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ON p-ADIC F-FUNCTIONS

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Abstract

We introduce the class of p-adic F-functions which contains both the p-adic E-functions and p-adic G-functions, as well as other functions. In this paper we obtain lower bounds for polynomials in the values at algebraic points of a class of p-adic F-functions defined over the completion of the algebraic closure of a p-adic field.

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1. Introduction

One speaks of the Siegel-Šidlovskii method for studying the arithmetic properties of E-functions and G-functions in virtue of Siegel's seminal paper [12] and Šidlovskii's far-reaching generalization [10] of this work. In the p-adic case, Flicker [4] considering a polynomial in p-adic G-functions, and Remmal [8] generalized a result of Bundschuh and Walliser [2] on the p-adic exponential functions by considering polynomials in p-adic E-functions defined over the completion of the algebraic closure of a p-adic field. Estimates at rational points are given. Remmal [8] also deals with the p-adic function $\sum_{h=0}^{\infty} h!z^h$ which is not a p-adic E-function or G-function. His work motivates us to consider a new class of p-adic functions. We name these functions p-adic F-functions and give estimates for values at algebraic points of a class of p-adic F-functions.

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2. Notations and results

As usual $\mathbf{Q}$ denotes the field of rational numbers. If p is a fixed prime then $\mathbf{Q}_p$ denotes the field of p-adic rationals and $\mathbf{C}_p$ a completion of an algebraic closure of $\mathbf{Q}_p$. We denote by $\mathbf{K}$ a subfield of $\mathbf{C}_p$ of degree d over $\mathbf{Q}$, and by $O_{\mathbf{K}}$ the domain of integers of $\mathbf{K}$. We use $|\cdot|$ to denote the archimedean valuation (that is, the ordinary absolute value), $|\cdot|_p$ the normalised non-archimedean valuation (that is, the p-adic valuation with $|p|_p = p^{-1}$), and use $\|\cdot\|$ to denote the size of an algebraic element of $\mathbf{C}_p$ (by which we mean the maximum of the absolute values of the element and its field conjugates).

A p-adic F-function is defined as an analytic function of the form

$$f(z) = \sum_{h=0}^{\infty} a_h z^h$$

where the a_h have the following properties:

(i) $a_h \in \mathbf{K}$, $h \geqslant 0$, and there exists a sequence of natural numbers $q_0, q_1, \ldots,$ and a function $\phi(h)$, which is an increasing function of h, such that

$$q_h a_j \in O_{\mathbf{K}} \quad \text{and} \quad \max(q_h, \|q_h a_j\|) \leqslant \phi(h), \qquad h \geqslant 0, 0 \leqslant j \leqslant h.$$

(ii) There are constants $a \geqslant 1$, $b \geqslant 1$, and $c > 0$ such that

$$|a_0|_p \leqslant a, \qquad |a_h|_p \leqslant a h^b c^h, \qquad h \geqslant 1.$$

So the series $f(z)$ converges in the subdisc of those z in $\mathbf{C}_p$ with $|z|_p < c^{-1}$.

Suppose now that we have a system of linear differential equations

$$(1) \qquad y_i' = Q_{i0}(z) + \sum_{h=1}^{m} Q_{ih}(z) y_h, \qquad 1 \leqslant i \leqslant m,$$

with the $Q_{ih}(z) \in \mathbf{K}(z)$. There is then no loss of generality in supposing that the $Q_{ih}(z)$ are rational functions with coefficients in $O_{\mathbf{K}}$ (see [10]). We denote by $T(z)$ a least common denominator for the rational functions $Q_{ih}(z)$. Thus $T(z)$ is a polynomial in $O_{\mathbf{K}}[z]$ such that all the $T(z)Q_{ih}(z)$ are in $O_{\mathbf{K}}[z]$.

Let

$$(2) \qquad g = \max_{i,h}\left(\deg T(z), \deg\left(T(z)Q_{ij}(z)\right)\right),$$

$$T = \max_{i,h}\left(\overline{|T(z)|}, \overline{|T(z)Q_{ih}(z)|}\right)$$

where $\overline{|T(z)|}$ denotes the height of polynomial $T(z)$ (that is, the maximum of the sizes of its coefficients).

If a set of p-adic F-functions $f_1(z), \ldots, f_m(z)$ satisfies (1) and (2) then we speak of them as belonging to the class $f(\mathbf{K}; \phi(h); a, b, c; g, T)$.

