for which $\mathscr{G}$ has r internally active edges and s externally active edges. The *dichromate* of $\mathscr{G}$ is then given by

$$(1.1)\quad \chi(\mathscr{G}; x, y) = \sum_{r,s} \chi_{rs} x^r y^s.$$

It can be shown [**2**; **6**] that this polynomial is independent of the labelling used in the definition of internal and external activities.

The dichromatic polynomial $Q(\mathscr{G}; x, y)$ [**8**; **9**] associated with $\mathscr{G}$ is related to the dichromate by the equation

$$Q(\mathscr{G}; x, y) = x\chi(\mathscr{G}; x + 1, y + 1)$$

and satisfies the identity

$$x^n Q(\mathscr{G}; x, x^{-1}) = (x + 1)^m.$$

Received September 9, 1976 and in revised form March 23, 1977.

It follows that the dichromate satisfies the identity

$$(1.2) \quad x^{n-1}\chi(\mathscr{G}; x^{-1} + 1, x + 1) = (x + 1)^m.$$

The *chromatic polynomial* $P(G; \lambda)$ is related to the dichromate by the equation

$$(1.3) \quad P(\mathscr{G}; \lambda) = \lambda(-1)^{n-1}\chi(\mathscr{G}; 1 - \lambda, 0).$$

With each edge e (including loops) there is associated two orientations. An *orientation* o of $\mathscr{G}$ is a selection of an orientation for each $e \in E$. The set of directed edges determined by o will be denoted by $A = E(o)$. The corresponding *digraph* $\mathscr{D} = \mathscr{D}(o) = (A, V)$ is an *oriented graph*. Although different orientations can lead to isomorphic oriented graphs [**3**] we have labelled the edges so that the 2^n orientations can be distinguished.

When referring to a digraph $\mathscr{D}$ we shall use the terms *cycle*, *circuit*, *cocircuit*, *elementary circuit*, etc. as in Berge [**1**] instead of the words *directed cycle*, *directed bond*, *directed elementary circuit*, etc. There is no ambiguity with these terms, with a different meaning, when used for $\mathscr{G}$. Following Berge a *directed edge* will be called an *arc*. Vector spaces Λ, Λ' called the *cycle space* and *cocycle space* are associated with the cycles and cocycles.

It was shown by Tutte [**6**] that every arc of A belongs to either a (directed) circuit or a (directed) cocircuit of $\mathscr{D}$, but no arc belongs to both. This theorem also follows from a theorem of Minty [**1**; **4**] on three-coloring the arcs of a digraph and can be restated as follows.

THEOREM 1.1. *There exists a unique partition $A = A_C \cup A_C'$ into disjoint subsets, and a corresponding partition $E = E_C \cup E_C'$ of the edges of $\mathscr{G}$ such that the arcs of A_C belong to circuits and the arcs of A_C' to cocircuits. In particular, if $\mathscr{D}$ is strongly connected, then $A = A_C$, $E = E_C$ and if $\mathscr{D}$ is acyclic, then $A = A_C'$, $E = E_C'$.*

Let $\mathscr{D}_C = (A_C, V)$ denote the digraph obtained from $\mathscr{D}$ by deleting the edges of A_C' and let $\mathscr{D}_C' = (A_C', V_C')$ denote the digraph obtained from $\mathscr{D}$ by contracting the edges of A_C. $\mathscr{D}_C$ is the union of the strongly connected components of $\mathscr{D}$ and $\mathscr{D}_C'$ is an acyclic graph which represents the cocircuit structures of $\mathscr{D}$. Let $\mathscr{G}_C = (E_C, V)$, $\mathscr{G}_C' = (E_C', V_C')$ denote the corresponding graphs. It is shown in [**1**] that for a strongly connected digraph $\mathscr{H}$ the corresponding cycle space has a basis consisting of ρ circuits where $\rho = \rho(\mathscr{H})$ is the cycle rank of $\mathscr{H}$ (or of the corresponding graph). It follows that Λ_C the cycle space of $\mathscr{D}_C$ has a basis consisting of $\rho(\mathscr{D}_C)$ circuits. Similarly the cocycle space Λ_C' of $\mathscr{D}_C'$ has a basis consisting of $\rho'(\mathscr{D}_C')$ cocircuits. In view of Theorem 1.1 these are the numbers of independent circuits and cocircuits of $\mathscr{D}$. The vector spaces Λ_C, Λ_C' will be called the *circuit space* and *cocircuit space* of $\mathscr{D}$.

