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LIE SOLVABLE GROUP RINGS

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Let $K[G]$ denote the group ring of G over the field K. One of the interesting problems which arises in the study of such rings is to find precisely when they satisfy polynomial identities. This has been solved for char $K = 0$ in [**1**] and for char $K = p > 0$ in [**3**]. The answer is as follows. If $p > 0$ we say that group A is p-abelian if A', the commutator subgroup of A, is a finite p-group. Moreover, for convenience, we say A is 0-abelian if and only if it is abelian.

THEOREM [**1**; **3**]. *$K[G]$ satisfies a polynomial identity if and only if G has a p-abelian subgroup of finite index where* char $K = p \geqq 0$.

While this solves the problem in general, it does not of course answer the question for any specific polynomial. In this paper we consider the polynomial identities which correspond to Lie nilpotence and Lie solvability.

Let R be any K-algebra (R need not have a 1). If A and B are two K-subspaces of R we define their Lie product $[A, B]$ to be the K-subspace of R spanned by all Lie products $[a, b] = ab - ba$ with $a \in A$, $b \in B$. We can then define inductively the Lie central and Lie derived series of R by

$$\gamma^0 R = R, \quad \gamma^{n+1} R = [\gamma^n R, R] \quad \text{(central series)}$$
$$\delta^0 R = R, \quad \delta^{n+1} R = [\delta^n R, \delta^n R] \quad \text{(derived series)}.$$

We say that R is Lie nilpotent if $\gamma^n R = 0$ for some integer n and similarly R is Lie solvable if $\delta^n R = 0$ for some integer n. It is apparent that R is Lie nilpotent or Lie solvable if and only if it satisfies certain multilinear polynomial identities. Thus for example $\delta^2 R = 0$ if and only if R satisfies the identity

$$[[\zeta_1, \zeta_2], [\zeta_3, \zeta_4]]$$

which is multilinear of degree 4.

Our main result is

THEOREM. *Let $K[G]$ be the group ring of G over the field K with* char $K = p \geqq 0$. *Then*

(i) *$K[G]$ is Lie nilpotent if and only if G is p-abelian and nilpotent;*

(ii) *for $p \neq 2$, $K[G]$ is Lie solvable if and only if G is p-abelian;*

(iii) *for $p = 2$, $K[G]$ is Lie solvable if and only if G has a 2-abelian subgroup of index at most* 2.

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Thus we see that the assumption of Lie nilpotence or solvability is quite restrictive. In fact this occurs in characteristic 0 if and only if G is abelian.

1. Matrix rings. Let S be a commutative K-algebra (S need not have a 1). Then we let $M_n(S)$ denote the ring of $n \times n$ matrices over S and $T_n(S)$ denotes those matrices of trace 0.

LEMMA 1.1. *With S as above we have*
(i) $[T_n(S), M_n(S)] = T_n(S^2)$;
(ii) $[T_n(S), T_n(S)] = T_n(S^2)$ *unless* $n = 2$ *and* char $K = 2$.

Proof. If $n = 1$ then $T_n(S) = 0$ so the result is clear. Thus, let $n \geqq 2$ and let $\{e_{ij}\}$ denote a set of matrix units. Clearly $[M_n(S), M_n(S)] \subseteq T_n(S^2)$ so we need only prove the reverse inclusions.

Let $i \neq j$. If $n \geqq 3$ we can choose $k \neq i, j$. Then for any $s, s' \in S$

$$[se_{ij}, s'(e_{jj} - e_{kk})] = ss'e_{ij}$$
$$[se_{ij}, s'e_{ji}] = ss'(e_{ii} - e_{jj}).$$

Since all these matrices have trace 0 and since the right hand matrices clearly span $T_n(S^2)$ part (ii) follows and so therefore does (i).

Now let $n = 2$. From the above and

$$[se_{ij}, s'e_{jj}] = ss'e_{ij}$$

we see that (ii) holds. Finally if char $K \neq 2$ then

$$[se_{ij}, s'(e_{jj} - e_{ii})] = 2ss'e_{ij}$$

implies that $ss'e_{ij} \in [T_n(S), T_n(S)]$ and the result follows.

LEMMA 1.2. *Assume that S is not nilpotent. Then*
(i) $M_n(S)$ *is Lie nilpotent if and only if* $n = 1$;
(ii) $M_n(S)$ *is Lie solvable if and only if* $n = 1$ *or* $n = 2$, char $K = 2$.

