ON THE ALMOST PERIODIC SOLUTION OF AN ABSTRACT DIFFFERENTIAL EQUATION

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Abstract. Under certain suitable conditions, the Stepanov-bounded solution of an abstract differential equation corresponding to a Stepanov almost periodic function is strongly (weakly) almost periodic.

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Suppose X is a Banach space and X^* is the dual space of X. Let J be the interval $-\infty < t < \infty$. A continuous function $f: J \to X$ is said to be (Bochner or strongly) almost periodic if, given $\varepsilon > 0$, there is a positive real number $\ell = \ell(\varepsilon)$ such that any interval of the real line of length ℓ contains at least one point τ for which

$$\sup_{t \in J} \|f(t+\tau) - f(t)\| \leqq \varepsilon. \tag{1.1}$$

We say that a function $f: J \to X$ is weakly almost periodic if the scalar-valued function $< x^*, f(t) > = x^*f(t)$ is almost periodic for each $x^* \in X^*$.

For $1 \leqq p < \infty$, a continuous function $f: J \to X$ is said to be Stepanov-bounded or S^p-bounded if

$$\|f\|_{S^p} = \sup_{t \in J} \left[\int_t^{t+1} \|f(s)\|^p ds \right]^{1/p} < \infty. \tag{1.2}$$

For $1 \leqq p < \infty$, a continuous function $f: J \to X$ is said to be Stepanov almost periodic or S^p-almost periodic if, given $\varepsilon > 0$, there is a positive real number $\ell = \ell(\varepsilon)$ such that any interval of the real line of length ℓ contains at least one point τ for which

$$\sup_{t \in J} \left[\int_t^{t+1} \|f(s+\tau) - f(s)\|^p ds \right]^{1/p} \leqq \varepsilon. \tag{1.3}$$

Let $\mathscr{L}(X, X)$ be the set of all bounded linear operators of X into itself. An operator-valued function $T: J \to \mathscr{L}(X, X)$ is called a (strongly) continuous group if

(1.4) $$T(0) = I = \text{the identity operator of } X;$$

(1.5) $$T(t_1 + t_2) = T(t_1)T(t_2) \qquad \text{for all } t_1, t_2 \in J;$$

(1.6) for each $x \in X$, $T(t)x$, $t \in J \to X$ is continuous.

Denote by A the infinitesimal generator associated with the continuous group $T(t)$, with domain of definition $D(A)$ (see Dunford-Schwartz [3]).

The group $T(t)$ is said to be almost periodic if $T(t)x$, $t \in J \to X$ is almost periodic for each $x \in X$.

Our main result is as follows.

THEOREM 1. *Suppose X is a reflexive Banach space, A is the infinitesimal generator of an almost periodic group $T(t)$, $t \in J \to \mathscr{L}(X, X)$, $f(t)$, $t \in J \to X$ is an S^p-almost periodic continuous function with $1 \leqq p < \infty$, and $u(t)$, $t \in J \to D(A)$ is a (strong) solution of the differential equation*

(1.7) $$u'(t) = Au(t) + f(t) \qquad \text{on } J.$$

Then, if $u(t)$ is S^p-bounded, it is almost periodic from J to X.

REMARK. For $T(t) \equiv I$, and hence $A = 0$, Theorem 1 reduces to a result which extends a theorem of Prouse [4, Theorem 5.1] for $p = 1$.

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We require the following lemmas

LEMMA 1. *Any solution of* (1.7) *has the representation*

(2.1) $$u(t) = T(t)u(0) + \int_0^t T(t-s)f(s)ds.$$

PROOF. For an arbitrary but fixed $t \in J$, an application of the operator $T(t-s)$ to (1.7) yields

(2.2) $$T(t-s)[u'(s) - Au(s)] = T(t-s)f(s), \qquad s \in J.$$

Integrating (2.2) from 0 to t, we obtain

$$\int_0^t T(t-s)[u'(s) - Au(s)]ds = \int_0^t T(t-s)f(s)ds,$$

that is,

$$\int_0^t \left[\frac{d}{ds} T(t-s)u(s)\right] ds = \int_0^t T(t-s)f(s)ds,$$

which gives the desired representation.

