A NOTE ON COMMUTATIVE SEMIGROUPS

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1. Introduction

In 1962, O. Frink [2] showed that in a pseudo-complemented semilattice $\langle P; \wedge, *, 0\rangle$, the closed elements form a Boolean algebra. We shall consider an extension of this result to arbitrary commutative semigroups with zero.

Let $S = \langle S; \cdot, 0\rangle$ be a commutative semigroup with zero. For a subset $A \subseteqq S$ we define $A^* = \{s \in S : sA \subseteqq r(S)\}$, where $r(S)$ denotes the set of all nilpotents of S (the radical). A^* is called the r-annihilator of A. If $A = \{a\}$ we write $\{a\}^* = (a)^*$ since $\{a\}^*$ coincides with $(a)^*$, where (a) denotes the principal ideal $(a) = aS$ generated by a.

A well-known congruence definable in semigroups with zero is the congruence R defined by

$$\langle a, b\rangle \in R \equiv_{Df} (a)^* = (b)^*.$$

Our main result is

THEOREM 1. *In a commutative semigroup* $S = \langle S; \cdot, 0\rangle$, S/R *is a Boolean algebra if and only if for all* $x \in S$ $(x)^{**} = (x')^*$ *for some* $x' \in S$.

We also consider when S/R is a Boolean algebra with a higher degree of (lattice) completeness, and determine the normal completion of S/R in a special case.

2. Proof of Theorem 1

We need some results on r-annihilators - the first result being straightforward has its proof omitted.

LEMMA 2.1. *For subsets* A *and* B *of* S *we have*

(i) $A^* = \bigcap_{a\in A} (a)^*$,

(ii) $A \subseteqq B$ *implies* $A^* \supseteqq B^*$ *and thus,* $A^{**} \subseteqq B^{**}$,

(iii) $A \subseteqq A^{**}$,

(iv) $A^* \cap A^{**} = r(S)$ *and* $A^{***} = A^*$.

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LEMMA 2.2. *For any two ideals I and J of S*

$$(I \cap J)^{**} = I^{**} \cap J^{**}$$

PROOF. Since $I \cap J \subseteqq I$ and $I \cap J \subseteqq J$ we have, by 2.1 (ii) $(I \cap J)^{**} \subseteqq I^{**}$ and $(I \cap J)^{**} \subseteqq J^{**}$ and so $(I \cap J)^{**} \subseteqq I^{**} \cap J^{**}$.

For the reverse inclusion, let $s \in I^{**} \cap J^{**}$ and $t \in (I \cap J)^*$, $i \in I$ and $j \in J$. Clearly $ij \in I \cap J$ and so $tij \in r(S)$ or $ti \in (j)^*$ for any $j \in J$. Thus $ti \in \bigcap_{j \in J}(j)^* = J^*$. This implies $sti \in r(S)$ since $s \in J^{**}$ and thus $st \in (i)^*$ for any $i \in I$.

We then have $st \in \bigcap_{i \in I} (i)^* = I^*$ and so $st \in I^{**} \cap I^* = r(S)$, or $s \in (t)^*$ $\forall t \in (I \cap J)^*$, which gives us the result $s \in (I \cap J)^{**}$ or

$$I^{**} \cap J^{**} \subseteqq (I \cap J)^{**}.$$

The reverse inclusion is now proved and the result follows.

COROLLARY. $(ab)^{**} = (a)^{**} \cap (b)^{**}$.

Rather than work with S/R. which is a semi-lattice by a result of R.S. Pierce [5], we prefer to consider the isomorphic semilattice $S^{**} = \langle S^{**}; \cap, (0)^{**} \rangle$ where $S^{**} = \{(a)^{**}: a \in S\}$.

LEMMA 2.3. $S/R \cong S^{**}$.

PROOF. If we let ρ denote the natural homomorphism existing between S and S/R we may define a map $\phi : S/R \to S^{**}$ by $a\rho\phi = (a)^{**}$. ϕ is well-defined, for if $a\rho = b\rho$, $(a)^* = (b)^*$ and hence $(a)^{**} = (b)^{**}$. This argument reverses to show ρ is an injection, and ϕ is obviously surjective. The corollary above shows that ϕ is a semigroup homomorphism and so the result follows.