In particular, the standard E-functions, $\Sigma_h a_h z^h/h!$, belong to the class $F(\mathbf{K}; C^{2(h+1)}h!; C, 0, Cp^{1/(p-1)}; g, T)$ for a suitable constant C, and the standard G-functions, $\Sigma_h a_h z^h$, belong to the class $F(\mathbf{K}; C^{2(h+1)}; C, 0, C; g, T)$ again for a suitable constant C. (Compare [8] and [4].)

We give some further examples: Let a be a non-negative integer. Consider a function

$$f_a(z) = \sum_{h=0}^{\infty} (a+1)\cdots(a+h)z^h.$$

Using a method similar to that of Lemma 3.1 in Chapter II of Bachman's book [1], it is easy to verify that for $h \geqslant 1$ we have

$$|(a+1)\cdots(a+h)|_p \leqslant p^{-h/(p-1)+\log(h+a)/\log p+1}$$
$$\leqslant p(a+1)h\left(p^{-1/(p-1)}\right)^h.$$

So the series $f_a(z)$ converges in the subdisc of those z in $\mathbf{C}_p$ with $|z|_p < p^{1/(p-1)}$. Moreover, the function $f_a(z)$ satisfies a linear differential equation

$$y' = -\frac{1}{z^2} + \frac{1-(a+1)z}{z^2}y.$$

Let $a_1, \ldots, a_m$ be m distinct non-negative integers. Put $a = \max(a_1, \ldots, a_m)$. Then $f_{a_1}(z), \ldots, f_{a_m}(z)$ belong to the class of p-adic F-functions

$$F\left(\mathbf{K}; (a+(h+1)/2)^h; p(a+1), 1, p^{-1/(p-1)}; 2, a+1\right).$$

In this paper we shall suppose that p-adic F-functions $f_1(z), \ldots, f_m(z)$ do not satisfy any algebraic equations of degree at most r, and with coefficients in $\mathbf{K}$. Let $P(x_1, \ldots, x_m) \not\equiv 0$ be any polynomial in $O_{\mathbf{K}}[x_1, \ldots, x_m]$ with degree $s \leqslant r$ and with height H, say

$$P(x_1, \ldots, x_m) = \sum_{0 \leqslant i_1 + \cdots + i_m \leqslant s} c_{i_1 \cdots i_m} x_1^{i_1} \cdots x_m^{i_m}, \tag{3}$$
$$c_{i_1 \cdots i_m} \in O_{\mathbf{K}} \quad \text{and} \quad \|c_{i_1 \cdots i_m}\| \leqslant H.$$

Put

$$u = \binom{r+m}{m}, \qquad v = \binom{r-s+m}{m}.$$

Suppose that $\xi \in \mathbf{K}$ with $\xi T(\xi) \neq 0$ and let q be the smallest natural number such that $q\xi^\sigma \in O_{\mathbf{K}}$, where the ξ^σ are the field conjugates of ξ. Let $Q = \max(q, \|q\xi\|)$. Clearly we have

$$Q^{-d} \leqslant \|q\xi\|^{-d} < |N(q\xi)|^{-1} \leqslant |q\xi|_p \leqslant |\xi|_p, \tag{4}$$

where $N(\cdot)$ denotes the norm of an element of $\mathbf{K}$. We assume

$$|\xi|_p \leqslant Q^{-d'} \tag{5}$$

where d' is a positive number with

$$0 < d' \leqslant d \quad \text{and} \quad d_0 = d'u - d(u - v) > 0. \tag{6}$$

Then we obtain

THEOREM 1. *Under the assumptions above, there exist positive constants* $\gamma_2, \gamma_2, \gamma_3$ *and* η_0, *independent of* Q *and* H, *and there is a non-zero function* $\Psi(\eta)$, *such that for any real number* $0 < \eta < \eta_0$ *and any* ξ *in* $\mathbf{K}$ *as above and with*

$$Q > \max(\gamma_1, \gamma_2 \Psi(\eta) H^\eta)$$

we have

$$| P(f_1(\xi), \ldots, f_m(\xi)) |_p > \gamma_3 Q^{-\lambda/\eta},$$

where

$$\lambda = 3d'duv/d_0. \tag{7}$$

In general, the constant $\gamma_1, \gamma_2, \gamma_3$ are effectively computable but the constant η_0 is not. $\gamma_1, \gamma_2, \gamma_3$ and $\Psi(\eta)$ will be detailed in the proof of the theorem.