LEMMA 2. *If $g(t)$, $t \in J \to X$ is an almost periodic function, and if $T(t)$, $t \in J \to \mathscr{L}(X, X)$ is an almost periodic group, then $T(t)g(t)$ is an almost periodic function.*

PROOF. See Zaidman [5].

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Proof of Theorem 1. By (2.1), we have

$$u(t) = T(t)u(0) + T(t)\int_0^t T(-s)f(s)ds \qquad \text{on } J. \tag{3.1}$$

Consider the function

$$f_h(t) = \frac{1}{h}\int_0^h f(t+s)ds \qquad \text{for any } h > 0.$$

Since f is S^p-almost periodic, and hence is S^1-almost periodic, it follows easily that $f_h(t)$ is almost periodic for each fixed $h > 0$. As shown for scalar-valued functions in Besicovitch [2], pp. 80–81, it can be proved that $f_h \to f$ as $h \to 0+$ in the S^1 sense, that is,

$$\sup_{t \in J} \int_t^{t+1} \| f(s) - f_h(s) \| ds \to 0 \qquad \text{as} \quad h \to 0+. \tag{3.2}$$

Obviously, $T(-s)$, $s \in J \to \mathscr{L}(X, X)$ is an almost periodic group. So, for each $x \in X$, the function $T(-s)x$ is almost periodic, and hence is bounded on J. Thus, by the uniform boundedness principle,

$$\sup_{s \in J} \| T(-s) \| = M < \infty. \tag{3.3}$$

Now we have

$$T(-s)f(s) = T(-s)[f(s) - f_h(s)] + T(-s)f_h(s), \tag{3.4}$$

and, by (3.3),

$$\sup_{t \in J} \int_t^{t+1} \| T(-s)[f(s) - f_h(s)] \| ds$$
$$\leqq M \sup_{t \in J} \int_t^{t+1} \| f(s) - f_h(s) \| ds \to 0 \quad \text{as} \quad h \to 0+. \tag{3.5}$$

By Lemma 2, the functions $T(-s)f_h(s)$ are almost periodic from J to X. Therefore it follows that $T(-s)f(s)$ is S^1-almost periodic from J to X.

We write

$$v(t) = \int_0^t T(-s)f(s)ds \qquad \text{on } J. \tag{3.6}$$

Then, by Theorem VIII, page 79, Amerio-Prouse [1], $v(t)$ is uniformly continuous on J. From (3.1) and (3.6), we obtain

$$T(-t)u(t) = u(0) + v(t). \tag{3.7}$$

We observe that the S^p-boundedness of $u(t)$ implies the S^1-boundedness of $u(t)$. Consequently, by (3.3) and (3.7), $v(t)$ is S^1-bounded.

By (3.6), we have

$$v'(t) = T(-t)f(t) = w(t), \text{ say.} \tag{3.8}$$

Now consider a sequence $\{\rho_n(t)\}_{n=1}^{\infty}$ of infinitely differentiable non-negative functions such that

$$\rho_n(t) = 0 \text{ for } |t| \geqq n^{-1}, \quad \int_{-n^{-1}}^{n^{-1}} \rho_n(t)dt = 1. \tag{3.9}$$

The convolution between v and ρ_n is defined by

$$(v * \rho_n)(t) = \int_{-\infty}^{\infty} v(t-s)\rho_n(s)ds = \int_{-\infty}^{\infty} v(s)\rho_n(t-s)ds. \tag{3.10}$$

From (3.8), it follows easily that

$$(v * \rho_n)'(t) = (w * \rho_n)(t) \qquad \text{on } J. \tag{3.11}$$

We set

$$C_{\rho_n} = \max_{|t| \leqq n^{-1}} \rho_n(t). \tag{3.12}$$

Then we have

$$\begin{aligned} \|(v * \rho_n)(t)\| &= \left\| \int_{-1}^{1} v(t-s)\rho_n(s)ds \right\| \\ &\leqq C_{\rho_n} \int_{t-1}^{t+1} \|v(\sigma)\| d\sigma \\ &\leqq 2C_{\rho_n} \|v\|_{S^1} \qquad \text{for all } t \in J. \end{aligned} \tag{3.13}$$

Similarly, we can show that $(w * \rho_n)(t)$ is almost periodic from J to X. So, X being a reflexive Banach space, it follows from (3.11) that $(v * \rho_n)(t)$ is almost periodic from J to X for all $n = 1, 2, \cdots$ (see Amerio-Prouse [1], page 55 and Authors' Remark on page 82).