Proof. If n = 1 then $M_n(S)$ is commutative and hence both Lie nilpotent and solvable. Let $n > 1$. Then by Lemma 1.1 (i) and induction we have for $k \geqq 1$, $\gamma^k M_n(S) = T_n(S^{2^k})$. Since S is not nilpotent, $T_n(S^{2^k}) \neq 0$ and $M_n(S)$ is not Lie nilpotent. Moreover by Lemma 1.1 (ii) unless $n = 2$ and char $K = 2$ we have $\delta^k M_n(S) = T_n(S^{2^k})$ for $k \geqq 1$. Thus $M_n(S)$ is not Lie solvable.

Finally let $n = 2$ and char $K = 2$. Then $\delta^1 M_2(S) = T_2(S)$ is spanned by S^2e_{12}, S^2e_{21} and $S^2(e_{11} + e_{22})$ where $\{e_{ij}\}$ denotes a set of matrix units. Thus since $e_{11} + e_{22}$, the identity matrix, is central $\delta^2 M_2(S)$ is spanned by

$$[S^2e_{12}, S^2e_{21}] = S^4(e_{11} + e_{22})$$

and hence $\delta^3 M_2(S) = 0$. Therefore $M_2(S)$ is Lie solvable if char $K = 2$.

Our proof requires some of the techniques of [**4**]. Unfortunately they are not stated in the generality we require so we restate the necessary lemmas below.

Let A be a normal abelian subgroup of G of finite index n and let $x_1 = 1, x_2, \ldots, x_n$ be a set of coset representatives. Let $\rho : K[G] \to M_n(K[A])$ be given by $\rho(\alpha) = [\alpha_{ij}]$ with $\alpha_{ij} \in K[A]$ where

$$x_i^{-1}\alpha = \sum_j \alpha_{ij}x_j^{-1} \quad \text{for} \quad i = 1, 2, \ldots, n.$$

LEMMA 1.3. *ρ is a monomorphism.*

Proof. This is just Lemma 1.1 of [**4**].

For each $i = 2, 3, \ldots, n$ the commutator

$$(A, x_i) = \{a^{-1}x_i^{-1}ax_i | a \in A\}$$

is a subgroup of A isomorphic to $A/\mathbf{C}_A(x_i)$ (see [**4**, Lemma 1.2]). Let S_i be the augmentation ideal of the group ring $K[(A, x_i)]$. Thus S_i is a K-subalgebra (without 1) of $K[A]$. Set

$$S = S_2S_3 \ldots S_n$$

so that S is also a K-subalgebra of $K[A]$.

LEMMA 1.4. *Let S_i be as above.*

(i) *If $[A : \mathbf{C}_A(x_i)] = \infty$ then S_i annihilates no nonzero element of $K[A]$.*

(ii) *If* char $K = p > 0$ *then S_i is nilpotent if and only if (A, x_i) is a finite p-group.*

Proof. Part (i) follows as in Lemma 1.3 of [**4**]. Consider part (ii). If $|(A, x_i)| = \infty$ then by (i) S_i cannot annihilate any element of $K[A]$ so certainly S_i is not nilpotent. On the other hand if (A, x_i) is finite then certainly S_i is nilpotent if and only if (A, x_i) is a p-group where char $K = p > 0$.

Observe that $K[A]$ is embedded in $M_n(K[A])$ in two different ways. First $K[A] \subseteq M_n(K[A])$ consists of all scalar matrices. Secondly $\rho(K[A]) \subseteq M_n(K[A])$. These are easily seen to be diagonal matrices but not necessarily scalars.

LEMMA 1.5. *Set $R = K[A] \cdot \rho(K[G]) \subseteq M_n(K[A])$. Then $R \supseteq M_n(S)$ where $S = S_2S_3 \ldots S_n$.*

Proof. This is just Lemma 1.4 of [**4**].

2. Characteristic $\neq 2$. We now study G under the assumption that $K[G]$ is Lie nilpotent or solvable. It is convenient to list three separate assumptions.

(*) $K[G]$ is Lie nilpotent, char $K = p$ ($p = 0$ allowed)

(**) $K[G]$ is Lie solvable, char $K = p \neq 2$ ($p = 0$ allowed)

(***) $K[G]$ is Lie solvable, char $K = 2$.

Recall that $\Delta(G)$ is the F.C. subgroup of G, that is $\Delta(G)$ is the set of all elements of G having only finitely many conjugates.