THEOREM 2. *Consider a set of p-adic E-functions* $f_1(z), \ldots, f_m(z)$ *defined as above. Under the assumptions of Theorem* 1, *there exists a positive constant* γ_4, *independent of* Q *and* H, *such that for any real number* $0 < \eta < \eta_0$ *and any* ξ *in* $\mathbf{K}$ *as above and with*

$$Q > \gamma_4 \eta^{-4d(u-v)/d_0} H^\eta,$$

we have

$$| P(f_1(\xi), \ldots, f_m(\xi)) |_p > Q^{-\lambda/\eta}$$

where η_0, λ *are as in Theorem* 1.

This result is a generalization of the theorem in Section 3 of Remmal [8].

A similar result can be obtained for a set of p-adic G-functions when $Q > \gamma_5 H^\eta$ (in place of the condition in Theorem 2). Additional hypotheses seem to be required to obtain more precise results of the type given by Flicker [4].

THEOREM 3. *Let* $\alpha_1, \ldots, \alpha_\mu$ *be* μ *distinct integers in* $O_{\mathbf{K}}$ *which are linearly independent over* $\mathbf{Q}$, *and let* $\beta_1, \ldots, \beta_\nu$ *be some other* ν *distinct non-zero integers in* $O_{\mathbf{K}}$ *with*

$$\alpha = \max_{i,j}(\|\alpha_i\|, \|\beta_j\|), \qquad \alpha_p = \max_{i,j}(|\alpha_i|_p, |\beta_j|_p).$$

If

$$u = \binom{r+\mu+\nu}{\mu+\nu}, \qquad d_0 = d'u - d(u-1) > 0,$$

then there exist positive constants γ_6, γ_7, *depending on* $p, \mu, \nu, d, d', \alpha, \alpha_p$ *and there is a non-zero function* $\Psi_1(\eta)$ *such that for any non-zero polynomial* $P(x_1, \ldots, x_{\mu+\nu})$ *in* $O_{\mathbf{K}}[x_1, \ldots, x_{\mu+\nu}]$ *of degree* r *and with height* H *and for any real number* $0 < \eta < \eta_0$ *and any* ξ *in* $\mathbf{K}$ *as above and with*

$$Q > \max(\gamma_6, \gamma_7, \Psi_1(\eta) H^{\eta}),$$

we have

$$|P(e^{\alpha_1 \xi}, \ldots, e^{\alpha_\mu \xi}, -\log(1-\beta_1 \xi), \ldots, -\log(1-\beta_\nu \xi))|_p > Q^{-3d'du/(d_0 \eta)}.$$

This result is a p-adic analogue of Theorem 2 of Čirskii [3].

THEOREM 4. *Let* $a_1, \ldots, a_m$ *be* m *distinct non-negative rational integers with* $a = \max(a_1, \ldots, a_m)$. *We consider p-adic functions*

$$f_{a_i}(z) = \sum_{h=0}^{\infty} (a_i + 1) \cdots (a_i + h) z^h, \qquad 1 \leqslant i \leqslant m.$$

If

$$u = \binom{r+m}{m}, \qquad d_0 = d'u - d(u-1) > 0,$$

then there exist positive constants γ_8, γ_9, *and there is a non-zero function* $\Psi_2(\eta)$ *such that for any non-zero polynomial* $P(x_1, \ldots, x_m) \in O_{\mathbf{K}}[x_1, \ldots, x_m]$ *of degree* r *and with height* H *and for any real number* $0 < \eta < \eta_0$ *and any* ξ *in* $\mathbf{K}$ *as above and with*

$$Q > \max(\gamma_8, \gamma_9 \Psi_2(\eta) H^{\eta}),$$

we have

$$|P(f_{a_1}(\xi), \ldots, f_{a_m}(\xi))|_p > Q^{-3d'du/(d_0 \eta)}.$$

This result is a generalization of the theorem of Remmal [8], Section 1.

