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LIMIT POINT AND LIMIT CIRCLE CRITERIA FOR A CLASS OF SINGULAR SYMMETRIC DIFFERENTIAL OPERATORS

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Introduction. For certain classes of singular symmetric differential operators L of order $2n$, this paper considers the problem of determining sufficient conditions for L to be of limit point type or of limit circle type. The operator discussed here is defined by

$$(0.1) \qquad L(y) = y^{(2n)} + Py \quad \text{on} \quad a \leqq t < \infty,$$

where P is a symmetric, $k \times k$ matrix of real measurable functions which are Lebesgue integrable on compact subintervals of (a, ∞) and y is a k-vector.

Let H be the Hilbert space of complex vector-valued functions f: $[a, \infty) \to \mathbf{C}^k$ such that f is Lebesgue measurable on $[a, \infty)$ and $\int_a^\infty f^*(s)f(s)ds < \infty$. In Sections 15–17 of [**11**] the basic theory is developed for the scalar case. The arguments for the vector case follow exactly as in the scalar problem except for Lemmas 1 and 2 of Section 17. There a slight but obvious modification is needed in order to carry through the argument. The emphasis in this paper is on the theory in Section 17 of Naimark. The details of the proofs for the modification are found in [**1**].

The classical arguments in [**11**] show that the number m of linearly independent solutions of $L(y) = \lambda y$ in H is the same for all non-real λ, and satisfies $nk \leqq m \leqq 2nk$. It is well known that any value of m between nk and $2nk$ can occur. For examples, see [**7**]. References to other examples are given in Section 17.5 of [**11**]. Here we are concerned only with the problem of finding conditions which imply that $m = nk$ (limit point type) or $m = 2nk$ (limit circle type).

Everitt [**4**; **5**], Everitt and Chandhuri [**6**], Walker [**12**; **13**], and others have given effective limit point criteria for scalar fourth-order operators and recently Hinton [**8**] gave sufficient conditions on growth rates of the coefficients, for the general even-order formally self adjoint scalar operator to be of limit point type. In [**10**] Lidskii studies the second-order version of (0.1) above and gives sufficient conditions for it to be of limit point type or of limit circle type.

In Sections 1 and 2 of this paper, a slight generalization of Hinton's techniques for the scalar equation are applied to the operator in (0.1) in order to generalize the first result of Lidskii. In Section 3, the second result of Lidskii is generalized.

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In a later paper I will give a thorough discussion of this operator for the second-order case.

1. Inequalities for a system of equations. Consider the system of differential equations

(1.1) $X' = wBX$

where $X = [x_1^T, x_2^T, \ldots, x_{2n}^T]^T$ is a column vector and each x_i is a k-vector, w is a positive continuous scalar valued function on $[a, \infty)$, and

$$B = \begin{bmatrix} B_{11} & \cdot \quad \cdot \quad \cdot & B_{1,2n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ B_{2n,1} & \cdot \quad \cdot \quad \cdot & B_{2n,2n} \end{bmatrix}$$

is a $2nk \times 2nk$ matrix where the B_{ij}'s are $k \times k$ block matrices of measurable, locally integrable, complex valued functions on $[a, \infty)$ satisfying

$$B_{ij} = \begin{Bmatrix} 0_k & \text{if } j > i+1 \\ \pm I_k & \text{if } j = i+1 \end{Bmatrix}.$$

THEOREM 1.1. *Suppose X is a solution of* (1.1) *and that on* $[a, \infty)|y^*B_{ij}x| \leqq K|y^*x|$ *for some constant K, $i \leqq n$, and all k-vectors x and y. Let*

$$m_i = m_i(t) = \max\left\{1, \int_a^t w(x_i^*x_i)ds\right\} \quad (i = 1, \ldots, 2n).$$

Suppose $m_1(\infty) < \infty$. Then for $i = 1, \ldots, n$ as $t \to \infty$

(1.2) $m_i = O(m_{i+1}^{(i-1)/i})$ *and* $x_i^*x_i = O(m_{i+1}^{(2i-1)/2i})$.

The proof is omitted here since the calculations are very similar to those for the scalar case (see [**8**]).