THEOREM 1.2. *The digraph $\mathscr{D}$ has $\rho(\mathscr{D}_C)$ independent circuits in A_C which is a basis for Λ_C and $\rho'(\mathscr{D}_C')$ independent cocircuits in Λ_C' which is a basis for Λ_C'.*

https://doi.org/10.4153/CJM-1977-095-1 Published online by Cambridge University Press

Let $O(\mathscr{G})$ denote the set of 2^m orientations of $\mathscr{G}$. Each orientation $o \in O(\mathscr{G})$ can be represented uniquely by one of the 2^m vectors

$$(1.4) \quad o = (\epsilon_1, \epsilon_2, \ldots, \epsilon_m), \quad \epsilon_i = \pm 1, i = 1, 2, \ldots, m$$

where ϵ_i represents the orientation of $e_i \in E$ in o. For each i the numbers $+1$, -1 are associated arbitrarily with the two orientations of e_i.

Let $\mathscr{D} = (A, V)$ correspond to any orientation $o \in O(\mathscr{G})$. Every cycle of A can be represented by one of the vectors.

$$(1.5) \quad u = (u_1, u_2, \ldots, u_m), \quad u_i = 0, \pm 1, i = 1, 2, \ldots, m$$

where $u_i = 0$ if the arc $e_i(o)$ (oriented arc corresponding to $e_i \in E$) is not in the cycle, $u_i = 1$ if the direction of $e_i(o)$ coincides with the direction in which the cycle is traversed and $u_i = -1$ otherwise. In particular, u represents a circuit (of $\mathscr{D}$) if every nonzero entry is the same. Every cocycle is associated with a nonempty subset $S \subset V$ and is also represented by one of the vectors (1.5), where $u_i = +1$ if $e_i(o)$ has only its initial endpoint in S; $u_i = -1$ if $e_i(o)$ has only its terminal endpoint in S, and $u_i = 0$ otherwise. In particular a cocircuit is represented by a vector u in which every nonzero entry is the same.

It follows that the circuits and cocircuits of $\mathscr{D}(o)$, which are bases for $\Lambda_C(o)$, $\Lambda_C'(o)$, can be obtained from an enumeration of the cycles and cocycles by selecting those cycles and cocycles whose representation (1.5) has all nonzero entries 1. The enumeration of all the cycles and cocycles of $\mathscr{D}(o)$ for any $o \in O(\mathscr{G})$ can easily be obtained from the enumeration for any one orientation, say $o_1 = (1, 1, \ldots, 1)$. For, let $\mathscr{D}_1 = \mathscr{D}(o_1)$ and let u denote a cycle of $\mathscr{D}_1$ then $ou = (\epsilon_1 u_1, \epsilon_2 u_2, \ldots, \epsilon_m u_m)$ is a cycle of $\mathscr{D}$, where o is represented by (1.4) and u by (1.5). Similarly in case u represents a cocycle.

In Section 2 orderings are defined for the sets of circuits and cocircuits of $\mathscr{D} = \mathscr{D}(o)$. This leads to a nest of subspaces of Λ_C, Λ_C' and a corresponding partitioning of A_C, A_C'. This in turn determines a partition of E. Internal and external activities r, s are defined for o and a set $O(o) \subset O(\mathscr{G})$ (containing o) of 2^{r+s} orientations defined having the same activities and determining the same partition of E. In Section 3 a 1-1 correspondence is shown between the sets $O(o)$ and sets of orientations corresponding to spanning trees of $\mathscr{G}$ with internal and external activities r, s (as defined by Tutte for spanning trees). This correspondence is applied in Section 4 to obtain results relating strongly connected graphs and acyclic graphs to the chromatic polynomial, as stated in the abstract.

2. External and internal activities of an orientation. Let $\mathscr{D} = \mathscr{D}(o) = (A, V)$ be the digraph of the orientation o of $\mathscr{G} = (E, V)$. The arc of A corresponding to $e \in E$ will be denoted by $\hat{e} = e(o)$. If S denotes any subset of A, let $\psi(S)$ denote the first edge of the set $\tilde{S} = \{e_i \in E, \hat{e}_i \in S\}$. The *min* of a collection of subsets $\{S_j\}$ of A is defined as follows: $S_m = \min \{S_j\}$ if

https://doi.org/10.4153/CJM-1977-095-1 Published online by Cambridge University Press

(i) $\psi(S_m) \leqq \psi(S_j)$, $j \neq m$, and

(ii) if $\psi(S_m) = \psi(S_j)$ for any j, then $\psi(S_m) - S_{mj} > \psi(S_j - S_{mj})$,

where S_{mj} denotes the intersection of the sets S_m, S_j and $\leqq$ denotes the ordering of E determined by the labelling.