LEMMA 2.1. *Consider the group ring* $K[G]$.
(i) $(*)$ *or* $(**)$ *imply that* $G = \Delta(G)$ *and* G' *is finite.*
(ii) $(***)$ *implies that* $[G : \Delta(G)] \leqq 2$ *and* $\Delta(G)'$ *is finite.*

Proof. Assume that $K[G]$ satisfies $(*)$, $(**)$ or $(***)$. Then $K[G]$ satisfies a polynomial identity, so by Theorem 2.2 of [**4**], $[G : \Delta(G)] < \infty$ and $|\Delta(G)'| < \infty$. Let $[G : \Delta(G)] = n$.

Suppose first that $\Delta(G) = A$ is abelian and consider (in the notation of Lemma 1.5) $R = K[A] \cdot \rho(K[G]) \subseteq M_n(K[A])$. Since $K[A] \subseteq M_n(K[A])$ is the set of scalar matrices we see that R satisfies the same nultilinear polynomial identities as does $K[G]$. Hence R is Lie nilpotent or Lie solvable accordingly as $K[G]$ is. By Lemma 1.5, $R \supseteq M_n(S)$ where $S = S_2S_3 \ldots S_n$ and by Lemma 1.4 (i) since $x_i \notin \Delta(G)$ for $i > 1$ we see that S is not nilpotent. Since $M_n(S)$ is Lie nilpotent or Lie solvable we see from Lemma 1.2 that $(*)$ or $(**)$ imply $n = 1$ and $(***)$ implies that $n = 1$ or 2.

Finally let $W = \Delta(G)'$. Since W is a finite normal subgroup of G we see that if $\bar{G} = G/W$ then $\Delta(\bar{G}) = \Delta(G)/W$ is abelian and $[G : \Delta(G)] = [\bar{G} : \Delta(\bar{G})]$. Since $K[\bar{G}]$ is also Lie nilpotent or Lie solvable, the result follows.

Let $JK[G]$ denote the Jacobson radical of $K[G]$.

LEMMA 2.2. *Let* $K[G]$ *be Lie nilpotent or Lie solvable and suppose* $\alpha \in K[G]$, $\alpha \notin JK[G]$. *Then there exists a homomorphism* $\sigma : K[G] \to M_n(E)$ *where* E *is some algebraically closed field extension of* K *such that* $\sigma(\alpha) \neq 0$. *Moreover*
(i) $(*)$ *or* $(**)$ *implies that* $n = 1$;
(ii) $(***)$ *implies that* $n = 2$.

Proof. Since $\alpha \notin JK[G]$ there exists an irreducible representation σ such that $\sigma(\alpha) \neq 0$. Then $\sigma(K[G]) = P$ is a primitive algebra satisfying a polynomial identity and thus by a theorem of Kaplansky [**2**, Theorem 6.4] P is finite dimensional over its center Z which is a field. Say

$$P = M_r(D) = D \underset{Z}{\otimes} M_r(Z)$$

where D is a finite dimensional division algebra over Z.

Let E be the algebraic closure of Z. Then $E \supseteq Z \supseteq K$, E is algebraically closed and Lemma 6.3 of [**2**] implies easily that $E \otimes_Z D \cong M_s(E)$ for some integer s. Thus

$$\begin{aligned} E \underset{Z}{\otimes} P &= E \underset{Z}{\otimes} D \underset{Z}{\otimes} M_r(Z) = M_s(E) \underset{Z}{\otimes} M_r(Z) \\ &= E \underset{Z}{\otimes} M_s(Z) \underset{Z}{\otimes} M_r(Z) = M_n(E) \end{aligned}$$

where $n = rs$. Since P is naturally contained in $E \otimes_Z P$ as $1 \otimes P$ we can extend σ to a map $\sigma : K[G] \to M_n(E)$ and certainly $\sigma(\alpha) \neq 0$.

Since $E \otimes_Z P$ is generated by $E \otimes 1$ and $1 \otimes P$ and since $E \otimes 1$ is central, we see that $E \otimes_Z P$ satisfies any multilinear polynomial identity satisfied by P. Thus $P = \sigma(K[G])$ implies that P and hence $M_n(E) = E \otimes_Z P$ is Lie

an irreducible representation of degree $\geq |P|$.

nilpotent or Lie solvable. Therefore we conclude from Lemma 1.2 that $(*)$ or $(**)$ implies $n = 1$ and $(***)$ implies $n = 1$ or 2. In the latter case if $n = 1$, we just embed $M_1(E)$ into $M_2(E)$ as the set of scalar matrices. Thus here we can always view σ as a map from $K[G]$ into $M_2(E)$.

We can now handle the easy cases.

