

ON DISTRIBUTIONS WHOSE MOMENTS ARE MAJORATED BY THE MOMENTS OF A KNOWN DISTRIBUTION*

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1. Summary

Let $F(x)$ be a distribution function and denote by $f(t) = \int_{-\infty}^{\infty} e^{itx} dF(x)$ its characteristic function and by $\alpha_k = \int_{-\infty}^{\infty} x^k dF(x)$ its moment of order k . H. Milicer-Grużewska [2] has derived the following theorem:

Suppose that $F(x)$ has moments of all orders and that they satisfy the relations

$$(1.1) \quad \begin{cases} \alpha_{2k-1} = 0 \\ \alpha_{2k} \leq C a^k \frac{(2k)!}{2^k k!} \end{cases}$$

where

$$(1.2) \quad 0 < a < 2,$$

while C is a positive constant. Then $f(t)$ is an entire function of order not exceeding two.

The proof given by H. Milicer-Grużewska is rather complicated, moreover the restriction (1.2) seems to be artificial and motivated only by the particular method of proof. We note that condition (1.1) means that the moments of $F(x)$ are majorated by the moments of a normal distribution and we use this remark to generalize the problem.

We prove the following theorem:

THEOREM. *Let $F_1(x)$ and $F_2(x)$ be two distribution functions and denote their moments by $\alpha_k^{(1)}$ and $\alpha_k^{(2)}$ and their characteristic functions by $f_1(t)$ and $f_2(t)$ respectively. Suppose that $f_2(t)$ is an analytic characteristic function and that $F_1(x)$ has moments of any order and that the relations*

$$(1.3) \quad |\alpha_k^{(1)}| \leq |\alpha_k^{(2)}|$$

are satisfied. Then $f_1(t)$ is also an analytic characteristic function; moreover

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the radius of convergence of $f_1(t)$ at the origin can not be inferior to the radius of convergence of $f_2(t)$ at the origin.

COROLLARY. Assume that $f_2(t)$ is an entire characteristic function and that (1.3) holds, then $f_1(t)$ is also an entire characteristic function. If $f_2(t)$ is an entire function of finite order ρ_2 , then $f_1(t)$ has also finite order ρ_1 and $\rho_1 \leq \rho_2$.

2. Proofs

Under the assumptions of the theorem the function $f_2(z)$ is also defined for complex values $z = t + iy$ and is regular in a neighborhood of $z = 0$ and admits the expansion

$$(2.1) \quad f_2(z) = \sum_{s=0}^{\infty} \frac{\alpha_s^{(2)}}{s!} (iz)^s.$$

The radius of convergence R_2 of this series is then given by

$$\limsup_{s \rightarrow \infty} \left| \frac{\alpha_s^{(2)}}{s!} \right|^{1/s} = \frac{1}{R_2}.$$

It follows then from (1.3) that the power series $\sum_{s=0}^{\infty} (\alpha_s^{(1)}/s!) (iz)^s$ converges in a circle of radius R_1 where $1/R_1 = \limsup_{s \rightarrow \infty} |\alpha_s^{(1)}/s!|^{1/s} \leq \limsup_{s \rightarrow \infty} |\alpha_s^{(2)}/s!|^{1/s} = 1/R_2$, hence

$$(2.2) \quad R_1 \geq R_2.$$

The power series $\sum_{s=0}^{\infty} (\alpha_s^{(1)}/s!) (iz)^s$ agrees with the characteristic function $f_1(t)$ in the interval $-R_1 < t < R_1$ of the real axis and is therefore, according to a well known result of R. P. Boas [1], an analytic characteristic function.

The first statement of the corollary follows immediately from (2.2) so that we must only prove its second part.

We denote by

$$M_j(r) = M(r; f_j) = \max_{|z| \leq r} |f_j(z)| \quad (j = 1, 2)$$

the maximum modulus of the characteristic function $f_j(z)$. It is known that

$$(2.3) \quad \frac{1}{2}[f_j(ir) + f_j(-ir)] \leq M_j(r) \leq [f_j(ir) + f_j(-ir)] \quad (j = 1, 2)$$

so that

$$(2.4) \quad M_j(r) = \lambda_j(r)[f_j(ir) + f_j(-ir)] \quad (j = 1, 2)$$

where

$$\frac{1}{2} \leq \lambda_j(r) \leq 1 \quad (j = 1, 2).$$

Using the power series expansions of $f_1(z)$ and $f_2(z)$ we see that

$$(2.5) \quad f_j(ir) + f_j(-ir) = 2 \sum_{s=0}^{\infty} \frac{\alpha_{2s}^{(j)}}{(2s)!} r^{2s}.$$

It follows easily from (1.3), (2.4) and (2.5) that

$$(2.6) \quad M_2(r) \geq \frac{\lambda_2(r)}{\lambda_1(r)} M_1(r).$$

The proof of the corollary is completed by noting that the quotient $\lambda_2(r)/\lambda_1(r)$ is bounded and that

$$\rho_j = \limsup_{r \rightarrow \infty} \frac{\log \log M_j(r)}{\log r}.$$

The result of H. Milicer-Grużewska is a consequence of the corollary.

3. Remark

The theorem of section 1 states that the circle of convergence of the minorant $f_1(z)$ at $z = 0$ contains the circle of convergence of the majorant $f_2(z)$ at $z = 0$. It is not possible to make a similar statement concerning the strips of regularity of the two analytic characteristic functions. We give an example which shows that these strips can actually overlap.

Let $f_2(t) = (1 - 2it)^{-1}$ be the characteristic function of an exponential distribution with parameter $\theta = \frac{1}{2}$. The function $f_2(z)$ has as its strip of regularity the half-plane $Im(z) > -\frac{1}{2}$ while $|z| < \frac{1}{2}$ is the circle of convergence of $f_2(z)$ at $z = 0$. It is easily seen that $\alpha_k^{(2)} = 2^k k!$. We choose $f_1(t) = 1/(1 + t_2)$ this is the characteristic function of the Laplace distribution. The strip of regularity of $f_1(z)$ is given by $|Im(z)| < 1$ and the circle $|z| < 1$ is the circle of convergence of $f_1(z)$ at $z = 0$. The moments of the Laplace distribution are $\alpha_{2k-1}^{(1)} = 0$, $\alpha_{2k}^{(1)} = (2k)!$. Relation (1.3) is satisfied and we see that $R_1 \geq R_2$, however, the strips of regularity of the analytic characteristic functions $f_1(z)$ and $f_2(z)$ overlap.

References.

- [1] R. P. Boas, Sur les séries et intégrales de Fourier à coefficients positifs. *C.R. Acad. Sci. Paris*, vol. 228 (1949), 1837–1838.
- [2] H. Milicer-Grużewska, On the law of probability and the characteristic function of the standardized sum of equivalent variables. *Soc. Sci. Lett. Varsovie, C.R. Cl. III, Sci. Math. Phys.* 42 (1949), (1952), 99–143.

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