Let $\mathscr{D}_C$ denote the digraph associated with the circuits of $\mathscr{D}$ as defined in section 1. We now define a sequence of independent circuits $\gamma_1, \gamma_2, \ldots, \gamma_q$ of Λ_C and a sequence of sets $\{\delta_j, \delta_j \subset \gamma_j\}$ which partition A_C into disjoint subsets. The circuits determine a *nest* of subspaces $\Lambda_C^1 \subset \Lambda_C^2 \subset \ldots \subset \Lambda_C^q = \Lambda_C$ where Λ_C^j is the space determined by $\gamma_1, \gamma_2, \ldots, \gamma_j$ for $j = 1, 2, \ldots, q$.

Let $\mathscr{D}_C^1 = \mathscr{D}_C$ and let $\{\gamma_j^1\}$ be the set of circuits of Λ_C^1. Set $\delta_1 = \min_j \{\gamma_j^1\}$. Let $\mathscr{D}_C^2$ denote the diagraph obtained from $\mathscr{D}_C^1$ by contracting δ_1 to a point, and let $\{\gamma_j^2\}$ be the set of circuits of Λ_C^2. Set $\delta_2 = \min \{\gamma_j^2\}$. Continue in this way. Let $\mathscr{D}_C^k$ denote the digraph obtained from $\mathscr{D}_C^{k-1}$ by contracting the circuit δ_{k-1} to a point, and let $\{\gamma_j^k\}$ be the set of circuits of Λ_C^k. Set

(2.1) $\delta_k = \min \{\gamma_j^k\}$.

Since $\mathscr{D}$ is a finite digraph this procedure finishes after q steps when $\mathscr{D}_C^q$ consists of a single circuit.

Notice that for each k the circuit γ_j^k of Λ_j^k corresponds to a circuit $\gamma_{i_j}^{k-1}$ of Λ_C^{k-1} for a unique i_j obtained by contracting $\delta_{k-1} \cap \gamma_j^{k-1}$ to a point. It follows that δ_j corresponds to a unique circuit γ_j of Λ_C' containing δ_j, and determines the space Λ_C^j. Further,

(2.2) $A_C = \delta_1 \cup \delta_2 \cup \ldots \cup \delta_q$

(2.3) $\delta_k = \gamma_k - \bigcup_{j<k} \delta_j, \quad k = 1, 2, \ldots, q$

and

(2.4) $\gamma_k \subset \delta_1 \cup \delta_2 \cup \ldots \cup \delta_k, \quad k = 1, 2, \ldots, q.$

Further, the circuits of the sequence $\gamma_1, \gamma_2, \ldots, \gamma_q$ are independent, forming a circuit basis for the cycles of Λ_C^j, $j = 1, 2, \ldots, q$. The undirected sets $D_j = \{e_i \in E | \hat{e}_i \in \delta_j\}$, $j = 1, 2, \ldots, q$ partition E_C into disjoint subsets corresponding to the partition (2.2) of A_C. This is summarized in the following theorem.

THEOREM 2.1. *Let $\gamma_1, \gamma_2, \ldots, \gamma_q$ be the sequence of circuits of Λ_C as defined above. Then $\{\gamma_j\}$ is a circuit basis of Λ_C, the sets $\{\delta_j\}$ which partition A_C into disjoint subsets satisfy* (2.3), (2.4) *and the sets $\{D_j\}$ partition E_C into disjoint subsets.*

We now consider the effect on the ordering of the circuits $\{\gamma_j\}$ and the partition $\{D_j\}$ of E_C, of reversing the orientation of any one of the sets δ_k. Consider two circuits $\gamma_a < \gamma_b$ (i.e., $a < b$), such that $\gamma_{ab} = \gamma_a \cap \gamma_b \neq \emptyset$. Let μ_a^1, μ_b^1 denote the corresponding circuits in which all the arcs of δ_j, $j < a$, have been contracted, so that μ_a^1, μ_b^1 are circuits of Λ_C^a and $\mu_a^1 = \delta_a = \min \{\gamma_j^a\}$. Let $\mu_{ab} = \mu_a^1 \cap \mu_b^1$, $\mu_{aa} = \mu_a - \mu_{ab}$ and $\mu_{bb} = \mu_b^1 - \mu_{ab}$. Then μ_{bb} contains δ_b and