We now proceed to the main part of our proof using the postulate set for Boolean algebras of O. Frink [1]. The postulates are in terms of semi-lattice meet ($\wedge$), and complement ($'$).

P1. $a \wedge b = b \wedge a$

P2. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

P3. $a \wedge a = a$

P4. $a \wedge b' = 0 \Leftrightarrow a \wedge b = a$.

Clearly P1, P2 and P3 are postulates for a semi-lattice, and P4 is the only postulate which needs considering in detail.

LEMMA 2.4. *If the commutative semigroup* $S = \langle S; \cdot, 0 \rangle$ *satisfies Condition* (*): *For any* $x \in S$, $(x)^{**} = (x')^*$ *for some* $x' \in S$, *then* $S^{**} = \langle S^{**}; \cap, r(S) \rangle$ *is a Boolean algebra.*

PROOF. In S^{**} the semi-lattice operation is set intersection $\cap$, the zero $r(S) = (0)^{**}$ and we define the complement of $(a)^{**} \in S^{**}$ by $(a)^{**'} = (a')^{**}$

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where a' is defined by Condition (*). Lemma 2.3 tells us that S^{**} is a semi-lattice so we need only consider P4. Suppose $(a)^{**} \cap (b)^{**\prime} = (0)^{**}$. Then $(a)^{**} \cap (b')^{**} = (ab')^{**} = (0)^{**}$ and hence $ab' \in r(S)$ by 2.1 (i). This implies $b' \in (a)^*$ and so $(b') \subseteqq (a)^*$, giving $(b')^* = (b)^{**} \supseteqq (a)^{**}$. Thus $(a)^{**} \cap (b)^{**} = (a)^{**}$ and the left-right implication of P4 is proved.

Next, suppose $(a)^{**} \cap (b)^{**} = (a)^{**}$. Then $(ab)^{**} = (a)^{**}$ and $(a)^{**} \cap (b)^{**\prime} = (a)^{**} \cap (b')^{**} = (ab)^{**} \cap (b')^{**} = (abb')^{**}$. Now $bb' \in (b)^{**} \cap (b')^{**} = r(S)$ and so $(abb')^{**} \subseteqq (bb')^{**} = r(S)$. Thus $(a)^{**} \cap (b)^{**} = r(S)$ and the right-left implication is proved.

Lemma 2.5. *Suppose $S = \langle S; \cdot, 0\rangle$ is a commutative semi-group with zero, and that $S^{**} = \langle S^{**}; \cap, (0)^{**}\rangle$ is a Boolean algebra. Then S satisfies Condition (*): For any $x \in S$, $(x)^{**} = (x')^*$ for some $x' \in S$.*

Proof. Since S^{**} is a Boolean algebra, P4 is satisfied; i.e. for any $(b)^{**}$ there is a $(b)^{**\prime}$ such that

$$(a)^{**} \cap (b)^{**\prime} = (0)^{**} \Leftrightarrow (a)^{**} \cap (b)^{**} = (a)^{**}$$

Defining $(b')^{**}$ by $(b')^{**} = (b)^{**\prime}$ we show that $(b)^* = (b')^{**}$ or, equivalently (2.1(iv)) $(b)^{**} = (b')^*$. Put $a = b$ in the above equivalence and, since the right side is clearly true, we deduce that $(b)^{**} \cap (b)^{**\prime} = (b)^{**} \cap (b')^{**} = (bb')^{**} = (0)^{**}$. Thus, by 2.1 (i) $bb' \in r(S)$ and so $b' \in (b)^*$, giving $(b') \subseteqq (b)^*$ or $(b')^* \supseteqq (b)^{**}$. Now take $a \in (b')^*$ and put it in the left side of the equivalence. For such an a, we see that

$$(a)^{**} \cap (b)^{**\prime} = (a)^{**} \cap (b')^{**} = (ab')^{**} = (0)^{**},$$

and so we deduce that $(a)^{**} \cap (b)^{**} = (a)^{**}$. This means, by 2.1 (ii) that $a \in (b)^{**}$ and we have thus proved $(b')^* \subseteqq (b)^{**}$. Combining this with the reverse inclusion obtained above gives us $(b')^* = (b)^{**}$ and the Lemma follows.