LEMMA 2.3. *Consider the group ring* $K[G]$.
(i) $(*)$ *implies that* G *is* p*-abelian and nilpotent.*
(ii) $(**)$ *implies that* G *is* p*-abelian.*

Proof. Assume that $K[G]$ satisfies $(*)$ or $(**)$. Then by Lemma 2.1, $|G'| < \infty$. Let $g \in G'$, $g \neq 1$ and suppose that $(1 - g) \notin JK[G]$. Then by the previous lemma, there exists a homomorphism $\sigma : K[G] \to M_1(E)$ with $1 - \sigma(g) = \sigma(1 - g) \neq 0$. Now $M_1(E) = E$ so σ yields a homomorphism $\sigma : G \to E - \{0\}$. But $E - \{0\}$ is a commutative group and $g \in G'$ so $\sigma(g) = 1$, a contradiction. We have therefore shown that the whole augmentation ideal of $K[G']$ is contained in $JK[G]$. Thus by [**2**, Lemma 16.9] this augmentation ideal is contained in $JK[G] \cap K[G'] \subseteq JK[G']$. This implies easily that if $\operatorname{char} K = 0$ then $G' = \langle 1 \rangle$ and if $\operatorname{char} K = p > 0$ then G' is a p-group. Thus G is p-abelian.

We need only show that if $K[G]$ is Lie nilpotent, then G is nilpotent. If $\operatorname{char} K = 0$ then G is abelian and hence certainly nilpotent. Thus we need only consider $\operatorname{char} K = p > 0$. If $\gamma^m K[G] = 0$ then certainly $\gamma^{m'} K[G] = 0$ for any $m' \geqq m$. Thus we may assume that $\gamma^m K[G] = 0$ where $m = 1 + p^n$ for some $n \geqq 1$. Let $x, y \in G$ and let r_y denote the operator on $K[G]$ which is right multiplication by y and let l_y denote left multiplication. Then $[x, y] = x(r_y - l_y)$ so

$$0 = [[\ldots[[x, y]y] \ldots]y] = x(r_y - l_y)^{p^n}.$$

Since right and left multiplication commute as operators and since $\operatorname{char} K = p > 0$ we have

$$0 = x(r_y - l_y)^{p^n} = x((r_y)^{p^n} - (l_y)^{p^n}) = xy^{p^n} - y^{p^n}x.$$

Thus y^{p^n} centralizes x for all $x \in G$, so $y^{p^n} \in \mathbf{Z}(G)$. This shows that $G/\mathbf{Z}(G)$ is a p-group. Moreover G' is finite so $G/\mathbf{Z}(G)$ is a p-group with a finite commutator subgroup. It follows easily that $G/\mathbf{Z}(G)$ is nilpotent and therefore so is G.

3. Characteristic = 2. In this section we complete the case $(***)$. It will be necessary to study the finite group $\Delta(G)'$ and we will need some facts which are well known to finite group theorists. Since these are not difficult to prove we include them for the sake of completeness.

LEMMA 3.1. [illegible]

$h_1, h_2 \in H - \{1\}$ and set $\alpha = (1 - h_1)(1 - h_2)$. Since $h_1, h_2 \neq 1$ and $|H|$ is odd, it is trivial to see that $\alpha \neq 0$. Thus there exists a σ as above with $\sigma(\alpha) \neq 0$

$x \in G$ either $P = P^x$ or $P \cap P^x = \langle 1 \rangle$. If P is not normal in G, then $L[G]$ has an irreducible representation of degree $\geqq |P|$.

Proof. Let $P_0 = P, P_1, P_2, \ldots, P_n$ be all the conjugates of P in G. Then G permutes these groups transitively and we construct a permutation module V for $L[G]$ as follows. V has an L-basis $\{v_0, v_1, \ldots, v_n\}$ and $x \in G$ acts on V by permuting the basis elements in the same way it permutes the groups in the set $\{P_0, P_1, \ldots, P_n\}$.

Let us consider V as an $L[P]$-module. It is certainly still a permutation module but now P is no longer transitive on the basis. In fact P normalizes $P_0 = P$ so P fixes v_0. Suppose the orbits under the action of P on $\{v_0, v_1, \ldots, v_k\}$ are $\mathcal{O}_0 = \{v_0\}, \mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$. Then clearly as an $L[P]$-module

$$V = V_0 \dot{+} V_1 \dot{+} \ldots \dot{+} V_r$$

where V_i is the permutation module for $L[P]$ corresponding to the orbit $\mathcal{O}_i$.