(2.5) $\mu_a^1 = \mu_{aa} \cup \mu_{ab}, \quad \mu_b^1 = \mu_{bb} \cup \mu_{ab}.$

If δ_a is replaced by $\bar{\delta}_a$ (reverse orientation) $\mu_a{}^1$ is replaced by $\mu_a{}^2$ and $\mu_b{}^1$ by $\mu_b{}^2$ where

(2.6) $\quad \mu_a{}^2 = \bar{\mu}_{aa} \cup \bar{\mu}_{ab}, \quad \mu_b{}^2 = \mu_{bb} \cup \bar{\mu}_{aa}.$

This can be obtained from the vector representation (1.5). Using the same notation as above, but with $+$ signs for vector addition we have the cycles $\mu_a{}^1 = \mu_{aa} + \mu_{ab}$, $\mu_b{}^1 = \mu_{bb} + \mu_{ab}$. It follows that $\bar{\mu}_a{}^1 = -\mu_{aa} - \mu_{ab} = \bar{\mu}_{aa} + \bar{\mu}_{ab}$ is a cycle, in fact the circuit containing $\bar{\delta}_a$ and $\mu_b{}^1 + \bar{\mu}_a{}^1 = \mu_{bb} + \bar{\mu}_{aa}$ is the circuit containing δ_b such that the arcs of $\delta_a = \gamma_a'$ have opposite orientation.

Now consider the effect on γ_b of reversing the orientation of γ_a. From the definition and (2.1)–(2.4) it is sufficient to consider $\mu_a{}^1$, $\mu_b{}^1$. There are two cases to consider i) $\hat{f}_a \in \mu_b{}^1$ and ii) $\hat{f}_a{}^1 \in' \mu_b{}^1$, where $f_a = \psi(\delta_a)$. In the first case, $\hat{f}_a \in \mu_{ab}$ so that $\psi(\mu_{aa}) > f_b = \psi(\delta_b)$, $\psi(\mu_a{}^2) = \psi(\mu_{ab}) = f_a$ and $\psi(\mu_b{}^2) = \psi(\mu_{bb}) = f_b$ implying $\mu_a{}^2 < \mu_b{}^2$ so that the order is preserved. In the second case, $\hat{f}_a \in \mu_{aa}$ so that $\mu_a{}^2 < \mu_b{}^2$ if and only if $\psi(\bar{\mu}_{ab}) = \psi(\mu_{ab}) > f_b$. This means that the order $\gamma_a < \gamma_b$ will be preserved under the transformation $\delta_a \rightarrow \bar{\delta}_a$ if and only if $\psi(\mu_{ab}) > f_b$.

The two cases can be combined by setting $\theta_{ab} = \delta_a\gamma_b$ if $\hat{f}_a \in' \delta_a\gamma_b$ and $\theta_{ab} - \delta_a\gamma_b$ otherwise. Then the order $\gamma_a < \gamma_b$ will be preserved under the transformation $\delta_a \rightarrow \bar{\delta}_a$ if $\psi(\theta_{ab}) > f_b$.

The order $\gamma_a < \gamma_b$ will also be preserved under the transformation $\delta_b \rightarrow \bar{\delta}_b$. To see this, first note that the ordering is preserved if the orientation on all the edges is reversed. Then reversing the orientations on $\bar{\delta}_1, \bar{\delta}_2, \ldots, \bar{\delta}_{q-1}$ is equivalent to reversing the orientation on δ_q in the original, i.e., the ordering is preserved under the transformation $\delta_q \rightarrow \bar{\delta}_q$ if $\psi(\theta_{aq}) < f_q$ for all $a < q$ for which $\delta_a\gamma_q \neq \emptyset$. Proceeding inductively it follows that the order is preserved under the transformation $\delta_b \rightarrow \bar{\delta}_b$ if $\phi(\theta_{ab}) > f_b$ for every $a < b$.

This suggests the following definitions. The circuit γ_k is *externally active* (in Λ_C relative to the underlying ordering of the edges of $\mathscr{G}$) if $\psi(\theta_{ak}) > f_k$ for every $a < k$ for which $\delta_a\gamma_k \neq \emptyset$. Let s be the number of circuits $\{\gamma_j\}$ which are externally active, then the orientation o has *external activity* s.

