REGULAR POLYTOPES AND HARMONIC POLYNOMIALS

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1. Introduction. In this paper we study the following problem originally proposed by Walsh (**8**). To determine the class of functions $f(x)$ continuous in a given n-dimensional region R and having the property that the value of $f(x)$ be equal to the average of $f(x)$ over the vertices of all sufficiently small regular polytopes similar to a given one, which are centred at x. This problem has been studied by several mathematicians (**1**; **6**; **8**) and has been completely solved except for the four-dimensional regular polytopes $\{3, 4, 3\}$, $\{3, 3, 5\}$, $\{5, 3, 3\}$ (see **3**, p. 129, for the meaning of these symbols) and the n-dimensional cube. In each case, the class of functions is identical with a class of harmonic polynomials which can be specified. In § 2, we solve the problem for the four-dimensional figures, thus leaving the problem open only for the n-dimensional cube.† We will give a detailed treatment for all regular polytopes simplifying the proofs of those cases already discussed in (**1**; **6**; **8**).

The problem of Walsh leads to a natural generalization. We observe that the groups of symmetries of the regular polytopes are generated by reflections, thus forming a subclass of the irreducible finite orthogonal reflection groups acting on E^n. We pose the problem of determining the functions $f(x)$, continuous in a given n-dimensional region R, and satisfying the mean value property

$$(1.1)\quad f(x) = \frac{1}{g}\sum_{\sigma\in G} f(x + t\sigma y), \quad x \in R, 0 < t < \epsilon_x, y \text{ a fixed non-zero vector,}$$

where g is the order of G, G being an irreducible finite orthogonal reflection group. If G is the group of symmetries of a regular polytope π_n centred at the origin and y is an arbitrary vertex of π_n, then G acts transitively on the vertices of π_n, and (1.1) becomes identical with Walsh's problem.

The study of the solution space to (1.1) turns out to be closely related to the invariant theory for the irreducible finite reflection groups. Chevalley (**2**, p. 778, Theorem A) has shown that for these groups, the algebra I of invariants is generated by n algebraically independent forms $I_1, \ldots, I_n$. Coxeter (**3**, Chapter 11) has classified these groups and has computed (**4**, p. 780, Table 3) the degrees $m_1 + 1, \ldots, m_n + 1$ of the forms $I_1, \ldots, I_n$. The m_js are distinct so that we may assume that $0 < m_1 < \ldots < m_n$. (This holds for all irreducible

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†*Added in proof.* Recently the conjecture has been solved in the affirmative (G. K. Haeuslein, *On the algebraic independence of symmetric functions*, to appear in Proc. Amer. Math. Soc.).

finite orthogonal reflection groups G which are not of type B_{2l} ($2l = n; l \geqq 2$). In this paper we shall always assume that $G \neq B_{2l}$. In particular, the symmetry groups of the regular polytopes are not of type B_{2l}.) Furthermore, $m_1 = 1$, $I_1 = \sum_{j=1}^{n} x_j^2$ (we use these facts later on). We have recently proved the following results concerning the solution space S of (1.1).

THEOREM 1.1 (**7**, Theorem 1.2). *Let G be an irreducible finite orthogonal reflection group acting on E^n. Let* $P_m(x, y) = \sum_{\sigma \in G} (x \cdot \sigma y)^m$ $(1 \leqq m < \infty)$ and

$$J(x, y) = \frac{\partial (P_{m_1+1}, \ldots, P_{m_n+1})}{\partial (x_1, \ldots, x_n)}.$$

Then

$$J(x, y) = \prod_{k=1}^{n} J_k(y) \prod_{k=1}^{r} L_k(x),$$

where the J_ks are homogeneous invariants ($\deg J_k = m_k + 1$) *forming an integrity basis for I and* $L_k(x) = 0$ $(1 \leqq k \leqq r)$ *are the reflecting hyperplanes corresponding to the reflections of G.*

Let S be the solution space of (1.1) and $D\Pi$ the linear span of partial derivatives of

$$\Pi(x) = \prod_{k=1}^{r} L_k(x).$$

$S = D\Pi$ if and only if $J_1(y) \ldots J_n(y) \neq 0$ or, equivalently, if and only if $P_{m_1+1}(x, y), \ldots, P_{m_n+1}(x, y)$ are algebraically independent as polynomials in x.

We say that y is an exceptional direction if $J_1(y) \ldots J_n(y) = 0$ and refer to $\mathscr{M} = \{y | J_1(y) \ldots J_n(y) = 0\}$ as the exceptional manifold. Let G again denote the group of symmetries of an n-dimensional polytope π_n centred at the origin. Walsh's problem will then be solved provided we can show that for such groups, $y \notin \mathscr{M}$, y denoting an arbitrary vertex of π_n. We verify this statement, referred to as the vertex conjecture, for all regular polyhedra with the exception of the n-dimensional cube. As explained at the end of § 2, we encounter a certain technical difficulty in this case which we cannot resolve.

Since the solution space S to (1.1) can be characterized for $y \notin \mathscr{M}$, it is natural to ask whether S can also be characterized for $y \in \mathscr{M}$. We solve this problem in § 3 for the dihedral and tetrahedral groups. The complexity of the solution in the latter case leads us to believe that the problem of characterizing S when $y \in \mathscr{M}$ for all irreducible finite orthogonal reflection groups is a rather hopeless one.

2. Verification of the vertex conjecture. In § 3 we describe explicitly the exceptional manifold $\mathscr{M}$ for the dihedral group D_n, which is the group of symmetries of the regular n-gon $\{n\}$. We will find in this case that the vertices of the polygon do not lie in $\mathscr{M}$, thus verifying the vertex conjecture for D_n.

This leaves us with the figures $\{3, 5\}$, $\{5, 3\}$, $\{3, 4, 3\}$, $\{3, 3, 5\}$, $\{5, 3, 3\}$, α_n, β_n, γ_n. The classification of the regular polytopes and the symbol $\{p_1, \ldots, p_{n-1}\}$ were discussed in (**3**, Chapter VII). $\{3, 5\}$ and $\{5, 3\}$ are three-dimensional; $\{3, 4, 3\}$, $\{5, 3, 3\}$, and $\{3, 3, 5\}$ are four-dimensional; α_n, β_n, and γ_n are n-dimensional. We consider separately these three classes. We will make constant use of the following result found in (**7**, formula 3.4).