Let ρ be a positive function on $[a, \infty)$ such that $\rho \in C^n[a, \infty)$. Consider the following conditions:

(1.3) For some $K > 0$, $(-1)^n\rho^{4n}P \geqq -KI$ (*i.e.*, $(-1)^n\rho^{4n}P + KI$ is positive semidefinite),

(1.4) $\rho\rho' = O(1)$ as $t \to \infty$,

(1.5) $\int_a^\infty \rho^{4n-2} = \infty$,

(1.6) For $j = 1, \ldots, n$ $[\rho^{4n-2}]^{(j)} = O(\rho^{4n-2-2j})$ and $[\rho^{4n}]^{(j)} = O(\rho^{4n-2j})$.

For (0.1) definite quasiderivatives $y^{[i]}$ by $y^{[i]} = y^{(i)}$ for $i = 1, \ldots, n$ and $y^{[n+i]} = -(y^{[n+i-1]})'$ for $i = 1, \ldots, n-1$ with $y^{[2n]} = L(y)$.

Then the equation $L(y) = \lambda y$ has the vector matrix formulation

(1.7) $Y' = AY$

where

$$Y = \begin{bmatrix} y^{[0]} \\ \cdot \\ \cdot \\ \cdot \\ y^{[2n-1]} \end{bmatrix}$$

and

$$A = \begin{bmatrix}
0_k I_k & \cdot\ \cdot\ \cdot & 0_k & 0_k & \cdot\ \cdot\ \cdot & 0_k \\
\ddots & & \cdot & \cdot & & \cdot \\
& \ddots & \cdot & & & \\
& & \cdot & \cdot & & \cdot \\
& & \ddots & & & \\
& & I_k & \cdot & & \cdot \\
& & 0_k & I_k & & 0_k \\
0_k & \cdot\ \cdot\ \cdot & 0_k & 0_k & -I_k \ \ldots & 0_k \\
\cdot & & \cdot & \cdot & \cdot & \cdot \\
\cdot & & \cdot & \cdot & \ddots & \cdot \\
\cdot & & \cdot & \cdot & & -I_k \\
P - \lambda I & \ldots & 0_k & 0_k & \cdot\ \cdot\ \cdot & 0_k
\end{bmatrix}$$

Transform equation (1.7) by the transformation $X = MY$ where $M =$ diagonal $[\rho I_k, \rho^3 I_k, \ldots, \rho^{4n-1} I_k]$.

Then

$$X = MY = \begin{bmatrix} \rho I_k & & & & \\ & \rho^3 I_k & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \rho^{4n-1} I_k \end{bmatrix} \begin{bmatrix} y^{[0]} \\ y^{[1]} \\ \cdot \\ \cdot \\ \cdot \\ y^{[2n-1]} \end{bmatrix},$$

$$X' = [MAM^{-1} + M'M^{-1}]X.$$

$$X' = (1/\rho^2)BX, \quad \text{where} \quad B = \rho^2[MAM^{-1} + M'M^{-1}].$$

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Then

$$
B = \begin{bmatrix}
\rho\rho' I_k & I_k & & 0_k & 0_k & & 0_k \\
& & \cdot & \cdot & \cdot & & \\
& 3\rho\rho' I_k & \cdot & \cdot & \cdot & & \\
& & \cdot & \cdot & \cdot & & \\
& & & I_k & \cdot & & \\
0_k & & & & I_k & & 0_k \\
0_k & & & 0_k & & -I_k & \\
\cdot & & & & \cdot & & \\
\cdot & & & & & \cdot & -I_k \\
\cdot & & & & & \cdot & \\
\rho^{4n}(P - \lambda I) & & & 0_k & & (4n-1)\rho\rho' I_k &
\end{bmatrix}.
$$

Hence condition (1.4) implies b_{ij} is bounded for $i \leqq nk$ and $b_{i,i+k} = 1$. As a matter of fact

$$|y^* B_{ij} x| \leqq K |y^* x| \quad \text{for some constant } K \text{ and } i \leqq n \text{ and } B_{i,i+1} = \pm I_k.$$