Corresponding to each of the s circuits γ_k which are externally active there is a subset $\tilde{\delta}_k$ of $\{\delta_j\}$ which contains δ_k and all sets δ_b, $b > k$, such that γ_b is inactive and contains an edge $\hat{f}_a \in \delta_a$ where $\delta_a \in \tilde{\delta}_k$. The sets $\tilde{\delta}_k$ can be constructed inductively, beginning with $\tilde{\delta}_a$ where $f_a = \psi(A_C)$. In terms of this notation, the above discussion can be restated.

THEOREM 2.2. *Let γ_k denote an externally active circuit and $\tilde{\delta}_k$ the corresponding subset of $\{\delta_j\}$ as defined above. Let o_k denote the orientation of $\mathscr{G}$ obtained by reversing the orientation on all the arcs of the sets $\tilde{\delta}_k$ of $\mathscr{D}(o)$. Let $\mu_j{}^{(k)}$ denote the circuit of $\mathscr{D}(o_k)$ corresponding to γ_j in $\mathscr{D}(o)$. Then the ordering of the circuits as determined by the original ordering of the edges of $\mathscr{G}$ is $\mu_1{}^{(k)}\ \mu_2{}^{(k)}, \ldots, \mu_q{}^{(k)}$. The nest of circuit spaces Λ_C and the corresponding partition of E_C is the same for o_k as for o.*

Analogous theorems can be obtained for $\mathscr{D}_C'$ which characterizes the cocircuits of $\mathscr{D}$. A sequence of independent cocircuits of Λ_C' and a corresponding partition of A_C' into disjoint subsets which leads to a definition of internal activity can be defined as follows.

Let $\mathscr{D}_C'^1 = \mathscr{D}_C'$. Let $\{\gamma_j'^k\}$ be the set of cocircuits of $\Lambda_C'^k$ and let $\delta_k' = \min \{\gamma_j'^k\}$; $\mathscr{D}_C'^{k+1}$ is obtained from $\mathscr{D}_C'^k$ by deleting the edges of δ_k'. Continue until $\{\gamma_j'^k\}$ consists of one cocircuit. Corresponding to δ_k' there is a unique cocircuit γ_k' of Λ_C' such that equations analogous to (2.1)–(2.4) hold and a Theorem 2.1′ analogous to Theorem 2.1 can be stated. The effect of reversing the orientation on a subset δ_k' is also analogous. The cocircuit γ_k' is *internally active* (in Λ_C' relative to the underlying ordering of the edges of $\mathscr{G}$) if $\psi(\theta_{ak}') > f_k$ for every $a < k$ such that $\delta_a'\gamma_k' \neq \emptyset$, where $\theta_{ak}' = \delta_a'\gamma_k$ if $\hat{e}_a \in \delta_a'\gamma_k'$ and $\theta_{ak}' = \delta_a' - \delta_a'\gamma_k'$ otherwise. The *internal activity* of o is the number r of internally active cocircuits of Λ_C'. Every internally active cocircuit γ_k' determines a subset $\tilde{\delta}_k'$ of $\{\delta_k'\}$ as in the case of externally active circuits. A Theorem 2.2′ analogous to Theorem 2.2 can be stated for cocircuits with primed symbols instead of unprimed symbols and the term internally active replacing externally active.

These theorems can be interpreted for $\mathscr{D}$ since the circuits of Λ_C are equivalent to the circuits of $\mathscr{D}$ and the cocircuits of Λ_C' are equivalent to the cocircuits of $\mathscr{D}$.

THEOREM 2.3. *Let o be an orientation of $\mathscr{G}$ with internal activity r and external activity s and let $\{\Lambda_C'^j\}$, $\{\Lambda_C^j\}$ be the corresponding nests of subspaces of Λ_C', Λ_C (relative to an ordering of the edges of E). Then there is a set $O(o) \subset O(\mathscr{G})$ containing 2^{r+s} orientations of $\mathscr{G}$ each of which determines these same nests of subspaces.*

A similar statement holds for the partition $\{D_j'\} \cup \{D_j\}$ of E relative to an ordering of the edges.