Let $f_1(z), \ldots, f_m(z)$ belong to $F(\mathbf{K}; \phi(h); a, b, c; g, T)$. We consider the set of functions

$$f_1^{h_1}(z) \cdots f_m^{h_m}(z), \qquad 0 \leqslant h_1 + \cdots + h_m \leqslant r,$$

and name them $F_1(z), \ldots, f_u(z)$, with the convention that $F_1(z) = 1$. As in [5], Lemma 7, we see that $F_1(z), \ldots, F_u(z)$ belong to

$$F(\mathbf{K}; 2^{r+h} \phi(h) \phi([h/2]) \cdots \phi([h/r]); a^r, rb, c; g, rT)$$

and satisfy a system of linear homogeneous differential equations

$$(8)\qquad y_i' = \sum_{h=1}^{u} Q_{ih}^*(z) y_h, \qquad 1 \leqslant i \leqslant u.$$

There are u polynomials

$$P_i(z) = \sum_{h=0}^{n-1} p_{ih} z^h, \qquad 1 \leqslant i \leqslant u,$$

in $O_{\mathbf{K}}[z]$ and not all zero, with the properties

(i) $\overline{|P_i(z)|} \leqslant C_1^n \Phi(n)$, where

$$(9)\qquad C_1 = (4^{rd+1}d)^{u^2/\omega},$$

and

$$(10)\qquad \Phi(n) = \{\phi(un)\phi([un/2]) \cdots \phi([un/r])\}^{du/\omega};$$

(ii) $R(z) = \sum_{i=1}^{u} P_i(z) F_i(z)$ satisfies ord $R(z) \geqslant un - [\omega n] - 1$, where ord $R(z)$ denotes the order of the zero of $R(z)$ at $z = 0$; and

(iii) $|R(z)|_p \leqslant (au^{rb})^n (c|z|_p)^{un-[\omega n]-1}$ for all z in $\mathbf{K}$ with $|z|_p < c^{-1}$. Here ω is a constant satisfying $0 < \omega \leqslant 1/2$.

To construct the required polynomials note that (ii) amounts to $M = un - [\omega n] - 1$ linear equations in the $N = un$ unknowns p_{ih}, the coefficients in the equations being in $O_{\mathbf{K}}$ and having sizes at most $A = 2^{r+un}\Phi(n)^{\omega/(du)}$. There is a solution of this system with

$$\| p_{ih} \| \leqslant (2^{M/2} d^N N^M A^{M+(d-1)N})^{1/(N-M)}$$

which gives (i). (The particular estimate used here follows from the proof of Lemma 1.3.1 in [15] and the remarks in [14].) Finally (iii) follows since the coefficients of the $F_i(z)$ have p-adic valuations not exceeding $a^r h^{rb} c^h$ and $a^r h^{rb} c^h |z|_p^h$ is a decreasing function of h.

Let $R_1(z) = R(z)$ and

$$(11)\qquad R_k(z) = T(z)\frac{d}{dz} R_{k-1}(z), \qquad k \geqslant 2.$$

It follows that

$$(12)\qquad R_k(z) = \sum_{i=1}^{u} P_{ki}(z) F_i(z), \qquad k \geqslant 1,$$

where the $P_{ki}(z)$ are in $O_{\mathbf{K}}[z]$ and satisfy the recurrence relation

$$P_{ki}(z) = T(z)\frac{d}{dz} P_{k-1,i}(z) + \sum_{h=1}^{u} T(z) Q_{hi}^*(z) P_{k-1,h}(z),$$
$$k \geqslant 2, \quad 1 \leqslant i \leqslant u,$$

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with $P_{i1}(z) = P_i(z)$ and $Q^*_{hi}(z)$ as in (8). Let the dimension of the vector space over $\mathbf{K}(z)$ generated by the $R_k(z)$ be l. From [6], Theorem 3,

$$\text{(13)} \qquad \operatorname{ord} R(z) \leqslant ln + \Omega(m)s^{\tau},$$

where $\tau = (m+1)^{m+1} + m + 1$ and $\Omega(m)$ is a constant depending on the functions $f_1(z),\dots,f_m(z)$.

Let $\Delta(z) = \det(P_{ki}(z))_{1 \leqslant i,k \leqslant u}$ and put

$$\text{(14)} \qquad t = [\omega n] + u(u-1)g/2,$$

$$\text{(15)} \qquad n_0 = 2\Omega(m)s^{\tau} + 2.$$

If $n > n_0$, we see, as in Lemma 8 of [11], that

$$\Delta(z) = z^{un - [\omega n] - 1}\Delta_1(z),$$

where $\Delta_1(z)$ is in $O_{\mathbf{K}}[z]$ and not identically zero and $\deg \Delta_1(z) \leqslant t$.