Now $X = MY$ yields $x_i = \rho^{2i-1} y^{[i-1]}$ hence $(1/\rho^2)(x_i^* x_i) = \rho^{4i-4}(y^{[i-1]*} y^{[i-1]})$ and we have the integral relations:

$$
(1.8) \quad
\begin{aligned}
\int_a^t \frac{1}{\rho^2}(x_i^* x_i) ds &= \int_a^t \rho^{4i-4}(y^{(i-1)*} y^{(i-1)}) ds \quad (i = 1, \ldots, n+1), \\
\int_a^t \frac{1}{\rho^2}(x_i^* x_i) ds &= \int_a^t \rho^{4i-4}(y^{[i-1]*} y^{[i-1]} ds \quad (i = n+2, \ldots, 2n).
\end{aligned}
$$

For quasi-derivatives the Lagrange bracket takes the form

$$(1.9) \quad [y, z] = \sum_{i=0}^{n-1} \{z^{[2n-i-1]*} y^{[i]} - z^{[i]*} y^{[2n-i-1]}\}.$$

Note that $L(y) = \lambda y$ and $L(z) = \bar{\lambda} z$ implies

$$[y, z]' = z^*(\lambda y) - (\bar{\lambda} z)^* y = 0.$$

For $L(y) = \lambda y$, by expanding $\{\sum_{i=0}^{n-1} y^{[i]*} y^{[2n-i-1]}\}'$ and using the facts that $y^{[2n-1]'} = (P - \lambda I) y$, and $y^{[n]} = y^{(n)}$ we have

$$(1.10) \quad -\lambda(y^* y) + (y^{(n)*} y^{(n)}) + y^* P y = \left\{ \sum_{i=0}^{n-1} y^{[i]*} y^{[2n-i-1]} \right\}'.$$

LEMMA 1.2. *Let y be in V_1, z be in V_2, where for λ fixed, not real, $V_1 = \{y | Ly = \lambda y, y \in H\}$ and $V_2 = \{y | Ly = \bar{\lambda} y, y \in H\}$, and assume* (1.4) *and* (1.6). *Define J_1 and J_2 by*

$$J_1(t) = \max \left\{ 1, \int_a^t \rho^{4n} [y^{(n)*} y^{(n)}] ds \right\},$$

$$J_2(t) = \max \left\{ 1, \int_a^t \rho^{4n} [z^{(n)*} z^{(n)}] ds \right\}.$$

Then for $i = n, \ldots, 2n-1$ *and* $(w_1, w_2) = (y, z)$ *or* (z, y)

$$\text{(i)} \int_a^t w_1^{[j]*} w_2^{[i]} \{(1 - s/t)^{n-1} \rho^{4n-2}\}^{(k)} = O([J_1 J_2]^{1/2})$$

as $t \to \infty$ *for all* j, k *such that* $j + k = 2n - i - 1$.

$$\text{(ii)} \int_a^t w_1^{[j]*} w_1^{[i]} \{(1 - s/t)^n \rho^{4n}\}^{(k)} ds = O(J_r^{(2n-1)/2n})$$

as $t \to \infty$ *for all* j, k *such that* $k \geqq 1$ *and* $i + j + k = 2n$ $(r = 1$ *if* $w_1 = y$, $r = 2$ *if* $w_1 = z)$.

LEMMA 1.3. (Hinton) *Let* F *be a nonnegative, continuous function on* $[a, \infty)$ *and define* $H(t) = \int_a^t (t-s)^n F(s) ds$. *If as* $t \to \infty$, $H(t) = O(t^n [H^{(n)}]^\alpha)$, *where* $\alpha = (2n-1)/2n$, *then* $\int_a^t F(s) ds = O(1)$ as $t \to \infty$.

For the proof, see [**8**].

2. Limit point criteria.

THEOREM 2.1. *Under the conditions* (1.3)–(1.6), *the equation* $L(y) = \lambda y$ *has exactly* nk *linearly independent solutions in* H, λ *not real.*

Proof. Let $\operatorname{Re}(\lambda) = 0$ and let y be in V_1, z be in V_2. Suppose J_1 and J_2 are as in Lemma 1.2. We first show $J_1(\infty) < \infty$.