3. Spanning trees. The circuit space Λ_C of $\mathscr{D}_C$ has a cycle basis consisting of the circuits $\gamma_1, \gamma_2, \ldots, \gamma_q$ where $q = \rho(\mathscr{D}_C)$. The equations (2.3) (2.4) imply that every (directed) circuit of $\mathscr{D}_C$ (and of $\mathscr{D}$) must contain one of the sets $\delta_j \subset \gamma_j$, $j = 1, 2, \ldots, q$, so that every (undirected) circuit of $\mathscr{G}_C$ must contain at least one of the edges $f_j = \psi(\delta_j)$, $j = 1, 2, \ldots, n$. Thus the set $E_C - F_C$, where $F_C = \{f_j\}$ cannot contain a circuit and must be a spanning forest of $\mathscr{D}_C$ which is the union of spanning trees of its components. Similarly $E_C' - F_C'$, where $F_C' = \{f_j'\}$, $f_j' = \psi(\gamma_j')$, $j = 1, 2, \ldots, p$, which is determined by the cocircuits of $\mathscr{D}_C'$, cannot contain a cocircuit. Since $\mathscr{G}_C'$ is connected, F_C' is a spanning tree of $\mathscr{G}_C'$. In terms of E, $T = (E_C - F_C) \cup F_C'$ is a spanning tree of $\mathscr{G}$.

THEOREM 3.1. *Let $o \in O(\mathscr{G})$ and let $\{D_j'\} \cup \{D_j\}$ be the corresponding partition of E. Then $T = (E_C - \{f_j\}) \cup \{f_j'\}$ is a spanning tree of $\mathscr{G}$ where f_j is* *the first edge in* E_C *not in* $\bigcup_{i<j} D_i$, $j = 1, 2, \ldots, q$ *and* f_j' *is the first edge of* E_C' *not in* $\bigcup_{i<j} D_i'$, $j = 1, 2, \ldots, p$.

Let $T(o)$ denote this tree associated with o. We now associate a set of orientations $O(T) \subset O(\mathscr{G})$ with any tree T in such a way that if $T = T(o)$, then $o \in O(T)$.

Let T denote any (spanning) tree of $\mathscr{G}$. We first show that T together with the ordering of E determines a decomposition of E into two disjoint subsets E_C, E_C' which will be identified with the decomposition determined by o as in Theorems 2.1, 2.1′. We do this inductively. For any set $S \subset E$ let $\psi(S)$ denote the first element of S. Set $g_1 = e_1$, $E_T^{(1)} = R_T g_1$, $E_T'^{(1)} = R_T'(g_1)$. Let $g_2 = \psi(E - E_T^{(1)} \cup E_T'^{(1)})$ and set $E_T^{(2)} = E_T^{(1)} \cup R_T g_2$, $E_T'^{(2)} = E_T'^{(1)} \cup R_T' g_2$. Continue in this way, setting

$$g_k = \psi(E - E_T^{(k-1)} \cup E_T'^{(k-1)})$$

and

$$E_T^{(k)} = E_T^{(k-1)} \cup R_T g_k, \quad E_T'^{(k)} = E_T'^{(k-1)} \cup R_T' g_k.$$

The process ends when $E = E_T^{(t)} \cup E_T'^{(t)}$. The sets $E_T^{(t)}$, $E_T'^{(t)}$ are disjoint, for at each stage one of the sets $R_T g_k$, $R_T' g_k$ is empty and the other set is disjoint with $E_T^{(k-1)} \cup E_T'^{(k-1)}$. Suppose, for example, $e \in R_T g_j$ and $e \in R_T' g_k$, $j \neq k$. Then $g_j \in R_T' e$, and $g_k \in R_T e$ which is impossible. Setting $E_C = E_T^t$, $E_C' = E_T^t$ we have

$$E = E_C \cup E_C', \quad E_C E_C' = \emptyset$$

where E_C is the union of a set of circuits of $\mathscr{G}$ and E_C' is the union of a set of cocircuits.

Let $f_1, f_2, \ldots, f_q$ denote the subsequence of $g_1, g_2, \ldots, g_t$ belonging to T' and let $f_1', f_2', \ldots, f_p'$ denote the subsequence belonging to T. Set $C_i = R_T f_i$, $i = 1, 2, \ldots, q$ and $C_i' = R_T' f_i'$, $i = 1, 2, \ldots, p$. Finally, set $D_1 = C_1$, $D_1' = C_1'$ and in general

$$(3.1) \quad D_k = C_k - \bigcup_{j<k} D_j, \quad D_k' = C_k' - \bigcup_{j<k} D_{j'}$$

so that

$$(3.2) \quad C_k \subset \bigcup_{j \leqq k} D_j, \quad C_k' \subset \bigcup_{j \leqq k} D_j'.$$

Equations (3.1), (3.2) are the analogues of (2.3) (2.4) (and their duals). We can define graphs $\mathscr{G}_C$, $\mathscr{G}_C'$ analogous to $\mathscr{D}_C$, $\mathscr{D}_C'$ by deleting the edges of E_C' and contracting the edges of E_C. Since the circuits $\{C_k\}$ of $\mathscr{G}_C$ are independent and the cocircuits $\{C_k'\}$ of $\mathscr{G}_C'$ are independent (by (3.1), (3.2)) we obtain theorems similar to 2.1, 2.1′.