Let ξ be given as above with $|\xi|_p < \min(1, c^{-1})$ and let

$$\text{(16)} \qquad u_0 = u(u-1)g/2 + u.$$

If $n > \max(n_0, u_0/\omega)$, then there are u distinct suffixes $j_1,\dots,j_u$ with $1 \leqslant j_1 < j_2 < \cdots < j_u \leqslant t + u$ such that the $u \times u$ determinant with entries

$$q_{ki} = q^{n + j_k g} P_{j_k, i}(\xi), \qquad 1 \leqslant i, k \leqslant u.$$

is non-zero. Further, the q_{ki} are in $O_{\mathbf{K}}$ and satisfy

$$\| q_{ki} \| \leqslant C_2^n \Phi(n) n^{2\omega n} Q_{n + j_k g},$$

where

$$\text{(17)} \qquad C_2 = 4^{g+1} r T C_1.$$

This follows from the argument of Lemma 6 of [13] and Lemma 7 of [10].

Finally we estimate $|R_{j_k}(\xi)|_p$. It is easily seen by induction that

$$|R_{j_k}(\xi)|_p \leqslant \max_{1 \leqslant j \leqslant j_k - 1} |R^{(j)}(\xi)|_p.$$

Moreover, since $\operatorname{ord} R^{(j)}(z) \geqslant M - j_k - 1$, we have

$$|R^{(j)}(\xi)|_p \leqslant \max_{h \geqslant M} \left(a^r h^{rb} c^h \left| h(h-1) \cdots (h-j+1)\xi^{h-j} \right|_p \right) \leqslant a^r M^{rb} c^M |\xi|_p^{M-j}.$$

Therefore

$$|R_{j_k}(\xi)|_p \leqslant C_3^n |\xi|_p^{un - 3\omega n},$$

with

$$\text{(18)} \qquad C_3 = (au)^{rb} c^u.$$

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3. The proofs of theorems

PROOF OF THEOREM 1. We consider the set of functions

$$f_1^{h_1}\cdots f_m^{h_m}P(f_1,\ldots,f_m),\qquad 0\leqslant h_1+\cdots+h_m\leqslant r-s,$$

and denote them by $\psi_1(z),\ldots,\psi_v(z)$. Then we have

$$\psi_k(z)=\sum_{i=1}^{u}c_{ki}F_i(z),\qquad 1\leqslant k\leqslant v$$

where the c_{ki} satisfy the conditions (3). We define

$$r_k(\xi)=q^{n+j_kg}R_{j_k}(\xi)=\sum_{i=1}^{u}q_{ki}F_i(\xi).$$

From the above construction, the linear forms $r_1(\xi),\ldots,r_u(\xi)$ are linearly independent. Since $\psi_1(\xi),\ldots,\psi_v(\xi)$ are linearly independent, we can select $w=u-v$ linear forms, indeed, without loss of generality, the first w forms, such that

$$r_1(\xi),\ldots,r_w(\xi),\psi_1(\xi),\ldots,\psi_v(\xi)$$

are u linearly independent linear forms. Denote the determinant of their coefficients by Δ. Clearly $\Delta\neq 0$ and $\Delta\in O_{\mathbf{K}}$. We have $|N(\Delta)|\geqslant 1$. By replacing the first column on the left by the sum of the ith column multiplied by $F_i(\xi)$, we get

$$\Delta=\begin{vmatrix} q_{11} & q_{12} & \cdots & q_{1u}\\ \cdots & \cdots & \cdots & \cdots\\ q_{w1} & q_{w2} & \cdots & q_{wu}\\ c_{11} & c_{12} & \cdots & c_{1u}\\ \cdots & \cdots & \cdots & \cdots\\ c_{v1} & c_{v2} & \cdots & c_{vu}\end{vmatrix}=\begin{vmatrix} r_1(\xi) & q_{12} & \cdots & q_{1u}\\ \cdots & \cdots & \cdots & \cdots\\ r_w(\xi) & q_{w2} & \cdots & q_{wu}\\ \psi_1(\xi) & c_{12} & \cdots & c_{1u}\\ \cdots & \cdots & \cdots & \cdots\\ \psi_v(\xi) & c_{v2} & \cdots & c_{vu}\end{vmatrix}. \tag{19}$$