From (1.10) and an integration by parts,

$$\begin{aligned} &\int_a^t [-\lambda(y^* y) + y^* P y + y^{*(n)} y^{(n)}](1 - s/t)^n \rho^{4n} ds \\ &\qquad = -\int_a^t \sum_{i=0}^{n-1} (y^{[i]*} y^{[2n-i-1]})\{(1 - s/t)^n \rho^{4n}\}' ds + O(1). \end{aligned} \tag{2.1}$$

By part (ii) of Lemma (1.2), the right side of (2.1) is $O(J_1^{(2n-1)/2n})$. We have by (1.3) that

$$\operatorname{Re} \int_a^t [y^*(P - \lambda I) y](1 - s/t)^n \rho^{4n} ds \geqq -K \int_a^t (y^* y) ds.$$

Then from (2.1) we have

$$\int_a^t (t-s)^n \rho^{4n} (y^{(n)*} y^{(n)}) ds = O(t^n J_1^{(2n-1)/2n}).$$

Lemma (1.3) with $F = \rho^{4n}(y^{(n)*} y^{(n)})$ now applies to yield $J_1(\infty) < \infty$. Similarly, $J_2(\infty) < \infty$.

Now $\dim V_1 = \dim V_2$. Suppose $\dim V_1 > nk$. By the argument in [**10**] or the one in [**8**] we may choose y in V_1 and z in V_2 such that $[y, z] \equiv 1$; hence

$$\int_a^t (1 - s/t)^{n-1} \rho^{4n-2} ds$$
$$= \int_a^t \sum_{i=0}^{n-1} [z^{[2n-i-1]*} y^{(i)} - z^{(i)*} y^{[2n-i-1]}](1 - s/t)^{n-1} \rho^{4n-2} ds,$$
$$= O([J_1 J_2]^{1/2})$$

by part (i) of Lemma 1.2. Now $J_1(\infty) < \infty$ and $J_2(\infty) < \infty$; thus

$$\limsup_{t\to\infty} \int_a^t (1 - s/t)^{n-1} \rho^{4n-2} ds < \infty,$$

contrary to (1.5), *i.e.*, $\int_a^\infty \rho^{4n-2} ds = \infty$. Therefore dim $V_1 \leqq nk$ and the proof is complete. For $Im\ \lambda \neq 0$, dim $V_1 =$ dim $V_2 = nk$.

By choosing $\rho = 1$ we get the following corollary.

COROLLARY 2.2. *If $(-1)^n P(t) \geqq 0$, then L is of limit point type.*

Remark. We have not been able to put the fourth-order scalar equation $y^{(iv)} - (qy') + py = 0$ into a system form where we could apply the above result to obtain any information, but it is conjectured that if $q \geqq 0$, $p \geqq 0$, the equation is of limit point type. This problem is discussed in [**9**].

If we choose $\rho = t^{-1/(4n-2)}$ we get the next corollary.

COROLLARY 2.3. *If $(-1)^n P + Kt^{2n/(2n-1)} I \geqq 0$, then L is of limit point type.*

Two special cases for which Corollary 2.3 is of interest are:

(1) *If* $P = \begin{bmatrix} \alpha & 0 \\ 0 & \gamma \end{bmatrix}$, $(-1)^{n+1}\alpha \leqq Kt^{2n/(2n-1)}$ *and* $(-1)^{n+1}\gamma \leqq Kt^{2n/(2n-1)}$, *then L is of limit point type.*

(2) *If* $P = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}$, *then $\beta \leqq Kt^{2n/(2n-1)}$ implies L is of limit point type.*

Note that if we let $\rho = q^{-1/4}$, $n = 1$, we get the second part of Lidskii's first theorem as a corollary to Theorem 2.1. Just for a point of interest we will now state the implications of Lidskii's first theorem (both parts) for the second order system.

THEOREM 2.4. *Consider*

$$L(y) = y'' + P(t)y = iy, \tag{2.2}$$

where $P = \begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix}$; *$\alpha$, γ, β are all continuous real valued functions on (a, ∞) (it is sufficient for them to be integrable on finite subintervals of (a, ∞)). If $\alpha(t)$, $\beta(t)$, $|\gamma(t)|$ are all $\leqq g(t)$ where $g^{-1/2}(t)$ is in L_1 (a, ∞) and*

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(i) $g(t)$ *is positive, continuous and monotone, or*

(ii) $g(t) \geqq \delta > 0$, *differentiable and* $\limsup_{t\to\infty} |g'(t)|/g^{3/2}(t) < \infty$, *then L is of limit point type on* (a, ∞).