THEOREM 2.1. *Let* $P_m(x, y) = \sum (x \cdot \sigma y)^m$, *G denoting an irreducible finite orthogonal reflection group. For fixed* y, $P_m(x, y)$ *is an invariant polynomial in* x. $P_{m_k+1}(x, y)$ $(1 \leqq k \leqq n)$ *has the representation*

$$(2.1) \quad P_{m_k+1}(x, y) = F_k(I_1(x), \ldots, I_{k-1}(x); I_1(x), \ldots, I_k(y)) + J_k(y) I_k(x),$$

where F_k *is a polynomial in* $I_1(x), \ldots, I_k(y)$ $(F_1 = 0)$ *and the* J_k*s are the basic invariants introduced in Theorem* 1.1.

In the following, y will denote an arbitrary vertex of the regular polytope π_n, G the group of symmetries of π_n. We define $P_m(x) = \sum_{k=1}^{N} (x \cdot y_k)^m$, where $y_k = (y_{k1}, \ldots, y_{kn})$ $(1 \leqq k \leqq N)$ denote the N vertices of π_n. Let H be the subgroup of G which fixes the vertex y; i.e., H is the stabilizer of y. Let h be the order of H. It is readily checked that $P_m(x, y) = hP_m(x)$. (2.1) then becomes

$$(2.2) \qquad P_{m_k+1}(x) = F_k^*(I_1(x), \ldots, I_{k-1}(x)) + a_k I_k(x), \qquad 1 \leqq k \leqq n,$$

where F_k^* is a polynomial in $I_1(x), \ldots, I_{k-1}(x)$ $(F_1^* = 0)$ and $a_k = (1/h) J_k(y)$ is a constant. The representation (2.2) is unique as $I_1(x), \ldots, I_n(x)$ are algebraically independent. We therefore have the following result.

COROLLARY. *The vertex conjecture is true if and only if* $a_k \neq 0$ $(1 \leqq k \leqq n)$, *the* a_k*s being the numbers occurring in the representation* (2.2).

We remark that the latter condition is equivalent to either of the following two:

(i) $P_{m_1+1}(x), \ldots, P_{m_n+1}(x)$ are algebraically independent;
(ii) $P_{m_1+1}(x), \ldots, P_{m_n+1}(x)$ form an integrity basis for the algebra I of invariants of G (see **7**, Lemma 2.5).

The polyhedra $\{3, 5\}$ *and* $\{5, 3\}$. The vertices of the icosahedron $\{3, 5\}$ may be chosen as the even permutations of $(0, \pm\tau, \pm 1)$, where $\tau = \frac{1}{2}(1 + \sqrt{5})$. The vertices of the dodecahedron $\{5, 3\}$ may be chosen as $(\pm\tau, \pm\tau, \pm\tau)$ and the even permutations of $(0, \pm 1, \pm\tau^2)$ (**3**, p. 52). These two figures have the same symmetry group which is denoted by [3, 5]. The degrees of the basic invariant forms of [3, 5] are 2, 6, 10. Let I_1, I_2, I_3 form a basic set of homogeneous invariants whose respective degrees are 2, 6, 10. Suppose that the vertex conjecture is false. It follows from the corollary to Theorem 2.1 that either $P_6 = c_1 I_1^3$ or $P_{10} = c_2 I_1^5 + c_3 I_1^2 I_2$, where the c_js are constants, $c_1 > 0$ as P_6 is positive-definite. Thus either $P_6 = c_1 I_1^3$ or $I_1 | P_{10}$. We show that this is not the case.

Let k_1, k_2 denote the coefficients of x_1^6 and $x_1^2x_2^2x_3^2$ in P_6, respectively. Thus

$$k_1 = \sum_{j=1}^{N} y_{j1}^6, \qquad k_2 = 90\sum_{j=1}^{N} y_{j1}^2 y_{j2}^2 y_{j3}^2,$$

where $y_j = (y_{j1}, y_{j2}, y_{j3})$ $(1 \leqq j \leqq N)$ denote the N vertices. Direct computations show that for $\{3, 5\}$, $k_1 = 4(\tau^6 + 1) = 8\sqrt{5}\tau^3$, $k_2 = 0$, and for $\{5, 3\}$, $k_1 = 8\tau^6 + 4(1 + \tau^{12}) = 80\tau^6$, $k_2 = 720\tau^6$. It is easily checked that in both cases $k_2 \neq 6k_1$ while the $x_1^2x_2^2x_3^2$ coefficient of $I_1^3 = 6$ (x_1^6 the coefficient of I_1^3). Hence $P_6 \neq c_1 I_1^3$.

We show next that $I_1 \nmid P_{10}$. Let T be an orthogonal transformation sending one of the vertices of $\{3, 5\}$ into a point along the x_3-axis. Let $Q_{10}(x) = P_{10}(T^{-1}x)$. Thus

$$Q_{10}(x) = \sum_{j=1}^{N} (T^{-1}x \cdot y_j)^{10} = \sum_{j=1}^{N} (x \cdot z_j)^{10},$$

where $z_j = Ty_j$ $(1 \leqq j \leqq N)$, and $Q_{10}(1, i, 0) = \sum_{j=1}^{N} (z_{j1} + z_{j2}i)^{10}$. For $\{3, 5\}$, two of the twelve numbers $z_{j1} + z_{j2}i$ $(1 \leqq j \leqq 12)$ are 0, the other ten being given by $\zeta^k w$ $(0 \leqq k \leqq 9)$, where $\zeta = e^{\pi i/5}$ and $w \neq 0$. For $\{5, 3\}$, the twenty numbers $z_{j1} + z_{j2}i$ $(1 \leqq j \leqq 20)$ are given by $\zeta^k w_1$, $\zeta^k w_2$ $(0 \leqq k \leqq 9)$, where $t = w_2/w_1 > 0$ (see **3**, p. 51, for the relevant diagrams). Thus $Q_{10}(1, i, 0) = 10w^{10} \neq 0$ for $\{3, 5\}$ and $Q_{10}(1, i, 0) = 10(1 + t^{10})w_1^{10} \neq 0$ for $\{5, 3\}$. Since $I_1(1, i, 0) = 0$, we conclude that $I_1 \nmid Q_{10}$ which is equivalent to $I_1 \nmid P_{10}$.