Let $h \in P$ and suppose that h fixes some v_j with $j \neq 0$. This means that h normalizes P_j so that $\langle P_j, h \rangle$ is a p-subgroup of G. Since P_j is a Sylow p-subgroup we conclude that $h \in P_j$ so $h \in P \cap P_j = \langle 1 \rangle$. This shows that for all orbits $\mathcal{O}_i$ with $i \neq 0$, P acts regularly on $\mathcal{O}_i$ so that V_i is essentially the regular representation of $L[P]$. In fact once we choose some $v_j \in \mathcal{O}_i$, the permutation basis for V_i is $\{v_j h | h \in P\}$. On the other hand V_0 is just 1-dimensional with basis $\{v_0\}$.

Now

$$\bar{V} = \{\sum a_k v_k | \sum a_k = 0\}$$

is certainly an $L[G]$-submodule of V and since P is not normal in G, $\bar{V} \neq 0$. Thus we can choose $0 < W \subseteq \bar{V}$ to be an irreducible submodule. Let $w \in W$, $w \neq 0$. Then by multiplying w by a suitable element of G if necessary we may assume that $w = \sum a_k v_k$ with $a_0 \neq 0$. This implies that $\sum_{k \neq 0} a_k \neq 0$ so $\sum a_k \neq 0$ over some orbit $\mathcal{O}_i$ with $i \neq 0$. For this i, the natural $L[P]$-projection $\tau_i : V \to V_i$ sends W to an $L[P]$-submodule of $V_i \cong L[P]$ which is not contained in the augmentation ideal. But P is a p-group and char $L = p$ so the augmentation ideal of $L[P]$ is the unique maximal submodule (that is, right ideal) of $L[P]$. Thus $\tau_i(W) = V_i \cong L[P]$ and

$$\dim_L W \geqq \dim_L \tau_i(W) = \dim_L V_i = |P|.$$

This completes the proof.

For the remainder of this section we assume that $K[G]$ satisfies (***).

LEMMA 3.2. *Let H be a finite normal abelian subgroup of G of odd order with $\langle 1 \rangle < H \subseteq G'$. Then $[G : \mathbf{C}(H)] = 2$ and $G/\mathbf{C}(H)$ acts in a dihedral manner on H.*

Proof. Since H is a finite group of odd order and char $K = 2$, Lemma 16.9 of [**2**] yields $JK[G] \cap K[H] \subseteq JK[H] = 0$. Thus if $\alpha \in K[H]$, $\alpha \neq 0$ then by Lemma 2.2 (ii) there exists a homomorphism $\sigma : K[G] \to M_2(E)$ where E is some algebraically closed extension of K with $\sigma(\alpha) \neq 0$. Now suppose

$h_1, h_2 \in H - \{1\}$ and set $\alpha = (1 - h_1)(1 - h_2)$. Since $h_1, h_2 \neq 1$ and $|H|$ is odd, it is trivial to see that $\alpha \neq 0$. Thus there exists a σ as above with $\sigma(\alpha) \neq 0$ and hence $\sigma(h_1) \neq 1$ and $\sigma(h_2) \neq 1$.

We first show that $\mathbf{C}(H) \neq G$. Since $H \neq \langle 1 \rangle$ choose $k \in H$, $k \neq 1$. By the above there exists $\sigma : K[G] \to M_2(E)$ with $\sigma(k) \neq 1$. Now $\sigma(k) \in M_2(E)$ has determinant 1 since $k \in H \subseteq G'$ so by taking similar matrices if necessary we can assume that $\sigma(k)$ is the diagonal matrix $\sigma(k) = \operatorname{diag}(a, a^{-1})$ for some $a \in E$, $a \neq 1$. Note that $a \neq a^{-1}$ since $a^2 = 1$ and char $E = 2$ imply that $a = 1$. Suppose $\mathbf{C}(H) = G$. Then $\sigma(G)$ centralizes $\sigma(k)$ and this must consist of diagonal matrices. This implies that $\sigma(G)$ is commutative so $\sigma(G') = \sigma(G)' = \langle 1 \rangle$, a contradiction since $k \in G'$ and $\sigma(k) \neq 1$.

Now let $x \in G - \mathbf{C}(H)$ and fix $k \in H$ not centralized by x. Then $k^x k^{-1}$ is a nonidentity element of H. We show now that $h^x = h^{-1}$ for all $h \in H$. If this is not the case choose $h \in H$ with $h^x h \neq 1$. By our above remarks there exists a homomorphism $\sigma : K[G] \to M_2(E)$ with $\sigma(k^x k^{-1}) \neq 1$ and $\sigma(h^x h) \neq 1$. Since

$$1 \neq \sigma(k^x k^{-1}) = \sigma(k)^{\sigma(x)} \sigma(k)^{-1}$$

we conclude that $\sigma(k) \neq 1$. As above we may assume that $\sigma(k) = \operatorname{diag}(a, a^{-1})$ with $a \neq 1$.