Proof. Let $q(t) = 2g(t)$ where $q(t)$ is as in Lidskii's theorem. Then for any 2-vector $h = [h_1, h_2]^T$,

$$\begin{aligned} h^*P(t)h &= \alpha(t)|h_1|^2 + \beta(t)|h_2|^2 + 2\,\mathrm{Re}(\gamma(t)h_1\bar{h}_2) \\ &\leqq g(t)(|h_1|^2 + |h_2|^2) + g(t)(|h_1|^2 + |h_2|^2) \\ &\leqq q(t)h^*h. \end{aligned}$$

3. Limit circle criteria. We will now generalize the limit circle criteria of Lidskii for the second-order system. First we need a few important lemmata.

LEMMA 3.1. *If $Q(t)$ is real, symmetric, differentiable $k \times k$ matrix such that $h^*Q(t)h > 0$ and $h^*Q'(t)h \geqq 0$ for any h and all $t > a$, then all solutions of the equation*

(3.1) $\quad y'' + Q(t)y = 0$

are bounded as $t \to \infty$.

Proof. Let y be a solution of (3.1) and consider

$$V(t) = y'(t)^*Q^{-1}(t)y'(t) + y(t)^*y(t).$$

Clearly $y'(t)^*Q^{-1}(t)y'(t) > 0$ and hence $V(t) > 0$.

Now $V'(t) = y'(t)^*(Q^{-1}(t))'y'(\mathrm{t})$ and $(Q^{-1}(t))' = -Q^{-1}(t)Q'(t)Q^{-1}(t)$, so $V'(t) = -(Q^{-1}(t)y'(t))^*Q'(t)Q^{-1}(t)y'(t)$, since Q is symmetric and hence Q^{-1} is symmetric. Thus $V'(t) = h^*Q'(t)h$ where $h = Q^{-1}(t)y'(t)$, and $V'(t) \leqq 0$. Hence $V(t) \leqq V(a)$ for all $t \geqq a$ and $y(t)^*y(t) \leqq V(a)$. So $y(t)$ is bounded as $t \to \infty$.

LEMMA 3.2. *If M has integrable norm and Q is as in Lemma 3.1, then all solutions of*

(3.2) $\quad y'' + [Q(t) + M(t)]y = 0$

are also bounded as $t \to \infty$.

Proof. We will use $|\cdot|$ to denote the matrix norm. Put equations (3.1) and (3.2) in vector-matrix form where

$$a = \begin{bmatrix} y \\ y' \end{bmatrix}, \quad A(t) = \begin{bmatrix} 0 & I \\ -Q(t) & 0 \end{bmatrix}, \quad b(t) = \begin{bmatrix} 0 \\ -M(t)y \end{bmatrix}.$$

We then have that (3.1) is equivalent to (3.3) and (3.2) is equivalent to (3.4) where

(3.3) $\quad z' = A(t)z,$

(3.4) $\quad z' = A(t)z + b(t).$

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Let $\Phi(t)$ be the fundamental solution matrix for (3.3) such that $\Phi(0) = I$. If we let K be the $2k \times 2k$ matrix $K = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$, then $\Phi^{-1}(t) = -K\Phi^T K$

To see that this is true note that $K^{-1} = K^T = -K$ and let $\Psi = -K\Phi^T K$. Then

$$\begin{aligned}\Psi'(t) &= -K(\Phi'(t))^T K = -\mathrm{K}(A(\mathrm{t})\Phi(t))^T K \\ &= K\Phi^T(t)A^T(t)K^T = -K\Phi^T(t)(KA(t)) \\ &= \Psi(t)A(t),\end{aligned}$$

since if Q is symmetric then $(KA)^T = KA$.