THEOREM 3.2. *The circuits* $\{C_j\}$ *are an independent set of circuits of* $\mathscr{G}_C$ *and the subsets* $\{D_j\}$ *(defined above) partition* E_C *into disjoint subsets such that if*

https://doi.org/10.4153/CJM-1977-095-1 Published online by Cambridge University Press

$f_k = \psi(E_C - \bigcup_{j<k} D_j)$, then $f_k \in D_k$, $k = 1, 2, \ldots, q$ and $T' = \{f_k\}$ is a cotree of $\mathscr{G}_C$ such that $C_k = R_T f_k$.

A similar Theorem 3.2′ holds for the cocircuits of $\mathscr{G}_C'$.

We now construct a set of orientations $O(T) \subset O(\mathscr{G})$ corresponding to any spanning tree T in the following way. Suppose $T = T(o)$ is the tree associated with o as in Theorem 3.1. If f_k is externally active relative to T and the ordering of the edges then f_k isthe first element of $R_T f_k$. But f_k corresponds to an element $\hat{f}_k$ of δ_k ($\psi(\delta_k) = f_k$) with the property that the decomposition $\{D_j\}$ was unchanged by reversing the orientation of the arcs of δ_k (which includes $\hat{f}_k$). Thus we associate an arbitrary orientation with the arcs of C_K. On the other hand, if f_j is not externally active the circuit $R_T f_j$ must contain an edge e_i corresponding to a set C_a with smallest subscript. If we have examined the edges in the order $f_1, f_2, \ldots$, then f_a has an orientation. Assign this same orientation to the edges of D_j. In this way we associate with each active element f_k a subset $\tilde{D}_k$ of $\{D_j\}$ all of whose edges have the same orientation. That is, we have s such sets where s is the external activity of T. Since the orientations can be assigned arbitrarily, we have 2^s different orientations of $\mathscr{G}_C$ corresponding to T in this way. Further, if o is one of these orientations, then δ_j as defined in Section 2 corresponds to D_j with the assigned orientation and $\psi(\delta_j) = f_j$, $j = 1, 2, \ldots, q$. Similarly for $\mathscr{G}_C'$.

THEOREM 3.3. *Let T be a spanning tree of $\mathscr{G}$ with internal activity r and external activity s. Let $\{\tilde{D}_{pj}\} \cup \{\tilde{D}_{qj}\}$ denote the corresponding partition of E as defined above and let $O(T)$ denote the set of 2^{r+s} orientations of $\mathscr{G}$ obtained by assigning arbitrary orientations to each of the sets $\tilde{D}_{pj}'$, $j = 1, 2, \ldots, r$; $\tilde{D}_{qj}$, $j = 1, 2, \ldots, s$. Then if $o \in O(T)$, $T(o) = T$ (where $T(o)$ is defined in Section 3) and each of the orientations $o \in O(T)$ has internal activity r and external activity s (as defined in Section 2).*

4. The dichromate. Let χ_{rs} denote the number of spanning trees of $\mathscr{G}$ with internal activity r and external activity s. By Theorem 3.3 associated with each of these trees there are 2^{r+s} different orientations with activities r, s, and orientations constructed from different trees cannot be the same. Let this set of $\chi_{rs} 2^{r+s}$ orientations be denoted by $O(\mathscr{G}, r, s)$, i.e.,

$$(4.1) \quad O(\mathscr{G}; r, s) = \bigcup{}' O(T)$$

where the union is taken over all trees with activities r, s. By (1.1), (1.2)

$$\sum_{r,s} \chi_{r,s} 2^{r+s} = \chi(\mathscr{G}, 2, 2) = 2^m$$

where $\chi(\mathscr{G}; x, y)$ is the dichromate of $\mathscr{G}$. Thus all 2^m orientations of $\mathscr{G}$ are accounted for, and we have a partition

$$(4.2) \quad O(\mathscr{G}) = \bigcup_{r,s} O(\mathscr{G}; r, s)$$

of $O(\mathscr{G})$ into disjoint subsets which are in correspondence with the terms of the dichromate. If we substitute (4.1) into (4.2) we obtain the decomposition

$$(4.3) \quad O(\mathscr{G}) = \bigcup_T O(T)$$

corresponding to the expansion of χ in the form

$$(4.4) \quad \chi(\mathscr{G}; x, y) = \sum_T x^{r(T)} y^{s(T)}$$

where $r(T)$, $s(T)$ are the activities of T.