The four-dimensional figures $\{3, 4, 3\}$, $\{3, 3, 5\}$, *and* $\{5, 3, 3\}$. We start our discussion with the figure $\{3, 4, 3\}$. The vertices of $\{3, 4, 3\}$ may be chosen as the permutations of $(\pm 1, \pm 1, 0, 0)$ (**3**, p. 156). The symmetry group of $\{3, 4, 3\}$ is denoted by $[3, 4, 3]$ and the degrees of the basic invariant forms are 2, 6, 8, 12.

Let I_1, I_2, I_3, I_4 be basic invariant forms whose respective degrees are 2, 6, 8, 12. Suppose that the vertex conjecture is false. Then either $P_6 = c_1 I_1^3$ or $P_8 = c_2 I_1^4 + c_3 I_1 I_2$ or $P_{12} = c_4 I_1^6 + c_5 I_1^3 I_2 + c_6 I_1^2 I_3 + c_7 I_2^2$, where the c_is are constants, $c_1 > 0$. Thus either $P_6 = c_1 I_1^3$ or $I_1 | P_8$ or $P_{12} \in (I_1, I_2)$. We show that these possibilities do not occur.

A direct computation yields

$$\tfrac{1}{4}P_6 = 3\sum_{j=1}^{4} x_j^4 + 15 \sum_{1 \leqq j < k \leqq 4} (x_j^4 x_k^2 + x_k^4 x_j^2).$$

The coefficient of $x_1^2x_2^2x_3^2$ in P_6 is zero, while the coefficient of $x_1^2x_2^2x_3^2$ in I_1^3 is not zero. Hence $P_6 \neq c_1 I_1^3$. Thus P_6 and I_1 are algebraically independent and we may choose I_2 to be P_6. We have $I_1(1, i, 0, 0) = 0$, $I_2(1, i, 0, 0) = 12(1^6 + i^6) + 60(i^2 + i^4) = 0$, $P_8(1, i, 0, 0) = 2[(i + 1)^8 + (i - 1)^8] + 16$, $P_2(1, i, 0, 0) = 2[(1 + i)^{12} + (1 - i)^{12}] + 16$. Since $1 + i = \sqrt{2}e^{\frac{1}{4}\pi i}$, $1 - i = \sqrt{2}e^{-\frac{1}{4}\pi i}$, we have $(1 + i)^8 = (1 - i)^8 = 16$, $(1 + i)^{12} = (1 - i)^{12} = -64$. Thus $P_8(1, i, 0, 0) = 80 \neq 0$, $P_{12}(1, i, 0, 0) = -240 \neq 0$. We conclude that $I_1 \nmid P_8$ and $P_{12} \notin (I_1, I_2)$.

We now treat the figures $\{3, 3, 5\}$ and $\{5, 3, 3\}$. The 120 vertices of $\{3, 3, 5\}$ may be chosen as the permutations of $(\pm 2, 0, 0, 0)$, $(\pm 1, \pm 1, \pm 1, \pm 1)$, and the even permutations of $(\pm\tau, \pm 1, \pm\tau^{-1}, 0)$, The 600 vertices of $\{5, 3, 3\}$ may be chosen as the permutations of $(\pm 2, \pm 2, 0, 0)$, $(\pm\sqrt{5}, \pm 1, \pm 1, \pm 1)$, $(\pm\tau, \pm\tau, \pm\tau, \pm\tau^{-2})$, $(\pm\tau^2, \pm\tau^{-1}, \pm\tau^{-1}, \pm\tau^{-1})$, along with the even permutations of $(\pm\tau^2, \pm\tau^{-2}, \pm 1, 0)$, $(\pm\sqrt{5}, \pm\tau^{-1}, \pm\tau, 0)$, $(\pm 2, \pm 1, \pm\tau, \pm\tau^{-1})$ (**3**, p. 157). The two figures have the same symmetry group denoted by $[3, 3, 5]$. The degrees of the basic invariant forms are 2, 12, 20, 30.

Let I_1, I_2, I_3, I_4 be basic invariant forms whose respective degrees are 2, 12, 20, 30. Suppose that the vertex conjecture is false. Then either $P_{12} = c_1 I_1^6$ or $P_{20} = c_2 I_1^{10} + c_3 I_1^4 I_2$ or $P_{30} = c_4 I_1^{15} + c_5 I_1^9 I_2 + c_6 I_1^3 I_2^2 + c_7 I_1^5 I_3$, where the c_is are constant, $c_1 > 0$. Thus $I_1 | P_{12}$ or $I_1 | P_{20}$ or $I_1 | P_{30}$. We show that these possibilities do not occur.

Let $x_1^{a_1}, \ldots, x_4^{a_4}$ be a monomial appearing in $P_m(x)$. Since the plus and minus sign can be chosen independently in the listing of vertices for $\{3, 3, 5\}$ and $\{5, 3, 3\}$, we conclude that the a_is must be even. Thus $P_m(x) = 0$ for m odd. For $m = 2k$ we have

$$(2.3)\quad P_{2k}(1, i, 0, 0) = \sum_{j=1}^{N} (y_{j1} + y_{j2} i)^{2k} = \sum_{a=0}^{k} \binom{2k}{2a} \sum_{j=1}^{N} (-1)^a y_{ji}^{2k-2a} y_{j2}^{2a}.$$

Let m be an odd prime, so that

$$\binom{m+1}{2a} \equiv 0 \pmod{m} \quad \text{for } a = 1, 2, \ldots, \tfrac{1}{2}(m-1).$$

Then

$$(2.4)\qquad P_{m+1}(1, i, 0, 0) \equiv \sum_{j=1}^{N} y_{j1}^{m+1} + (-1)^{\frac{1}{2}(m+1)} \sum_{j=1}^{N} y_{j2}^{m+1} \pmod{m}.$$

Since $\sum_{j=1}^{N} y_{j1}^{m+1} = \sum_{j=1}^{N} y_{j2}^{m+1}$, we have

$$(2.5)\qquad P_{m+1}(1, i, 0, 0) \equiv \{1 + (-1)^{\frac{1}{2}(m+1)}\} \sum_{j=1}^{N} y_{j1}^{m+1} \pmod{m}.$$