But by uniqueness of solutions to initial value problems and since $\Psi(0) = I$ we have that $\Phi^{-1}(t) = \Psi(t)$ is the only solution to $\Psi' = \Psi A(t)$, $\Psi(0) = I$. Hence

$$\Phi^{-1}(t) = -K\Phi^T(t)K.$$

Now any solution $z(t)$ of (3.4) can be written as

$$z(t) = z_h(t) + \int_a^t \Phi(t)\ \Phi^{-1}(s)b(s)ds,$$

where $z_h(t)$ is a solution to the homogeneous problem (3.3). We are interested in a solution $y(t)$ to the problem (3.2), hence only in the first k components of $z(t)$.

Now write $\Phi(t)$ in block form

$$\Phi(t) = \begin{bmatrix} \Phi_{11}(t) & \Phi_{12}(t) \\ \Phi_{21}(t) & \Phi_{22}(t) \end{bmatrix}, \quad \text{where } \Phi_{ij} \text{ is a } k \times k \text{ matrix.}$$

Then

$$\Phi(t)\Phi^{-1}(s)b(s) = -\Phi(t)K\Phi^T(s)Kb(s),$$

$$\Phi(t)\Phi^{-1}(s)b(s) = \begin{bmatrix} \Phi_{11}(t)\Phi_{12}{}^T(s)M(s)y(s) - \Phi_{12}(t)\Phi_{11}{}^T(s)M(s)y(s) \\ \Phi_{21}(t)\Phi_{12}{}^T(s)M(s)y(s) - \Phi_{22}(t)\Phi_{11}{}^T(s)M(s)y(s) \end{bmatrix}.$$

Hence

$$y(t) = y_h(t) + \int_a^t [\Phi_{11}(t)\Phi_{12}{}^T(s) - \Phi_{12}(t)\Phi_{11}{}^T(s)]M(s)y(s)ds.$$

Now the columns of Φ_{11} and Φ_{12} are solutions of (3.1), and by Lemma 3.1, Φ_{11} and Φ_{12} are bounded. Let C be a constant such that $|y_h(t)| \leqq C$ and $|\Phi_{11}(t)\Phi_{12}{}^T(s) - \Phi_{12}(t)\Phi_{11}{}^T(s)| \leqq C$ for all $a \leqq s \leqq t$ as $t \to \infty$. Then

$$|y(t)| \leqq C + \int_a^t C|M(s)|\ |y(s)|ds \quad \text{for all } t \geqq a$$

and by Gronwall's inequality

$$\begin{aligned}|y(t)| &\leqq C\exp\left[C\int_a^t |M(s)|ds\right] \\ &\leqq C\exp\left[C\int_a^\infty |M(s)|ds\right] \quad \text{for all } t \geqq a.\end{aligned}$$

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Since $M(s)$ has integrable norm, $y(t)$ is bounded as $t \to \infty$.

LEMMA 3.3. *Suppose g is positive and has two continuous derivatives on $[a, \infty)$. If $g^{-3/2}g'' \in L[a, \infty)$, then so is $g^{-3/2}g'' - 9/4g^{-5/2}(g')^2$.*

Proof. Let $p = g^{5/2}$, $\mu = g^{-3/2}$. Then $\mu(p\mu')' = g^{-3/2}(-3/2g')' = -3/2g^{-3/2}g''$ so the hypothesis of Lemma 5 [**3**, p. 119] holds. Hence $(p\mu')^2 = 9/4g^{-5/2}(g')^2 \in L[a, \infty)$.

THEOREM 3.4. *Consider the system*

$$y'' + P(t)y = 0, \tag{3.5}$$

where $P(t)$ is a $k \times k$ matrix of real differentiable functions on $[a, \infty)$. Suppose there exists a positive function g with two continuous derivatives on $[a, \infty)$ such that

(i) $g^{-3/2}g'' \in L[a, \infty)$,
(ii) $P > 0$,
(iii) $(P/g)' \geqq 0$, *and*
(iv) $g^{-1/2} \in L[a, \infty)$.