Now H is abelian so $\sigma(H)$ centralizes $\sigma(k)$ and hence $\sigma(H)$ consists of diagonal matrices with determinant 1. Since $\sigma(k^x) \neq \sigma(k)$ we then have $\sigma(k^x) = \operatorname{diag}(b, b^{-1})$ with $b \neq a$. Suppose

$$\sigma(x) = \begin{bmatrix} r & s \\ t & u \end{bmatrix}.$$

From $\sigma(x)\sigma(k^x) = \sigma(k)\sigma(x)$ we have

$$\begin{bmatrix} r & s \\ t & u \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \begin{bmatrix} r & s \\ t & u \end{bmatrix}$$

so $a \neq b$ yields easily $r = u = 0$, and

$$\sigma(x) = \begin{bmatrix} 0 & s \\ t & 0 \end{bmatrix}.$$

Finally, $\sigma(h) = \operatorname{diag}(c, c^{-1})$ for some $c \in E$, so

$$\begin{aligned} \sigma(h^x) = \sigma(x)^{-1}\sigma(h)\sigma(x) &= \begin{bmatrix} 0 & t^{-1} \\ s^{-1} & 0 \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix} \begin{bmatrix} 0 & s \\ t & 0 \end{bmatrix} \\ &= \begin{bmatrix} c^{-1} & 0 \\ 0 & c \end{bmatrix} = \sigma(h)^{-1}. \end{aligned}$$

Thus $\sigma(h^x h) = 1$, a contradiction.

We have therefore shown that every element of $x \in G - \mathbf{C}(H)$ acts in a dihedral manner on H. This clearly yields $[G : \mathbf{C}(H)] = 2$.

We now obtain all needed information when G is finite. For finite groups G

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we let $\mathbf{O}_2(G)$ denote their maximal normal 2-subgroup and we let $\mathbf{O}_{2'}(G)$ denote their maximal normal subgroup of odd order.

LEMMA 3.3. *Let G be a finite group.*
(i) *G is solvable.*
(ii) *If $|G|$ is odd, then G is abelian.*
(iii) *If $\mathbf{O}_2(G) = \langle 1 \rangle$ then $\mathbf{C}_G(\mathbf{O}_{2'}(G)) = \mathbf{O}_{2'}(G)$.*

Proof. (i) Since $(***)$ is inherited by subgroups and quotient groups we need only show that G cannot be a finite nonabelian simple group. Thus by way of contradiction assume that G is such a group and let P be a Sylow 2-subgroup of G. There are three cases to consider accordingly as $|P| = 1, 2$ or $\geqq 4$. Let L be the algebraic closure of K. Then $L[G]$ also satisfies $(***)$. If σ is an irreducible representation of $L[G]$, then since G is finite $\sigma(L[G]) = M_n(L)$ for some n. By Lemma 1.2 (ii) we have only $n = 1$ or 2. Moreover since $G = G'$, $\sigma(G)$ is contained in the matrices of determinant 1. Thus if $n = 1$ then for all $g \in G$, $\sigma(g) = 1$. Now G is not a 2-group so $JL[G]$ is not the augmentation ideal. Thus for some $g \in G$, $g \neq 1$ there exists σ with $\sigma(g) \neq 1$. Since G is simple this implies that σ is an isomorphism on G so G is isomorphic to an irreducible subgroup of $\mathrm{SL}_2(L)$, the group of 2×2 matrices over L with determinant 1.

Suppose $|P| = 1$ so that G has odd order. Then by Maschke's theorem [**2**, Theorem 15.3], $L[G]$ is completely reducible. Now $G \subseteq \mathrm{SL}_2(L) \subseteq M_2(L)$ so G acts by conjugation on the 4-dimensional L-space $M_2(L)$ and $T_2(L)$ is an $L[G]$-submodule of codimension 1. Thus by complete reducibility $M_2(L) = T_2(L) + W$ where W is a 1-dimensional space acted upon by G by conjugation. But G acts trivially on all 1-dimensional spaces so $W \subseteq M_2(L)$ is centralized by G. Since G is an irreducible subgroup of $\mathrm{SL}_2(L)$ this implies that W consists of scalar matrices and hence since char $L = 2$ we have $W \subseteq T_2(L)$, a contradiction.