In particular,

$$(2.6)\qquad P_{m+1}(1, i, 0, 0) \equiv 2 \sum_{j=1}^{N} y_{j1}^{m+1} \pmod{m} \quad \text{for } m = 11, 19.$$

We obtain a similar congruence for P_{30}.

$$(2.7)\qquad P_{30}(\sqrt{2}, i, i, 0) = \sum_{j=1}^{N} (\sqrt{2}\, y_{j1} + y_{j2} i + y_{j3} i)^{30}.$$

Since 29 is prime, the multinomial coefficients behave like the above binomial coefficients, and we have

$$(2.8)\qquad P_{30}(\sqrt{2}, i, i, 0) \equiv 2(2^{14} - 1) \sum_{j=1}^{N} y_{j1}^{30} \equiv -4 \sum_{j=1}^{N} y_{j1}^{30} \pmod{29}.$$

We now show that

$$\sum_{j=1}^{N} y_{j1}^{m+1} \not\equiv 0 \pmod{m} \quad \text{for } m = 11, 19, 29.$$

(2.5) then implies that $P_{m+1}(1, i, 0, 0) \neq 0$ for $m = 11, 19$ and (2.8) implies that $P_{30}(\sqrt{2}, i, i, 0) \neq 0$. Since $I_1(1, i, 0, 0) = I_1(\sqrt{2}, i, i, 0)$, we conclude that $I_1 \nmid P_{m+1}$ for $m = 11, 19, 29$.

We first consider $\{3, 3, 5\}$. A direct computation yields

$$\sum_{j=1}^{N} y_{j1}^{m+1} = 16 + 2 \cdot 2^{m+1} + 24(\tau^{m+1} + \tau^{-(m+1)} + 1). \tag{2.9}$$

It is known that, when n is even,

$$\tau^n + \tau^{-n} = f_{n-1} + f_{n+1}, \tag{2.10}$$

where $\{f_n\}$ is the Fibonacci sequence, i.e.,

$$f_1 = f_2 = 1, \qquad f_{n+2} = f_{n+1} + f_n \tag{2.11}$$

(**5**, pp. 166–167, equations 11.42 and 11.48). It is easily checked that the Fibonacci sequence modulo m is periodic. In particular, $f_{n+m-1} \equiv f_n \pmod{m}$ for $m = 11, 19, 29$. Thus we have

$$\tau^{m+1} + \tau^{-m-1} = f_m + f_{m+2} \equiv f_1 + f_3 = 1 + 2 = 3 \pmod{m}. \tag{2.12}$$

Also, m being an odd prime, we have

$$2^{m-1} \equiv 1 \pmod{m}. \tag{2.13}$$

It follows from (2.9), (2.12), and (2.13) that

$$\sum_{j=1}^{N} y_{j1}^{m+1} \equiv 16 + 8 + 72 + 24 = 120 \not\equiv 0 \pmod{m}.$$

We consider next the figure $\{5, 3, 3\}$. A direct computation yields

$$\begin{aligned} \sum_{j=1}^{N} y_{j1}^{m+1} &= 12 \cdot 2^{m+1} + (16 \cdot 5^{\frac{1}{2}(m+1)} + \ldots) \\ &= 240 \cdot 2^{m-1} + 40 \cdot 5^{\frac{1}{2}(m+1)} + 40(\tau^{2(m+1)} + \tau^{-2(m+1)}) \\ &\qquad + 120(\tau^{m+1} + \tau^{-(m+1)}) + 120 \\ &\equiv 240 + 200 \cdot 5^{\frac{1}{2}(m-1)} + 40(f_3 + f_5) \\ &\qquad + 120(f_1 + f_3 + 1) \pmod{m}. \end{aligned} \tag{2.14}$$

A direct check yields

$$5^{\frac{1}{2}(m-1)} \equiv 1 \pmod{m} \quad \text{for } m = 11, 19, 29.$$

Thus

$$\sum_{j=1}^{N} y_j^{m+1} \equiv 1200 \not\equiv 0 \pmod{m}.$$

The n-dimensional polytopes. The vertex problem has been discussed for these figures in (**6**, pp. 264–266). We discuss the problem again here for the sake of completeness and give a more direct treatment for the n-dimensional simplex α_n.

Let $y_1, \ldots, y_{n+1}$ denote the $n + 1$ vertices of the n-dimensional simplex α_n. Any n of the y_js are then linearly independent. The symmetry group of α_n is denoted by S_{n+1}. The degrees of the basic invariants of S_{n+1} are $2, \ldots, n + 1$. We proceed to show that the polynomials

$$P_m(x) = \sum_{j=1}^{n+1} (x \cdot y_j)^m \qquad (2 \leqq m \leqq n + 1)$$

are algebraically independent. Let $\xi_j = x \cdot y_j$ $(1 \leqq j \leqq n + 1)$. Then $\xi_j = x \cdot y_j$ $(1 \leqq j \leqq n)$ is a non-singular transformation from $(x_1, \ldots, x_n)$ to $(\xi_1, \ldots, \xi_n)$. Since $\sum_{j=1}^{n+1} y_j = 0$, $\sum_{j=1}^{n+1} (x \cdot y_j) = 0$ so that $\xi_{n+1} = -\sum_{j=1}^{n} \xi_j$. Translating the problem into the $(\xi_1, \ldots, \xi_n)$ variables we must show that the polynomials $P_k = \sum_{j=1}^{n+1} \xi_j^k$ $(2 \leqq k \leqq n + 1)$, where $\xi_{n+1} = -\sum_{j=1}^{n} \xi_j$, are algebraically independent. We do this as follows. Let $\bar{P}_k = \bar{P}_k(\xi_1, \ldots, \xi_{n+1}) = \sum_{j=1}^{n+1} \xi_j^k$ $(1 \leqq k \leqq n + 1)$. Let

$$J = \frac{\partial(P_2, \ldots, P_{n+1})}{\partial(\xi_1, \ldots, \xi_n)}, \qquad \bar{J} = \frac{\partial(\bar{P}_1, \ldots, \bar{P}_{n+1})}{\partial(\xi_1, \ldots, \xi_{n+1})}.$$