Then (3.5) *is in the limit circle case.*

Proof. First make a transformation on Equation (3.5). Let $y = \omega\eta$, $\xi = \int_a^t \omega^{-2}(s)ds$ where ω is a scalar function and $\eta(\xi)$ is a vector function. Then system (3.5) becomes

$$\frac{d^2\eta}{d\xi^2} + \left(\frac{d^2\omega}{dt^2} I + \omega P\right)\omega^3\eta = 0. \tag{3.6}$$

If we choose $\omega(t) = g^{-1/4}(t)$ we have

$$\frac{d\omega}{dt} = -1/4g^{-5/4}(t)g'(t), \quad \frac{d^2\omega}{dt^2} = 1/4[5/4g^{-9/4}(t)(g'(t))^2 - g^{-5/4}(t)g''(t)],$$

or

$$\begin{aligned}\omega^3\frac{d^2\omega}{dt^2} &= 1/4[5/4g^{-3}(t)(g'(t))^2 - g^{-2}(t)g''(t)]\\ &= \frac{g^{-3}(t)}{4}[5/4(g'(t))^2 - g(t)g''(t)]\\ &= \frac{g^{-3}(t)}{4}[9/4(g'(t))^2 - (g(t)g'(t))'].\end{aligned}$$

Let

$$M(\xi) = \omega^3\frac{d^2\omega}{dt^2} I,$$

then since $d\xi = g^{-1/2}(t)dt$ by (i) $M(\xi)$ has integrable norm. Let $Q(\xi) = \omega^4P(t)$. Then if $Q(\xi)$ satisfies the hypothesis of Lemma 3.1, all solutions $\eta(\xi)$ of (3.6) are bounded. But $\eta(\xi)$ bounded and $\omega(t) \in L_2[a, \infty)$ implies $y(t) \in L_2[a, \infty)$. Hence we only need to verify that $h^*Q(\xi)h > 0$ and $h^*Q'(\xi)h \geqq 0$ for all h and all $\xi \geqq 0$.

To this end,

$$h^*Q(\xi)h = h^*(\omega^4P(t))h > 0$$

by (ii).

Now

$$\begin{aligned} h^*Q'(\xi)h &= h^*\left(\omega^4P'(t)\frac{dt}{d\xi}\right)h + h^*\left(4\omega^3\frac{d\omega}{dt}\frac{dt}{d\xi}P(t)\right)h \\ &= h^*(g^{-3/2}(t)P'(t))h - h^*(g^{-5/2}(t)g'(t)P(t))h \\ &= g^{-5/2}(t)[h^*(g(t)P'(t) - g'(t)P(t)h] \\ &\geqq 0 \end{aligned}$$

by (iii). This completes the proof.

The referee has kindly pointed out that Lemma 3.1 can also be obtained from Theorem 5, p. 61 of [**3**]. I would also like to thank the referee for his helpful suggestions for improvements in the first manuscript.

References

1. R. L. Anderson, *Singular symmetric differential operators on a Hilbert Space of vector valued functions*, Unpublished doctoral dissertation, The University of Tennessee, Knoxville, Tennessee, 1973.

2. R. Bellman, *Stability theory of differential equations* (Dover Publications, Inc., New York, 1969).

3. W. A. Coppel, *Stability and asymptotic behavior of differential equations* (D. C. Heath and Company, Boston, 1965).

4. W. N. Everitt, *Some positive definite differential operators*, J. London Math. Soc. *43* (1968), 465–473.

5. ——— *On the limit-point classification of fourth-order differential equations*, J. London Math. Soc. *44* (1969), 273–281.

6. W. N. Everitt and J. Chandhuri, *On the square of a formally self-adjoint expression*, J. London Math. Soc. *44* (1969), 661–673.

7. I. M. Glazman, *On the theory of singular differential operators*, Uspekhi Mat. Nauk. *5*, *6*(40) (1950) 102–35,. (Russian).

8. D. Hinton, *Limit point criteria for differential equations*, *Can. J. Math. 24* (1972), 293–305.

9. ——— *Limit point criteria for positive definite fourth-order differential operators*, to appear.

10. V. B. Lidskii, *On the number of solutions with integrable square of the system $y'' + Py = \lambda y$*, Doklady Akad. Nauk. SSSR (N.S.) *95* (1954), 217–220. (Russian).

11. M. A. Naimark, *Linear differential operators*, *Part I* (Ungar, New York, 1967).

12. P. Walker, *Asymptotica of the solutions to $[(ry'')' - py']' + qy = \sigma y$*, J. Differential Equations *9* (1971), 108–132.

13. ——— *Deficiency indices of fourth-order singular differential operators*, J. Differential Equations *9* (1971), 133–140.

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