Suppose $|P| = 2$. Then as is well-known G has a normal subgroup of index 2, a contradiction. The argument here is to embed G in the symmetric group on its elements and then to observe that if $x \in P$ has order 2 then x is an odd permutation. Thus G intersected with the alternating group is a proper normal subgroup of G.

Finally let $|P| \geqq 4$ and choose $x \in \mathbf{Z}(P)$ with x of order 2. Then by taking similar matrices if necessary we can assume that

$$x = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in \mathrm{SL}_2(L).$$

Then we see easily that $\mathbf{C}(x)$, the centralizer of x in $\mathrm{SL}_2(L)$, is given by

$$\mathbf{C}(x) = \left\{ \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \middle| a \in L \right\}$$

and this is an elementary abelian 2-group. Since $x \in \mathbf{Z}(P)$ this implies that P is elementary abelian and then that $P = \mathbf{C}_G(y)$ for any $y \in P$, $y \neq 1$. Now P

is not normal in G. If $y \in P \cap P^g$ and $y \neq 1$ then $P = \mathbf{C}_G(y) = P^g$. Thus by Lemma 3.1 with $p = 2$, $L[G]$ has an irreducible representation of degree at least $|P| \geqq 4$, a contradiction. Thus G is solvable.

(ii) Let $|G|$ be odd. By the above G is solvable. If G is not abelian we can choose H to be the last nonidentity term in the derived series for G and have $H \subseteq G'$. By Lemma 3.2 since H is abelian we have $[G : \mathbf{C}(H)] = 2$, a contradiction since G has odd order.

(iii) Now $H = \mathbf{O}_{2'}(G)$ is abelian by (ii) so $\mathbf{C}(H) \supseteq H$. Suppose $\mathbf{C}(H) > H$. Since $H = \mathbf{O}_{2'}(G)$ we see that $\mathbf{O}_{2'}(\mathbf{C}(H)/H) = \langle 1 \rangle$ so $\mathbf{O}_2(\mathbf{C}(H)/H) \neq \langle 1 \rangle$ since $\mathbf{C}(H)/H$ is solvable. Let $G \supseteq N > H$ with $N/H = \mathbf{O}_2(\mathbf{C}(H)/H)$. If P is a Sylow 2-subgroup of N, then $N = HP$ and since H is central in N, $N = H \times P$. Thus $P = \mathbf{O}_2(N) \neq \langle 1 \rangle$. Now $N \lhd G$ so $\mathbf{O}_2(N) \lhd G$ and $\mathbf{O}_2(G) \neq \langle 1 \rangle$, a contradiction. Thus $\mathbf{C}(H) = H$ and the lemma is proved.

Finally we obtain

LEMMA 3.4. *G has a 2-abelian subgroup of index $\leqq 2$.*

Proof. By Lemma 2.1 (ii), $[G : \Delta(G)] = 1$ or 2 and $\Delta(G)'$ is finite. We assume first that $\mathbf{O}_2(\Delta(G)') = \langle 1 \rangle$ and show that G has an abelian subgroup of index $\leqq 2$. This is clear if $\Delta(G)' = \langle 1 \rangle$ so suppose $\Delta(G)' \neq \langle 1 \rangle$. Since this group is solvable by Lemma 3.3 (i) $H = \mathbf{O}_{2'}(\Delta(G)') \neq \langle 1 \rangle$ and then by Lemma 3.3 (ii) H is a finite normal abelian subgroup of G of odd order with $H \subseteq \Delta(G)' \subseteq G'$. We conclude from Lemma 3.2 that $[G : \mathbf{C}_G(H)] = 2$ and $[\Delta(G) : \mathbf{C}_{\Delta(G)}(H)] = 2$. Note that $\Delta(G)/\mathbf{C}_{\Delta(G)}(H)$ has order 2 so it is abelian and $\Delta(G)' \subseteq \mathbf{C}_{\Delta(G)}(H)$. Thus by Lemma 3.3 (iii) since $\mathbf{O}_2(\Delta(G)') = \langle 1 \rangle$ we see that $H = \Delta(G)'$.

Let $A = \mathbf{C}_{\Delta(G)}(H)$ so that $[\Delta(G) : A] = 2$. Then $A' \subseteq \Delta(G)' = H$ so A' is central and of odd order in A. By Lemma 3.2, $A' = \langle 1 \rangle$ and A is abelian. Since $[G : \Delta(G)] = 1$ or 2 and $[\Delta(G) : A] = 2$ we see that A is a normal abelian subgroup of G of index 2 or 4. We assume that $[G : A] = 4$ and derive a contradiction.