Then

$$\bar{J} = \begin{vmatrix} 1 & \cdot & \cdot & & 1 \\ \dfrac{\partial \bar{P}_2}{\partial \xi_1} & \cdot & \cdot & \cdot & \dfrac{\partial \bar{P}_2}{\partial \xi_{n+1}} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \dfrac{\partial \bar{P}_{n+1}}{\partial \xi_1} & \cdot & \cdot & \cdot & \dfrac{\partial \bar{P}_{n+1}}{\partial \xi_{n+1}} \end{vmatrix} \tag{2.15}$$

$$= (n+1)! \begin{vmatrix} 1 & \cdot & \cdot & \cdot & 1 \\ \xi_1 & \cdot & \cdot & \cdot & \xi_{n+1} \\ \cdot & & \cdot & \cdot & \cdot \\ \xi_1^{n+1} & \cdot & \cdot & \cdot & \xi_{n+1}^{n+1} \end{vmatrix}$$

$$= (n+1)! \prod_{1 \leqq j < k \leqq n+1} (\xi_k - \xi_j).$$

Now

$$\frac{\partial P_r}{\partial \xi_s} = \frac{\partial \bar{P}_r}{\partial \xi_s} - \frac{\partial \bar{P}_r}{\partial \xi_{n+1}}, \qquad 2 \leqq r \leqq n + 1,\ 1 \leqq s \leqq n. \tag{2.16}$$

Subtracting the $(n+1)$st column in $\bar{J}$ from all other columns we conclude from (2.16) that

$$(2.17)\qquad \bar{J} = \left|\begin{array}{cccc|c} 0 & \cdot & \cdot & 0 & 1 \\ \hline & & & & \frac{\partial \bar{P}_2}{\partial \xi_{n+1}} \\ & & J & & \cdot \\ & & & & \frac{\partial \bar{P}_{n+1}}{\partial \xi_{n+1}} \end{array}\right| = (-1)^{n+1} J.$$

Thus

$$(2.18)\qquad J = (-1)^{n+1}(n+1)! \prod_{1 \leqq j < k \leqq n+1} (\xi_k - \xi_j).$$

Since $\xi_{n+1} = -\sum_{j=1}^{n} \xi_j$, we obtain

$$(2.19)\qquad J = (-1)^{2n+1}(n+1)! \prod_{1 \leqq j < k \leqq n} (\xi_k - \xi_j) \prod_{j=1}^{n} (\xi_1 + \ldots + \xi_n + \xi_j).$$

Hence $J \not\equiv 0$ so that $P_2, \ldots, P_{n+1}$ are algebraically independent.

The $2n$ vertices of the n-dimensional cross polytope ("octahedron") β_n may be chosen to be the permutations of $(\pm 1, 0, \ldots, 0)$. β_n and γ_n have the same symmetry group denoted by C_n. The degrees of the basic invariants of C_n are given by $2k$ $(1 \leqq k \leqq n)$. Now

$$P_{2k}(x) = \sum_{j=1}^{2n} (x \cdot y_j)^{2k} = 2 \sum_{j=1}^{n} x_j^{2k} \qquad (1 \leqq k \leqq n).$$

We have just shown that the polynomials $\sum_{j=1}^{n} x_j^k$ $(1 \leqq k \leqq n)$ are algebraically independent. Substituting x_j^2 for x_j, we conclude that the polynomials $P_{2k}(x)$ $(1 \leqq k \leqq n)$ are algebraically independent.

We note that for the figures α_n, β_n, the fact that $P_{m_1+1}, \ldots, P_{m_n+1}$ are algebraically independent is equivalent to the well-known fact that the power sums $\sum_{j=1}^{n} x_j^k$ $(1 \leqq k \leqq n)$ are algebraically independent. This is no longer the case for the n-dimensional cube. We may thus say that the difficulty of the vertex problem in this case stems from the abundance of vertices of γ_n. In (**6**, pp. 266–267) the vertex problem is settled in the affirmative for $n \leqq 7$ but we have no proof yielding the result for all n.

3. The mean value problem for the exceptional directions. As shown in § 2, the solution space S to (1.2) can be completely described provided $y \notin \mathcal{M}, \mathcal{M}$ denoting the exceptional manifold. We now take up the problem of describing S when y is an exceptional direction. We do this in detail for the dihedral group D_n and for the group S_4 of the tetrahedron α_3.

The dihedral group D_n. This group has a simple description if we introduce the complex coordinate $z = x_1 + ix_2$. We then choose D_n to be the group generated by the linear transformations $z' = \zeta z$, $z' = \bar{z}$, where $\zeta = e^{2\pi i/n}$. D_n is thus the symmetry group of the regular n-gon $\{n\}$ with the vertices $re^{i(\pi/n+2k\pi/n)}$ $(0 \leqq k < n)$, $r > 0$. The mirrors for the reflections of D_n have the equations $\theta = \arg z = k\pi/n$ $(0 \leqq k < 2n)$. These lines divide the plane into $2n$ fundamental regions. The degrees of the basic invariants are 2 and n, $|z|^2$ and $\mathrm{Re}(z^n)$ forming a basic set. Thus $P_2(x_1, x_2, y) = J_1(y)(x_1^2 + x_2^2)$, $P_n(x_1, x_2, y) = a(y)(x_1^2 + x_2^2)^{\frac{1}{2}n} + J_2(y)[(x_1 - ix_2)^n + (x_1 + ix_2)^n]$, where $a(y)$ is a polynomial in y, $J_1(y)$ and $J_2(y)$ being the basic invariants introduced in Theorem 1.1. Since $J_1(y) = c(y_1^2 + y_2^2)$, where $c \neq 0$, $\mathscr{M} = \{y | J_2(y) = 0\}$. $P_n(1, i, y) = 2^n J_2(y)$, so that we conclude that $y \in \mathscr{M}$ if and only if $P_n(1, i, y) = 0$. Now $P_n(1, i, y) = \sum_{j=1}^{2n} (y_{j1} + y_{j2}i)^n$, where the vectors $y_j = (y_{j1}, y_{j2})$ $(1 \leqq j \leqq 2n)$ denote the $2n$ vectors σy $(\sigma \in D_n)$. Let $Y = y_1 + y_2 i$. The $2n$ numbers $(y_{j1} + y_{j2}i)$ $(1 \leqq j \leqq 2n)$ are identical with the $2n$ numbers $\zeta^j Y$, $\zeta^j \bar{Y}$ $(0 \leqq j < n)$. Thus

$$(3.1)\quad P_n(1, i, y) = \sum_{j=0}^{n-1} (\zeta^j Y)^n + \sum_{j=0}^{n-1} (\zeta^j \bar{Y})^n = n(Y^n + \bar{Y}^n) = 2n|Y|^n \cos n\theta,$$

where $Y = |Y|e^{i\theta}$.