We now estimate the size of Δ using the determinant on the left of (19). Since $1\leqslant j_1+\cdots+j_w\leqslant 2w\omega n$ we obtain by (3)

$$\|\Delta\|\leqslant u!H^vC_2^{wn}(\Phi(n))^w n^{2w\omega n}Q^{wn+2wg\omega n},$$

and so

$$\begin{aligned}|\Delta|_p&\geqslant|N(\Delta)|^{-1}\geqslant\|\Delta\|^{-d}\\ &\geqslant u^{-dn}C_2^{-dwn}H^{-dv}(\Phi(n))^{-dw}n^{-dwn}Q^{-dwn-2dwg\omega n}.\end{aligned} \tag{20}$$

By (4) and (5) we have

$$|r_k(\xi)|_p\leqslant C_3^nQ^{-d'un+3\omega n}. \tag{21}$$

We now take

$$d_1=3d+2dg(u-v),\quad d^*=\min(d_0,d_1),\quad \omega_0=d^*/(2d_1). \tag{22}$$

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Clearly $0 < \omega_0 \leqslant 1/2$. We choose

(23) $$\gamma_1 = \min\left(C_3^{-1/(3d\omega_0)}, (2c)^{1/d'}\right),$$

(24) $$\gamma_2 = u^{2d/d_0} C_2^{2d(u-v)/d_0} C_3^{2/d_0} (3dv/d_0)^{2d(u-v)/d_0},$$

(25) $$\gamma_3 = \left(\max\left(a, a(b/(e\log 2))^b\right)\right)^{s-r},$$

(26) $$\eta_0 = \min(2dv/(d_0 n_0), 2dv\omega_0/(d_0 u_0)),$$

(27) $$\omega = \omega_0.$$

For any real number $0 < \eta < \eta_0$ we set

(28) $$n = [2dv/(d_0\eta)] + 1.$$

Then for any Q with $Q > \gamma_2 \Psi(\eta) H^{\eta}$, where

(29) $$\Psi(\eta) = \Phi(3dv/(d_0\eta))^{(u-v)\eta/v} \eta^{-2d(u-v)/d_0},$$

we have

(30) $$u^{dn} C_2^{dwn} C_3^n (\Phi(n) n^n)^{dw} H^{dv} < Q^{d_0 n/2} < Q^{(d'u - dw - (3d+2dwg)\omega)n},$$

so that $|r_k(\xi)|_p < |\Delta|_p$ by (20) and (21). Finally, we use the determinant on the right of (19). This gives

$$\Delta = \sum_{k=1}^{w} r_k(\xi)\Delta_k + \sum_{k=1}^{v} \psi_k(\xi)\delta_k,$$

where Δ_k and δ_k are certain minors of the determinant. Clearly $|\Delta_k|_p \leqslant 1$, $|\delta_k|_p \leqslant 1$. Since $Q > \gamma_1 \geqslant (2c)^{1/d'}$ we see that $|\xi|_p < (2c)^{-1}$ and that for $1 \leqslant i \leqslant m$,

$$\begin{aligned}|f_i(\xi)|_p &\leqslant \max_{h\geqslant 0}\left(|a_{ih}|_p |\xi|_p^h\right) \leqslant \max_{h\geqslant 1}\left(a, ah^b c^h (2c)^{-h}\right)\\ &\leqslant \max\left(a, a(b/(e\log 2))^b\right).\end{aligned}$$

It follows that

$$\begin{aligned}|\Delta|_p &\leqslant \max_k\left(|\psi_k(\xi)|_p\right) \leqslant \max\left(a, a(b/(e\log 2))^b\right)^{r-s} |P(f_1(\xi),\ldots,f_m(\xi))|_p\\ &\leqslant \gamma_3^{-1} |P(f_1(\xi),\ldots,f_m(\xi))|_p.\end{aligned}$$

Noting $Q > \gamma_1 \geqslant C_3^{-1/(3d\omega_0)}$, we have

$$|P(f_1(\xi),\ldots,f_m(\xi))|_p > \gamma_3 Q^{-d'un} = \gamma_3 Q^{-\lambda/\eta}$$

by (7), (20), (21), (30), completing the proof of Theorem 1.

