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BOUNDEDNESS IN A QUASI-UNIFORM SPACE

BY

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1. **Introduction.** Although a nontopological concept, boundedness seems to be of considerable importance in a topological space. 'There are many topological problems in which it is essential to be able to make this distinction' (between bounded and unbounded sets) [1]. Boundedness and in particular 'boundedness-preserving' uniform spaces appear to have applications to topological dynamics [4].

In spite of this importance, there have been only isolated attempts at developing the concept. Alexander [1] and Hu [7] tried the axiomatic approach. Hu, for example, calls a nonempty family $\mathscr{B}$ of sets a boundedness if $\mathscr{B}$ is hereditary and closed under finite union.

An entirely different (and direct) approach is due to Bourbaki [2], who defines boundedness in a uniform space and later [3] improves on his earlier definition. Bushaw [4] has proved some interesting results in the particular context of 'boundedness-preserving' uniform spaces, but for some inexplicable reason, he chooses Bourbaki's earlier definition. The choice is rather unfortunate because this boundedness lacks some of the desirable properties. For instance, total boundedness does not imply boundedness; even compactness does not imply boundedness; union of two bounded sets is not necessarily bounded. Bourbaki's second definition eliminates many of these deficiencies, but the fact remains that it is usable in the very restrictive class of uniformizable (and hence completely regular) spaces only. Hejcman [6] has used Bourbaki's second definition and generalized some of Bushaw's results.

Since Császár has proved [5] that every topological space is quasiuniformizable, it seems logical to extend Bourbaki's definition to quasiuniform spaces. This will define boundedness in a general topological space (in contrast with Bourbaki's definition) and besides it will provide an analogy with metric spaces (in contrast with the approach of Alexander and Hu).

For terminology, notation and basic definitions, the reader is referred to [8]. The identities listed in Chapter 0 of [8] are also used extensively in the proofs.

2.1 Definition. A set A in a quasi-uniform space $(X, \mathscr{Q})$ is said to be BOUNDED if given an entourage Q, there exists a positive integer n and a finite set $F \subset X$ such that

$$A \subset Q^n[F].$$

If X is bounded, the space $(X, \mathscr{Q})$ is said to be bounded.

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It is obvious that in the definition, 'entourage' may be replaced by 'basis-entourage'. It is less obvious that 'finite set F' may be replaced by 'compact set F'. Let F be a compact set with $A\subset Q^n[F]$. Since $\{\text{Int } Q[x] \mid x\in F\}$ is an open cover of F, it will contain a finite subcover $\{\text{Int } Q[x_i]\}$, $i=1$ to p. Therefore $F\subset Q[x_1, x_2, \ldots, x_p]$ and $A\subset Q^{n+1}[x_1, x_2, \ldots, x_p]$. This proves the equivalence of the two definitions.

As immediate consequences of the definition we mention the following:

(2.1.1) Finite sets and more generally compact sets, are bounded.

(2.1.2) Finite unions of bounded sets are bounded.

(2.1.3) Subsets of bounded sets are bounded.

We note in passing that bounded sets in our sense satisfy Hu's axioms for a boundedness.

(2.1.4) Boundedness is contractive; that is, if A is bounded in $(X, \mathscr{Q})$ and $\mathscr{Q}'\subset\mathscr{Q}$, then A is bounded in $(X, \mathscr{Q}')$.

(2.1.5) Every precompact set is bounded. It has been shown [7] that compactness, total boundedness and precompactness are successively weaker concepts. Thus:

$$\text{Compact} \Rightarrow \text{totally bounded} \Rightarrow \text{precompact} \Rightarrow \text{bounded}.$$

We now prove some further properties of boundedness.

2.2 Theorem. *Boundedness is preserved under every quasi-uniformly continuous function and hence a fortiori a quasi-uniform invariant.*

Proof. Let A be bounded in $(X, \mathscr{Q})$ and $f\colon (X, \mathscr{Q})\to(Y, \mathscr{R})$ be a quasi-uniformly continuous function. If $V\in\mathscr{R}$, $U=f_2^{-1}(V)\in\mathscr{Q}$. Now $A\subset U^n[F]$ for some $n>0$ and a finite set F. Hence

$$\begin{aligned} f(A) &\subset f(U^n[F]) \\ &\subset V^n[f(F)]. \end{aligned}$$

Since $f(F)$ is finite, the boundedness of $f(A)$ is proved.

2.3 Theorem. *The pre-image of a bounded set is bounded. More precisely, if B is bounded in $(Y, \mathscr{R})$ and f is any function from X onto Y, then $f^{-1}(B)$ is bounded in the pre-image structure $f^{-1}(\mathscr{R})$.*

Proof. Let $f^{-1}(R)$, $R\in\mathscr{R}$ be a given entourage in the pre-image structure. Then $B\subset R^n[F]$, for some integer $n>0$ and some finite $F=\{y_1, y_2, \ldots, y_p\}\subset Y$. Let $G=\{x_1, x_2, \ldots, x_p\}$ where x_i are so chosen that $f(x_i)=y_i$. Since

$$\begin{aligned} f^{-1}(B) &\subset f^{-1}(R^n[F]) \\ &\subset (f_2^{-1}(R))^n[G], \end{aligned}$$

the assertion is proved.