The exceptional directions are thus obtained by setting $\cos n\theta = 0$. This occurs for $\theta = \pi/2n + k\pi/n$ $(0 \leqq k < 2n)$. Thus $\mathscr{M}$ consists of the angle bisectors of the $2n$ fundamental regions determined by the mirrors for the reflections of D_n. Since $\mathrm{Im}(z^n)$ is a homogeneous nth degree polynomial vanishing on the reflecting lines $\theta = k\pi/n$ $(0 \leqq k < 2n)$, we may choose $\Pi(x) = \mathrm{Im}(z^n)$. Hence $S = D[\mathrm{Im}(z^n)]$ provided $\mathrm{Re}(Y^n) \neq 0$. If y is a vertex, then $\mathrm{Re}(Y^n) = -r^n \neq 0$. The vertex of the n-gon $\{n\}$ is therefore not an exceptional direction so that the vertex problem is solved for D_n.

If $y \in \mathscr{M}$, then $\arg Y = \pi/2n + k\pi/n$ $(0 \leqq k < 2n)$. y is thus a vertex of the regular $2n$-gon $\{2n\}$ whose symmetry group is D_{2n}. Hence $S = D[\mathrm{Im}(z^{2n})]$. We summarize the above discussion in the following result.

THEOREM 4.1. *Let* D_n *be the dihedral group generated by* $z' = e^{2\pi i/n}z$, $z' = \bar{z}$. *Let* $Y = y_1 + y_2 i$. *If* $\mathrm{Re}(Y^n) \neq 0$, *then the solution space* S *to* (1.1) $(G = D_n)$ *is given by* $D[\mathrm{Im}(z^n)]$. *If* $\mathrm{Re}(Y^n) = 0$, *then* $S = D[\mathrm{Im}(z^{2n})]$.

The symmetric group S_4. The vertices of the tetrahedron α_3 may be chosen to be $(1, 1, 1)$, $(1, -1, -1)$, $(-1, -1, 1)$, $(-1, 1, -1)$. Let S_4 be the group of 24 linear transformations $x_j' = \epsilon_j x_{\sigma(j)}$ $(1 \leqq j \leqq 3)$, where $\sigma(j)$ is an arbitrary permutation of 1, 2, 3 and either all $\epsilon_j = 1$ or one ϵ_j equals 1, the other two being equal to -1. It is easily seen that S_4 is the group of symmetries of α_3. The degrees of the basic invariants of S_4 are 2, 3, 4. The basic invariants I_1, I_2, I_3 may be chosen as

$$\sum_{j=1}^{3} x_j^2, \quad x_1 x_2 x_3, \quad \sum_{1 \leqq j < k \leqq 3} x_j^2 x_k^2.$$

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A direct computation yields

$$c_1P_2(x, y) = I_1(y)I_1(x), \qquad c_2P_3(x, y) = I_2(y)I_2(x),$$

(3.2)
$$\begin{aligned} c_4P_4(x, y) &= \sum_{j=1}^{3} y_j^4 \cdot \sum_{j=1}^{3} x_j^4 + 6 \sum_{1 \leqq j<k \leqq 3} x_j^2 x_k^2 \sum_{1 \leqq j<k \leqq 3} y_j^2 y_k^2 \\ &= \left(\sum_{j=1}^{3} y_j^4\right) I_1^2(x) + \left[-2\sum_{j=1}^{3} y_j^4 + 6 \sum_{1 \leqq j<k \leqq 3} y_j^2 y_k^2\right] I_3(x), \end{aligned}$$

where the c_js are non-zero constants.

Let

$$J_1(y) = I_1(y), \quad J_2(y) = I_2(y), \quad J_3(y) = -2\sum_{j=1}^{3} y_j^4 + 6 \sum_{1 \leqq j<k \leqq 3} y_j^2 y_k^2.$$

Since $J_1(y) = \sum_{j=1}^{3} y_j^2$, we have $\mathscr{M} = \{y |\ J_2(y)J_3(y) = 0\}$. A sketch of $\mathscr{M}$ is to be found in (**9**, p. 68). We now describe the solution space S to (1.1) when $y \in \mathscr{M}$. For each fixed y, we form the ideal $\mathscr{P}_y$ generated by $P_m(x, y)$ $(1 \leqq m < \infty)$. We will obtain a finite basis for $\mathscr{P}_y$. We first prove two lemmas which will be useful in the remainder of the paper.

LEMMA 3.1. *Let S_4 be the group of symmetries of α_3 and let $P_m(x, y) = \sum_{\sigma \in S_4} (x \cdot \sigma y)^m$ $(1 \leqq m < \infty)$. Then*

$$P_m(x, y) = \sum_{|a|=m} \frac{m!}{a!} x^a J_a(y),$$

where $J_a(y) = \sum_{\sigma \in S_4} (\sigma y)^a$. If $a = (a_1, a_2, a_3)$, then $J_a(y) = 0$ unless all a_js have the same parity.

Proof.

$$P_m(x, y) = \sum_{\sigma \in S_4} (x \cdot \sigma y)^m = \sum_{\sigma \in S_4} \sum_{|a|=m} \frac{m!}{a!} x^a (\sigma y)^a = \sum_{|a|=m} \frac{m!}{a!} x^a J_a(y).$$

Let T be the group of permutations σ of (y_1, y_2, y_3). Then

(3.3)
$$J_a(y) = [1 + (-1)^{a_1+a_2} + (-1)^{a_1+a_3} + (-1)^{a_2+a_3}] \sum_{\sigma \in T} (\sigma y)^m.$$

If all the a_is do not have the same parity, then two of the three numbers $a_1 + a_2$, $a_1 + a_3$, $a_2 + a_3$ are odd and one is even. Thus $1 + (-1)^{a_1+a_2} + (-1)^{a_1+a_3} + (-1)^{a_2+a_3} = 0$, proving the lemma.