REMARK 1. If the p-adic functions $f_1(z),\ldots,f_m(z)$ are algebraically independent over $\mathbf{K}$, then for any non-zero polynomial $P(x_1,\ldots,x_m) \in O_{\mathbf{K}}[x_1,\ldots,x_m]$ with

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degree s, we can choose $r = s$. S in this case we have $v = 1$, $\gamma_3 = 1$ and the theorem takes a simpler form.

REMARK 2. The constant $\Omega(m)$ in Theorem 1 is not effectively computable. However, suppose that $f_1(z), \ldots, f_m(z)$ constitute an irreducible set of functions. That is, the functions satisfy a system of linear homogeneous differential equations and an equation

$$\sum_{k=1}^{m} P_k(z) y_k = 0, \qquad P_k(z) \in \mathbf{C}_p[z],\ 1 \leqslant k \leqslant m,$$

where $y_1, \ldots, y_m$ is some solution of the system of differential equations, occurs only when $P_k(z) y_k = 0$, $1 \leqslant k \leqslant m$, identically in z. In this case the constant $\Omega(m)$ is effectively computable. However we should note that proofs of the irreducibility of sets of functions are very complicated (for example see [11]).

If the functions $F_1(z), \ldots, F_u(z)$ constitute an irreducible set of functions then we can compute the constant n_0 in (15) by using methods similar to those of Lemma 6 of [10], and Lemma 3 of [11]. This gives

$$n_0 = 2\sigma + u(u-1)g, \tag{31}$$

where σ is the least order of a zero of the functions $F_i(z)$ at $z = 0$.

PROOF OF THEOREM 2. Using a similar method we can compute the constant γ_4 as follows: Let

$$D_1 = 4^{g+1} r T \left(4^d r^{2d-1} C^{(2d-1)r}\right)^{u^2/\omega_0},$$
$$D_2 = \left(r C^r p^{1/(p-1)}\right)^u.$$

Then

$$\gamma_4 = u^{2d/d_0} D_1^{2d(u-v)/d_0} D_2^{2/d_0} (2dv/d_0)^{4d(u-v)/d_0},$$

where the constant ω_0 is given by (22). This completes the proof of Theorem 2.

Similarly, to obtain the remark about G-functions, let

$$D_3 = 4^{g+1} r T \left(4^d r^{2d-1} C^{(2d-1)r}\right)^{u^2/\omega_0},$$

and

$$\gamma_5 = u^{2d/d_0} D_3^{2d(u-v)/d_0} \left(r C^r\right)^{2u/d_0}.$$

PROOF OF THEOREM 3. It is easy to verify that the functions

$$e^{\alpha_1 z}, \ldots, e^{\alpha_\mu z}, -\log(1 - \beta_1 z), \ldots, -\log(1 - \beta_\nu z)$$

satisfy the system of linear differential equations

$$y_i' = \alpha_i y_i, \qquad 1 \leqslant i \leqslant \mu,$$
$$y_j' = \beta_j/(1-\beta_j z), \quad 1 \leqslant j \leqslant \nu,$$

and belong to the class of p-adic F-functions

$$F\left(\mathbf{K};\, \alpha^h h^h;\, 1, 0, \alpha_p p^{1/(p-1)};\, \nu, (2\alpha)^{\nu+1}\right).$$

We can see from an analogue of the theorem in [7] that these functions are algebraically independent over $\mathbf{C}_p(z)$. Clearly $v = 1$, $\gamma_3 = 1$ (see Remark 1), and

$$u = \binom{r+\mu+\nu}{\mu+\nu}.$$

Hence

$$\eta_0 = \min(2d/(d_0 n_0), 23 d\omega_0/(d_0 u_0))$$

and we can compute the constant γ_6 from (18), (22), (23), (27); γ_7 from (9), (17), (18), (22), (24), (27), and the function $\Psi_1(\eta)$ is given by (10), (27), (29).

Proof of Theorem 4. From our example of p-adic F-functions in Section 1 we see that the functions $f_{a_1}(z), \ldots, f_{a_m}(a)$ belong to

$$F\left(\mathbf{K};\, (a + (h+1)/2)^h;\, p(a+1), 1, p^{-1/(p-1)};\, 2, a+1\right).$$

It is easy to verify that the functions $f_{a_1}(z), \ldots, f_{a_m}(z)$ are algebraically independent over $\mathbf{C}_p(z)$. Using a method similar to the proof of Theorem 3 we can compute the constants γ_8, γ_9 and η_0 and the function $\Psi_2(\eta)$.

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