2.4 THEOREM. *Boundedness is projective and productive. That is, a set in a product space is bounded if and only if each of its projections is bounded.*

Proof. The projectivity is a consequence of (2.2). To prove productivity, let $A = \mathsf{P}_{i \in I} A_i$ with A_i bounded in $(X_i, \mathcal{Q}_i)$ for each index i. Let $Q = \bigcap_{i \in K} (p_i)_2^{-1}(Q_i)$, K a finite subset of I, be an arbitrary basis-entourage. Then for each $i \in K$ there is an $n_i > 0$ and a finite $F_i \subset X$ such that $A_i \subset Q_i^{n_i}[F_i]$. Define $n = \max n_i$ and $F = \mathsf{P}_{i \in I} M_i$, where M_i is the set F_i for $i \in K$ and an arbitrary singleton m_i for $i \notin K$. F is clearly finite. It therefore suffices to show that $A \subset Q^n[F]$. To this end, let $z = (z_i)$ be an arbitrary point of A. For $i \in K$, $z_i \in Q_i^n[F_i]$ and hence there exists a chain of points

$$x_i = x_{i,1}, x_{i,2}, \ldots, x_{i,n}, x_{i,n+1} = z_i$$

such that $x_i \in F_i$ and $(x_{i,\lambda}, x_{i,\lambda+1}) \in Q_i$, $\lambda = 1$ to n. (Such a chain will be called a Q_i-chain from x_i to z_i.) Consider the points $t_1, t_2, \ldots, t_n$, $t_{n+1} = z$ defined as follows:

$$\text{for } 1 \le \lambda \le n, \qquad p_i(t_\lambda) = \begin{cases} x_{i,\lambda}, & \text{if } i \in K \\ m_i, & \text{if } i \notin K. \end{cases}$$

It is easily verified that these points form a Q-chain from $t_1 \in F$ to z. This is equivalent to saying $A \subset Q^n[F]$.

We now mention some 'negative' properties and give counter-examples.

A set which is bounded in a quasi-uniform space $(X, \mathcal{Q})$ is not necessarily bounded in the conjugate space $(X, \mathcal{Q}^{-1})$. Let X be the set of real numbers and $\mathcal{Q}$ be the quasi-uniformity generated by the basis $\{W\}$, W being the set $\{(x, y) \mid x \le y\}$. The right rays (a, ∞) are bounded in $\mathcal{Q}$, but not in $\mathcal{Q}^{-1}$.

The same example shows that, unlike in a uniform space, closure of a bounded set is not necessarily bounded. For, $\bar{a} = \bigcap_{Q \in \mathcal{Q}} Q^{-1}[a] = (-\infty, a)$ which is not bounded in $\mathcal{Q}$.

However, we have the following theorems.

2.5 THEOREM. *If A is bounded in $(X, \mathcal{Q})$, so is $\bar{A}^*$, where $\bar{A}^*$ denotes the closure of A in the conjugate topology induced by $\mathcal{Q}^{-1}$.*

Proof. For a given entourage Q, $A \subset Q^n[F]$ for some $n > 0$ and a finite F.

$$\bar{A}^* = \bigcap_{U \in \mathcal{Q}} U[A] \subset Q^{n+1}[F].$$

That closures are bounded in a uniform space follows as a corollary.

2.6 THEOREM. *In an R_0-space, closures of finite sets are bounded.*

Proof. It suffices to prove that point-closures are bounded. It is known [7] that in an R_0-space

$$\bar{x} = \bigcap_{U \in \mathcal{Q}} U[x] \quad \text{for each } x \in X.$$

Hence $\bar{x} \subset Q[x]$ for every entourage Q, and $\bar{x}$ is bounded.

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The supremum of bounded structures is not necessarily bounded. Let X be the interval [0, 1] with the quasi-metric

$$d(x, y) = \begin{cases} y-x, & \text{if } x \leq y \\ 1, & \text{if } x > y. \end{cases}$$

If $\mathscr{Q}$ and $\mathscr{Q}^{-1}$ are the quasi-uniformities generated by d and its conjugate respectively, then both $(X, \mathscr{Q})$ and $(X, \mathscr{Q}^{-1})$ are bounded. But $(X, \mathscr{Q} \vee \mathscr{Q}^{-1})$ is the discrete uniform space and hence not bounded.

When a quasi-uniformity derives from a quasi-metric it would be pertinent to compare boundedness defined here with the boundedness in the original quasi-metric. It is easy to see that if a set is bounded in the sense of 2.1, it is also bounded in the metric sense. To see that the converse implication does not hold, consider the set J of integers with the zero-one metric. The corresponding quasi-uniformity is discrete and hence J is unbounded in the sense of 2.1, but obviously bounded in the metric.

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