It follows from Lemma 3.1 that for odd m, $J_a(y) = 0$ unless a_1, a_2, a_3 are all odd. In this case, $x_1x_2x_3 | x^a$, $y_1y_2y_3 | J_a(y)$. We thus obtain the following result.

LEMMA 3.2. *For odd m, we have $I_2(x)I_2(y) | P_m(x, y)$.*

We now find a finite basis for $\mathscr{P}_y$; we assume throughout the discussion that $y \neq 0$. We distinguish several cases. The required computations prove to be rather lengthy and the final results are incorporated in a table at the end of this section.

Case I. $J_2(y) \neq 0, J_3(y) = 0$. A direct computation yields $(I_1(x), I_2(x)) = (P_2(x, y), P_3(x, y))$. Because of Lemma 3.2, $P_m \in (I_2) \subset (P_2, P_3)$ for m odd. Since $J_3(y) = 0$, (3.2) shows that $P_4 \in (I_1) \subset (P_2, P_3)$. Writing $P_6(x, y)$ and $P_8(x, y)$ as polynomials in $I_1(x)$, $I_2(x)$, $I_3(x)$ we obtain $P_6(x, y) \in (I_1, I_2)$ and

$$P_8(x, y) = Q(x, y) + b_0(y)I_3^2(x), \tag{3.4}$$

where $Q(x, y) \in (I_1) \subset (I_1, I_2) = (P_2, P_3)$.

We show next that $b_0(y) \neq 0$. It then follows from (3.4) that $(P_2, P_3, P_8) = (I_1, I_2, I_3^2)$. Writing out $P_m(x, y)$ as a polynomial in $I_1(x), I_2(x), I_3(x)$, we see that for $m \geqq 8$, $P_m \in (I_1, I_2, I_3^2)$. It follows that $\mathscr{P}_y = (I_1, I_2, I_3^2)$. Now $I_1(1, i, 0) = 0$, $I_3(1, i, 0) = -1$. It follows from (3.3) that $P_8(1, i, 0, y) = b_0(y)$. Using Lemma 3.1 to compute $P_8(1, i, 0, y)$ we obtain

$$\frac{b_0(y)}{16} = \sum_{j=1}^{3} y_j^8 - 14 \sum_{1\leqq j<k\leqq 3} (y_j^6 y_k^2 + y_j^2 y_k^6) + 35 \sum_{1\leqq j<k\leqq 3} y_j^4 y_k^4. \tag{3.5}$$

Suppose that $b_0(y) = 0$. Let $u_j = y_j^2$ $(1 \leqq j \leqq 3)$. Since $J_2(y) \neq 0$, $u_j > 0$ $(1 \leqq j \leqq 3)$. $J_3(y) = 0$ and $b_0(y) = 0$ become

$$\begin{aligned} &\sum_{j=1}^{3} u_j^2 - 3 \sum_{1\leqq j<k\leqq 3} u_j u_k = 0, \\ &\sum_{j=1}^{3} u_j^4 - 14 \sum_{1\leqq j<k\leqq 3} (u_j^3 u_k + u_j u_k^3) + 35 \sum_{1\leqq j<k\leqq 3} u_j^2 u_k^2 = 0, \end{aligned} \tag{3.6}$$

respectively. We solve (3.6) by introducing the new variables

$$\xi_1 = \sum_{j=1}^{3} u_j, \qquad \xi_2 = \sum_{1\leqq j<k\leqq 3} u_j u_k, \qquad \xi_3 = u_1 u_2 u_3.$$

We note that the polynomials appearing in (3.6) are symmetric in u_1, u_2, u_3. They can thus be rewritten as polynomials in ξ_1, ξ_2, ξ_3. A straightforward but rather lengthy computation shows that (3.6) becomes transformed into

$$\xi_1^2 - 5\xi_2 = 0, \qquad \xi_1^4 - 18\xi_1^2\xi_2 - 52\xi_1\xi_3 + 65\xi_2^2 = 0. \tag{3.7}$$

Substituting the first equation of (3.7) into the second, we obtain $\xi_1\xi_3 = 0$. Since $\xi_3 = u_1u_2u_3 \neq 0$, we have $\xi_1 = 0$. (3.7) then implies that $\xi_2 = 0$. u_1, u_2, u_3 are the three roots of the equation $u^3 - \xi_1 u^2 + \xi_2 u - \xi_3 = 0$ so that $u_j^3 = \xi_3$ $(1 \leqq j \leqq 3)$. This is impossible since ξ_3 must have complex cubic roots. Thus $b_0(y) \neq 0$ and $\mathscr{P}_y = (I_1, I_2, I_3^2)$.

We observe that $\xi_1^2 - 5\xi_2 = 0$, $\xi_3 = 0$ yield a solution to (3.7). Thus $J_2(y) = 0$ and $J_3(y) = 0$ imply that $b_0(y) = 0$. We use this fact later on in Case III.

Case II. $J_2(y) = 0, J_3(y) \neq 0$. It follows from (3.2) that $(P_2, P_4) = (I_1, I_3)$ and that $P_3 = 0$. Since $J_2(y) = 0$, Lemma 3.2 implies that $P_m(x, y) = 0$ for m odd. Now $P_6(x, y) = R(x, y) + b_1(y)I_2^2(x)$, where $R(x, y) \in (I_1)$ and $b_1(y)$ is a polynomial in y. If $b_1(y) \neq 0$, then $(P_2, P_4, P_6) = (I_1, I_2^2, I_3)$. Writing $P_m(x, y)$ as a polynomial in $I_1(x)$, $I_2(x)$, $I_3(x)$, we observe that for $m \geqq 6$, we have $P_m \in (I_1, I_2^2, I_3)$. It follows that $\mathscr{P}_y = (I_1, I_2^2, I_3)$. Since $I_1(\sqrt{2}, i, i) = 0$, $I_2(\sqrt{2}, i, i) = -\sqrt{2}$, we have $P_6(\sqrt{2}, i, i, y) = 2b_1(y)$. Using Lemma 3.1 to compute $P_6(x, y)$, we obtain