Let $\rho : K[G] \to M_4(K[A])$ be as in section 1 and let $R = K[A] \cdot \rho(K[G])$. Then R is also Lie solvable. Since $[G : \Delta(G)] = 2$ and $[\Delta(G) : A] = 2$, the coset representatives $x_1 = 1, x_2, x_3, x_4$ can be so labeled that $x_2 \in \Delta(G) - A$ and $x_3, x_4 \in G - \Delta(G)$. By Lemma 1.4 (i), S_3 and S_4 do not annihilate any nonzero element of $K[A]$. Furthermore, x_2 acts in a dihedral manner on the group H of odd order so $(A, x_2) \supseteq (H, x_2) = H$. Thus, Lemma 1.4 (ii) implies that S_2 is not nilpotent and therefore by all of the above $S = S_2S_3S_4$ is not nilpotent. Now $R \supseteq M_4(S)$ by Lemma 1.5 and since R is Lie solvable and S is not nilpotent Lemma 1.2 (ii) yields a contradiction. Thus G has an abelian subgroup of index at most 2.

Finally we consider the general case. Let $W = \mathbf{O}_2(\Delta(G)')$. Then W is a finite normal subgroup of G so if $\bar{G} = G/W$ then $\Delta(\bar{G}) = \Delta(G)/W$, $\Delta(\bar{G})' = \Delta(G)'/W$ and thus $\mathbf{O}_2(\Delta(\bar{G})') = \langle 1 \rangle$. By the above $\bar{G}$ has an abelian subgroup

$\bar{A}$ with $[\bar{G} : \bar{A}] \leqq 2$. If A is the complete inverse image of $\bar{A}$ in G, then $[G : A] \leqq 2$ and $A' \subseteq W$ so A is 2-abelian.

4. Conclusion. It is now a simple matter to prove our main theorem.

Proof of the Theorem. Suppose $K[G]$ is Lie nilpotent or Lie solvable. Then by Lemmas 2.3 and 3.4 we see that G has the appropriate structure. We need only show that if G has the appropriate properties then so does $K[G]$.

(i) Suppose G is p-abelian and nilpotent. If char $K = 0$, then G is abelian so $K[G]$ is certainly Lie nilpotent. Thus let char $K = p > 0$. We show by induction on $|G'|$ that $K[G]$ is Lie nilpotent. The result is clear if $G' = \langle 1 \rangle$. Suppose $G' \neq \langle 1 \rangle$. Then since G is nilpotent, $G' \cap \mathbf{Z}(G) \neq \langle 1 \rangle$ so choose $x \in G' \cap \mathbf{Z}(G)$ of order p. Let $\bar{G} = G/\langle x \rangle$ and consider the natural map $\tau : K[G] \rightarrow K[\bar{G}]$. By induction $K[\bar{G}]$ is Lie nilpotent so say $\gamma^n K[\bar{G}] = 0$. This implies that $\gamma^n K[G]$ is contained in $(1 - x)K[G]$, the kernel of τ. Since $(1 - x)$ is central this yields easily $\gamma^{2n-1}K[G] \subseteq (1 - x)^2 K[G]$ and continuing in this manner $\gamma^{pn-1}K[G] \subseteq (1 - x)^p K[G]$. But $(1 - x)^p = 1 - x^p = 0$ so $\gamma^{pn-1}K[G] = 0$ and $K[G]$ is Lie nilpotent.

(ii) Again if char $K = 0$ then G is abelian and certainly $K[G]$ is Lie solvable. Let char $K = p > 0$ and consider the natural map $\tau : K[G] \rightarrow K[G/G']$. Then the image of τ is commutative and the kernel of τ is nilpotent. Since $\delta^1 K[G]$ is nilpotent, it is Lie solvable and hence so is $K[G]$.

(iii) Finally let char $K = 2$ and suppose first that G has an abelian subgroup A of index 2. Then $\rho : K[G] \rightarrow M_2(K[A])$ is a monomorphism by Lemma 1.3 and $M_2(K[A])$ is Lie solvable by Lemma 1.2. Therefore $K[G]$ is Lie solvable. Now suppose G has a 2-abelian subgroup A of index 2 and let τ be the natural map $\tau : K[G] \rightarrow K[G/A']$. Then again the image of τ is Lie solvable and the kernel is nilpotent so $K[G]$ is Lie solvable. This completes the proof.

We remark that the degree of Lie nilpotence or Lie solvability bounds the size of the commutator subgroup of the p-abelian group by the main result of [**3**]. Presumably in this case a much sharper bound can be obtained.

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