Further, by the uniform continuity of $v(t)$ on J, the sequence of convolutions $(v * \rho_n)(t)$ converges uniformly to $v(t)$ on J. Therefore $v(t)$ is almost periodic from J to X. Hence, by (3.7), $T(-t)u(t)$ is almost periodic from J to X.

Consequently, again by Lemma 2, $T(t)[T(-t)u(t)] = u(t)$ is almost periodic from J to X, which completes the proof of the theorem.

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THEOREM 2. *Suppose X is a Banach space, A is the infinitesimal generator of a continuous group $T(t)$, $t \in J \to \mathscr{L}(X, X)$, with the group of adjoint operators $T^*(t)$, $t \in J \to \mathscr{L}(X^*, X^*)$ being almost periodic, $f(t)$, $t \in J \to X$ is an S^p-almost periodic (or weakly almost periodic) continuous function with $1 \leqq p < \infty$. If $u(t)$, $t \in J \to D(A)$ is an S^p-bounded solution of the equation* (1.7), *then it is weakly almost periodic.*

To prove Theorem 2, we require the following lemma.

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LEMMA 3. *If $\alpha(t)$ is weakly almost periodic from J to X, and if $\beta(t)$ is almost periodic from J to the dual space X^*, then $\beta(t)\alpha(t)$ is almost periodic from J to the scalars.*

PROOF. See Amerio-Prouse [1, Page 72].

Proof of Theorem 2. The application of an arbitrary but fixed $x^* \in X^*$ to (3.7) gives

$$x^*T(-t)u(t) = x^*u(0) + \int_0^t x^*T(-s)f(s)ds \quad \text{on } J. \tag{4.1}$$

By the assumption made on T^*, $x^*T(-s) = T^*(-s)x^*$ is almost periodic from J to X^*, and so is bounded on J. We set

$$\sup_{s \in J} \| x^*T(-s) \| = K < \infty. \tag{4.2}$$

Again consider the function f_h defined in the proof of Theorem 1. By Lemma 3, $x^*T(-s)f_h(s)$ is almost periodic from J to the scalars. So it follows from (3.2) and (4.2) that $x^*T(-s)f(s)$ is S^1-almost periodic from J to the scalars.

Also, by (4.2), $x^*T(-t)u(t)$, $t \in J \to$ the scalars is S^1-bounded.

Now, proceeding on the lines of the proof of Theorem 1, we can show that $x^*T(-t)u(t)$ is almost periodic from J to the scalars. Thus it follows that $T(-t)u(t)$ is weakly almost periodic from J to X. Again by Lemma 3, $T(t)[T(-t)u(t)] = u(t)$ is weakly almost periodic from J to X.

If $f(t)$ is weakly almost periodic from J to X, then the proof is obviously simpler.

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References

[1] L. Amerio, and G. Prouse, *Almost periodic functions and functional equations*, (Van Nostrand Reinhold Company (1971)).

[2] A. S. Besicovitch, *Almost periodic functions*, (Dover Publications, Inc. (1954)).

[3] N. Dunford and J. T. Schwartz, *Linear operators*, part I, (Interscience Publishers, Inc., New York (1958)).

[4] G. Prouse, 'Sulle equazioni differenziali astratte quasi-periodiche secondo Stepanoff', *Ricerche di Matematica* 11 (1962), 254–270.

[5] S. Zaidman, 'Sur la perturbation presque périodique des groupes et semi-groupes de transformations d'un espace de Banach', *Rendiconti di Matematica e delle sue Applicazioni* (Rome) 16 (1957), 197–206.

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