THEOREM 4.1. *There are $\chi_{rs}2^{r+s}$ elements in the set $O(\mathscr{G}; r, s)$ of orientations of $\mathscr{G}$ with activities r, s, where χ_{rs} is the coefficient of $x^r y^s$ in the dichromate of $\mathscr{G}$. These sets are in* 1-1 *correspondence with the terms of the dichromate* (1.1) *and partition $O(\mathscr{G})$ into disjoint subsets.*

If $\mathscr{D}(o)$ is an acyclic graph, o is an *acyclic orientation* of $\mathscr{G}$. In this case $\mathscr{D} = \mathscr{D}_C'$ and the sets $O(\mathscr{G}; r, s)$, $s \neq 0$ are empty. It follows that the set of acyclic orientations $O_C'(\mathscr{G})$ is given by

$$O_C'(\mathscr{G}) = \bigcup_r O(\mathscr{G}; r, 0)$$

and by Theorem 4.1 the number of acyclic orientations of $\mathscr{G}$ is given by

$$|O_C'(\mathscr{G})| = \sum_r |O(\mathscr{G}; r, 0)| = \sum_r \chi_{r0}2^r = \chi(\mathscr{G}; 2, 0).$$

This number can also be expressed in terms of the chromatic polynomial, for setting $\lambda = -1$ in 1.3 we get $P(\mathscr{G}, -1) = (-1)^n\chi(\mathscr{G}; 2, 0)$.

Analogously, if $\mathscr{D}(o)$ is a strongly connected graph, then $\mathscr{D} = \mathscr{D}_C$ and the set of these orientations is $O_C(\mathscr{G}) = \bigcup_s O(\mathscr{G}; 0, s)$ so that $|O_C(\mathscr{G})| = \chi(\mathscr{G}; 0, 2)$. If $\mathscr{G}$ is planar this can be interpreted in terms of the chromatic polynomial of the dual graph evaluated at -1.

THEOREM 4.2. *The number of acyclic orientations of $\mathscr{G}$ is $\chi(\mathscr{G}; 2, 0) = |P(\mathscr{G}, -1)|$, where $\chi(\mathscr{G}; x, y)$ is the dichromate and $P(\mathscr{G}; \lambda)$ is the chromatic polynomial of $\mathscr{G}$. The number of orientations of $\mathscr{G}$ such that $\mathscr{D}(o)$ is strongly connected is $\chi(\mathscr{G}; 0, 2)$, and if $\mathscr{G}$ is planar this is also given by $|P(\mathscr{G}'; -1)|$ where $\mathscr{G}'$ is the planar dual of $\mathscr{G}$.*

A proof of this result employing enumeration techniques was given by Stanley [**5**]. Michel Las Vergnas has obtained analogous results involving orientable matroids.

REFERENCES

1. Claude Berge, *Graphs and hypergraphs* (North Holland, Amsterdam, 1973).

2. Norman Biggs, *Algebraic graph theory* (Cambridge U. Press, London, 1974).

3. F. Harary and E. Palmer, *On the number of orientations of a given graph*, Bull. de L'Acad. Polonaise des Sciences *14* (1966), 125–128.

4. G. J. Minty, *Monotone networks*, Proc. Roy. Soc. *A257* (1960), 194–208.

5. R. P. Stanley, *Acyclic orientations of graphs*, Discrete Math. *5* (1973), 171–178.
6. W. T. Tutte, *A contribution to the theory of chromatic polynomials*, Can. J. Math. *6* (1953), 80–91.
7. ——— *A class of abelian groups*, Can. J. Math. *8* (1956), 13–28.
8. ——— *On dichromatic polynomials*, J. Comb. Theory *2* (1967), 301–320.
9. ——— *The dichromatic polynomial*, Proc. Fifth British Combinatorial Conference, University of Aberdeen (1975), 605–635.

University of Waterloo,
Waterloo, Ontario