ASYMPTOTICALLY QUASI-COMPACT PRODUCTS OF BOUNDED LINEAR OPERATORS

Dedicated to the memory of Hanna Neumann

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1. Introduction

It is known (see, for instance, [1] p. 64, [6] p. 264) that, if A and B are bounded linear operators on a Banach space $\mathfrak{X}$ into itself (or, more generally, if A is a bounded linear operator on $\mathfrak{X}$ into a Banach space $\mathfrak{Y}$ and B is a bounded linear operator on $\mathfrak{Y}$ into $\mathfrak{X}$), then AB and BA have the same spectrum except (possibly) for zero. In the present note, it is shown that AB is asymptotically quasi-compact if and only if BA is asymptotically quasi-compact, and that then any Fredholm determinant for AB is a Fredholm determinant for BA and vice versa.

2. Preliminaries

We first recall some properties of asymptotically quasi-compact bounded linear operators on a Banach space $\mathfrak{X}$ into itself. For convenience, we shall where necessary restrict our attention to *complex* Banach spaces. In such cases, the conclusions can be extended to *real* Banach spaces by considering their complexifications (cf. [3] §3, pp. 373–374).

A bounded linear operator K on a Banach space $\mathfrak{X}$ into itself is said to be *asymptotically quasi-compact* if $\kappa(K^n)^{1/n} \to 0$ as $n \to \infty$, where, for any bounded linear operator T on $\mathfrak{X}$ into itself, $\kappa(T) = \inf \| T - C \|$, the infimum being taken over all compact linear operators C on $\mathfrak{X}$ into itself (cf. [4] Definitions 3.1 and 3.2, p. 322).

A scalar integral (entire) function $\Delta_\upsilon(.)$ is called a *Fredholm determinant* for an asymptotically quasi-compact bounded linear operator K on a Banach space

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$\mathfrak{X}$ into itself if it is not identically zero and the power series

$$\sum_{r=0}^{\infty} \Delta_n^r \lambda^r \qquad (n = 0, 1, 2, \cdots),$$

defined as in [5] Definition 2.1, pp. 26–27, all have infinite radius of convergence.

In practice, we shall find it convenient to use the criterion in Theorem 2.1 below. We use the following notation.

Let K be an asymptotically quasi-compact bounded linear operator on a complex Banach space $\mathfrak{X}$ into itself. We define the *multiplicity* $m_K(\lambda)$ of the complex number λ with respect to K to be the maximum number of (complex) dimensions of the kernel $\mathfrak{M}_n(K;\lambda) = (I - \lambda K)^{-n}[\{\Theta\}]$ of $(I - \lambda K)^n$ for all non-negative integers n. This is finite by [4] Lemmas 2.1 and 2.2, pp. 320–321.

THEOREM 2.1. *Let K be an asymptotically quasi-compact bounded linear operator on a complex Banach space $\mathfrak{X}$ into itself. A scalar integral function $\Delta_0(.)$ will be a Fredholm determinant for K if and only if it is not identically zero and the order of each complex number λ as a zero of $\Delta.(.)$ is at least the multiplicity $m_k(\lambda)$ of λ with respect to K.*

PROOF. The sufficiency of the condition is (in effect) remarked on in [3] §5, pp. 375–376; it can be proved by adapting the proof of [3] Theorem 2.2, pp. 369–372. The necessity is remarked on in a Note in [4] (p. 321); see also [5] Theorem 3.3 and remark following (p. 38).

We shall again find it convenient to use the polynomial Φ_n (where n is a non-negative integer) given by

$$\Phi_n(\mu) = \sum_{r=1}^{n} \binom{n}{r} (-\mu)^{r-1} = \{1 - (1-\mu)^n\}/\mu$$
$$= 1 + (1-\mu) + (1-\mu)^2 + \cdots + (1-\mu)^{n-1}$$

(cf. [5] p. 43; the last expression, which I have overlooked in the past, is particularly useful in finding $\Phi_n(\lambda K)$, since we are in any case interested in powers of $I - \lambda K$).

3. The main theorems

THEOREM 3.1. *Let A and B be bounded linear operators on a Banach space $\mathfrak{X}$ into itself (or, more generally, let A be a bounded linear operator on $\mathfrak{X}$ into a Banach space $\mathfrak{Y}$, and let B be a bounded linear operator on $\mathfrak{Y}$ into $\mathfrak{X}$). Then AB is asymptotically quasi-compact if and only if BA is asymptotically quasi-compact.*

PROOF. This follows at once from the obvious inequalities

$$\kappa((AB)^{n+1}) \leqq \| A \| \kappa((BA)^n) \| B \|$$

and

$$\kappa((BA)^{n+1}) \leqq \| B \| \kappa((AB)^n) \| A \|.$$

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THEOREM 3.2. *Let A and B be as in Theorem* 3.1, *and such that AB* (*and so BA*) *is asymptotically quasi-compact. Then a scalar integral function $\Delta_0(.)$ is a Fredholm determinant for AB if and only if it is a Fredholm determinant for BA.*

PROOF. We shall assume that $\mathfrak{X}$ (and $\mathfrak{Y}$ in the more general situation) is a *complex* Banach space (cf. remark above).

We shall use the obvious facts that, if ϕ is any polynomial, then

$$B\phi(AB) = \phi(BA)B \qquad \text{and} \qquad A\phi(BA) = \phi(AB)A$$

(these could, in fact, be extended to more general functions ϕ using Dunford's operational calculus, cf. [2] §VII.3, pp. 566–577).

Now let λ be any complex number, and let n be any non-negative integer. We consider the kernels $\mathfrak{M}_n(AB;\lambda)$ and $\mathfrak{M}_n(BA;\lambda)$ of $(I-\lambda AB)^n$ and $(I-\lambda BA)^n$ respectively.

Let x be any point of $\mathfrak{M}_n(AB;\lambda)$. Then

$$(I-\lambda BA)^n Bx = B(I-\lambda AB)^n x = \Theta,$$

and so $Bx \in \mathfrak{M}_n(BA;\lambda)$; moreover $(I-\lambda AB)^n x = \Theta$, and so

$$x = \lambda AB\Phi_n(\lambda AB)x = \lambda A\Phi_n(\lambda BA)(Bx).$$

On the other hand, if y is any point of $\mathfrak{M}_n(BA;\lambda)$ and we put $x = \lambda A\Phi_n(\lambda BA)y$, then we get

$$(I-\lambda AB)^n x = \lambda A\Phi_n(\lambda BA)(I-\lambda BA)^n y = \Theta,$$

and so $x \in \mathfrak{M}_n(AB;\lambda)$, and $Bx = \lambda BA\Phi_n(\lambda BA)y = y$. Thus B maps $\mathfrak{M}_n(AB;\lambda)$ linearly and bijectively onto $\mathfrak{M}_n(BA;\lambda)$, and so these spaces have the same number of dimensions. In particular, $m_{AB}(\lambda) = m_{BA}(\lambda)$.

The theorem now follows from Theorem 2.1.

NOTE. It follows further, since in the above argument n can be *any* non-negative integer, that the "spot diagram" (for a particular scalar λ), as described in [5] (pp. 31–32), will be the same for AB as for BA.

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