Theorem 1. *Let $S = \langle S; \cdot, 0\rangle$ be a commutative semi-group with zero. Then S/R is a Boolean algebra if and only if Condition (*) holds in S.*

Proof. Lemmas 2.3, 2.4, and 2.5.

Remark. A commutative semigroup with zero is called a Baer semigroup if for each $s \in S$ there exists an idempotent $e \in S$ such that

$$\{t : st = 0\} = Se.$$

J. Kist [4] has shown that in a commutative Baer semigroup $r(S) = (0)$ and so for any $s \in S$, $(s)^* = Se$ for some idempotent $e \in S$. This enables us to give a new proof of Theorem 7.3 of J. Kist [4].

Corollary 1. *If $S = \langle S; \cdot, 0\rangle$ is a commutative Baer semigroup, then S/R is a Boolean algebra.*

Proof. For $s \in S$, $(s)^* = Se$. We then show $(e)^* = (s)^{**}$ and so we may take $s' = e$ in Condition (*). Observe that $(s)^{**} = (Se)^*$. Now if $tSe = (0)$, then $tee = te^2 = te = 0$ and so $t \in (e)^*$. Further, if $t \in (e)^*$, then $tse = 0$ for $s \in S$, and so $t \in (Se)^*$. Thus $(Se)^* = (e)^*$ and the Corollary is proved.

Corollary 2. (*O. Frink* [2]) *If* $S = \langle S; \wedge, *, 0\rangle$ *is a pseudo-complemented semi-lattice, then* S^{**} *is a Boolean algebra.*

Proof. A pseudo-complemented semi-lattice is a commutative Baer semigroup and so the result follows from Lemma 2.3 and Corollary 1 above.

3. Completeness of S/R

In this section we generalise Condition (*) to the following ($\mathfrak{m}$ denotes an arbitrary cardinal)

Condition $\mathfrak{m}(*)$. For any $A \subseteqq S$ with $|A| \leqq \mathfrak{m}$, $A^{**} = (a')^*$ for some $a' \in S$.

Remark. Condition $\mathfrak{m}(*)$ implies Condition $\mathfrak{n}(*)$ for $\mathfrak{n}$ a cardinal, $\mathfrak{n} \leqq \mathfrak{m}$.

Theorem 2. *Let* $S = \langle S; \cdot, 0\rangle$ *be a commutative semi-group with zero. Then* $S^{**} = \langle S^{**}; \cap, (0)^{**}\rangle$ *is an* $\mathfrak{m}$*-complete Boolean algebra if and only if* S *satisfies Condition* $\mathfrak{m}(*)$

Proof. Assume S satisfies Condition $\mathfrak{m}(*)$ and take $\{a_\gamma : \gamma \in \Gamma\} \subseteqq S$ with $|\Gamma| \leqq \mathfrak{m}$. Then $\bigcap_{\gamma \in \Gamma} (a_\gamma)^{**} = \bigcap_{\gamma \in \Gamma} (a_\gamma)^* = A^* = (a')^{**}$ where $A = \{a'_\gamma : \gamma \in \Gamma\}$ and a'_γ, a' exist because of Condition $\mathfrak{m}(*)$. Thus S^{**} is closed under intersections of $\mathfrak{m}$ elements, and by Condition (*), it is complemented.

This implies $S^{**} = S/R$ is an $\mathfrak{m}$-complete Bolean algebra, and the first half of our proof is complete.

Next we assume S^{**} is an $\mathfrak{m}$-complete Boolean algebra. Then S^{**} is closed under intersections of $\mathfrak{m}$ elements, and satisfies Condition (*), by Theorem 1. Take $A = \{a_\gamma : \gamma \in \Gamma\}$, $|\Gamma| \leqq \mathfrak{m}$.

$$A^* = \bigcap_{\gamma \in \Gamma} (a_\gamma)^* \text{ by 2.1 (i)}$$

and so

$$= \bigcap_{\gamma \in \Gamma} (a'_\gamma)^{**} \text{ by Condition (*)}$$

$$= (a')^{**} \text{ since } S^{**} \text{ is } \mathfrak{m}\text{-complete.}$$

Thus $A^{**} = (a')^*$ and our theorem is proved.

Corollary. S^{**} *is a complete Boolean algebra if and only if for* $A \subseteqq S$, $A^{**} = (a')^*$ *for some* $a' \in S$.

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4. The normal completion of S/R

We next consider the normal completion of $S/R = S^{**}$. Our construction applies to the class of commutative semi-groups without radical for which the mapping $a \to (a)$ is injective. A wide class of semigroups satisfying this condition is the class of semi-lattices. The result is in fact mainly of interest in the case of semi-lattices. For this reason we shall formulate our results for semi-lattices, although the extension to the class of semi-groups mentioned above is immediate.

LEMMA 4.1. *If* $E = \langle E; \wedge, 0\rangle$ *is a semi-lattice with zero, then the semi-lattice of ideals,* $\mathscr{I}(E) = \langle I(E); \cap, (0)\rangle$ *is a pseudo-complemented semi-lattice. The pseudo-complement of* $J \in I(E)$ *is simply* J^*. *Further,* $\mathscr{I}(E)^{**}$ *is a complete Boolean algebra.*

PROOF. Only the last statement really needs checking. Suppose $\mathscr{A} = \{I_\alpha : \alpha \in A\}$ is an arbitrary family of ideals of E. Then

$$\mathscr{A}^{**} = (\bigcup_\alpha I_\alpha)^{**} = (\bigcap_\alpha I_\alpha^*)^* = I^*$$

where $I = \bigcap_\alpha I_\alpha^*$.

We see that the conditions of the preceding Corollary are satisfied and so the result follows.

Next we note, by the comments above, that there is a faithful copy of E embedded in $\mathscr{I}(E)$. More important is that this implies E^{**} is a subsemi-lattice of $\mathscr{I}(E)^{**}$, since $\{a\}^{**} = (a)^{**}$. A subset Q of a semi-lattice with zero $\langle P; \wedge, 0\rangle$ is said to be dense if for any $p \in P$, $p \neq 0$, there is $q \in Q$ with $0 < q \leqq p$.

LEMMA 4.2. E^{**} *is a dense subsemi-lattice of* $\mathscr{I}(E)^{**}$.

PROOF. We must show that for any $I^{**} \in \mathscr{I}(E)^{**}$ such that $I^{**} \neq (0)$ there is $(a)^{**} \in E^{**}$, $(0) \subset (a)^{**} \subseteqq I^{**}$. This follows readily since $I^{**} \neq (0)$ implies $(i)^{**} \neq (0)$ for some $i \in I$. Clearly then $(0) \subset (i)^{**} \subseteqq I^{**}$ and our result is proved.

An immediate consequence of 4.2 is

THEOREM 3. *Let* $E = \langle E; \wedge, 0\rangle$ *be a semi-lattice with zero. If* E^{**} *is a Boolean algebra, then* $\mathscr{I}(E)^{**}$ *is the normal completion of* E^{**}.

PROOF. E^{**} as a Boolean algebra is a dense subsemi-lattice of $\mathscr{I}(E)^{**}$. It is well known that under these conditions $\mathscr{I}(E)^{**}$ is the normal completion of E^{**}. See R. Sikorski [6] p. 153.

5. Concluding remarks

In this note our method of proof of the main theorem follows that of O. Frink [2] using the postulates of O. Frink [1]. In the author's thesis these results followed (in the case of distributive lattices and semi-lattices) from theorems regarding the space of minimal prime ideals. Condition (*) was introduced by M. Henrikson and M. Jerison [3] and was related to the congruence R via distributive lattices.

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