$$(3.8)\quad \frac{1}{4}P_6(x, y) = 2\sum_{j=1}^{3} x_j^6 \cdot \sum_{j=1}^{3} y_j^6 + \frac{6!}{4!\,2!} \sum_{1 \leqq j<k \leqq 3} (x_j^4 x_k^2 + x_k^4 x_j^2) \cdot \sum_{1 \leqq j<k \leqq 3} (y_j^4 y_k^2 + y_k^4 y_j^2) + 6\frac{6!}{2!\,2!\,2!} x_1^2 x_2^2 x_3^2 y_1^2 y_2^2 y_3^2,$$

$$(3.9)\quad \frac{1}{24}P_6(\sqrt{2}, i, i, y) = 2\sum_{j=1}^{3} y_j^6 - 15 \sum_{1 \leqq j<k \leqq 3} (y_j^4 y_k^2 + y_k^4 y_j^2) + 180 y_1^2 y_2^2 y_3^2.$$

Suppose now that $b_1(y) = 0$. Since $J_2(y) = 0$, $y_j = 0$ for some j $(1 \leqq j \leqq 3)$. If $y_3 = 0$, then we conclude from (3.9) that

$$(3.10)\quad 2(y_1^6 + y_2^6) - 15(y_1^4 y_2^2 + y_1^2 y_2^4) = 0.$$

Set $z = y_2^2/y_1^2$. (3.10) then becomes

$$(3.11)\quad (z + 1)(2z^2 - 17z + 2) = 0$$

so that $y_2/y_1 = \pm\sqrt{(\frac{1}{4}(17 \pm \sqrt{273}))} = \pm \frac{1}{4}(\sqrt{42} \pm \sqrt{26})$. Setting in turn $y_2 = 0$ and $y_1 = 0$ we find that the common solutions to $b_1(y) = 0$, $J_2(y) = 0$ are given by the 24 rays $y_{\sigma(1)} = t$, $y_{\sigma(2)} = \pm \frac{1}{4}(\sqrt{42} \pm \sqrt{26})t$, $y_{\sigma(3)} = 0$, where $t > 0$ and $\sigma(j)$ denotes an arbitrary permutation of 1, 2, 3. Let L_1 denote the union of these rays. We have shown that $\mathscr{P}_y = (I_1, I_2^2, I_3)$ if $J_2(y) = 0, J_3(y) \neq 0, y \notin L_1$.

We now investigate the case $y \in L_1$. A direct computation shows that $J_3(y) \neq 0$ for $y \in L_1$ so that we have again $(P_2, P_4) = (I_1, I_3)$. Since $b_1(y) = 0$, we have $P_6 \in (I_1) \subset (I_1, I_3)$. Writing out $P_8(x, y)$, $P_{10}(x, y)$, and $P_{12}(x)$ as polynomials in $I_1(x)$, $I_2(x)$, and $I_3(x)$, we find that $P_m(x, y) \in (I_1, I_3)$ for $m = 8$, 10 and $P_{12}(x, y) = S(x, y) + b_2(y)I_2^4(x)$, where $S(x, y) \in (I_1, I_3)$ and $b_2(y)$ is a polynomial in y. We show that $b_2(y) \neq 0$ for $y \in L_1$. Thus $(P_2, P_4, P_{12}) = (I_1, I_2^4, I_3)$ for $y \in L_1$. Writing out $P_m(x, y)$ as a polynomial in $I_1(x)$, $I_2(x)$, $I_3(x)$, we find that for $m \geqq 12$, $P_m \in (I_1, I_2^4, I_3)$. It thus follows that for $y \in L_1$, $\mathscr{P}_y = (I_1, I_2^4, I_3)$. Let $\zeta = e^{\frac{1}{3}\pi i}$. Since $I_1(1, \zeta, \zeta^2) = I_3(1, \zeta, \zeta^2) = 0$, $I_2(1, \zeta, \zeta^2) = -1$, we have $P_{12}(1, \zeta, \zeta^2, y) = b_2(y)$. Since

$y \in L_1, y_j = 0$ for some j $(1 \leqq j \leqq 3)$. Suppose that $y_3 = 0$. Using Lemma 3.1 we have

$$(3.12) \quad \tfrac{1}{4}P_{12}(x_1, x_2, x_3, y_1, y_2, 0) = 2(x_1^{12} + x_2^{12} + x_3^{12})(y_1^{12} + y_2^{12}) + \binom{12}{2} \sum_{1 \leqq j < k \leqq 3} (x_j^{10}x_k^{12} + x_k^{10}x_j^{2}) \cdot (y_1^{10}y_2^{2} + y_1^{2}y_2^{10}) + \binom{12}{4} \sum_{1 \leqq j < k \leqq 3} (x_j^{8}x_k^{4} + x_k^{8}x_j^{4}) \cdot (y_1^{8}y_2^{4} + y_1^{4}y_2^{8}) + 2\binom{12}{6} \sum_{1 \leqq j < k \leqq 3} x_j^{6}x_k^{6} \cdot y_1^{6}y_2^{6}.$$

Hence $b_2(y_1, y_2, 0) = 0$ becomes

$$(3.13) \quad 2(y_1^{12} + y_2^{12}) - \binom{12}{2}(y_1^{10}y_2^{2} + y_1^{2}y_2^{10}) - \binom{12}{4}(y_1^{8}y_2^{4} + y_1^{4}y_2^{8}) + 2\binom{12}{6}y_1^{6}y_2^{6} = 0.$$

Since $(y_1, y_2, 0) \in L_1$, (3.10) holds. We claim that these two equations have no common solutions. It follows that $b_2(y) \neq 0$ for $(y_1, y_2, 0) \in L_1$. The same reasoning holds if we assume that $y_1 = 0$ or $y_2 = 0$, so that $b_2(y) \neq 0$ for $y \in L_1$. To see that (3.10) and (3.13) are incompatible, we let $z = y_2^2/y_1^2$. (3.10) and (3.13) then become (3.11) and

$$(3.14) \quad 2(z^6 + 1) - \binom{12}{2}(z^5 + z) - \binom{12}{4}(z^4 + z^2) + 2\binom{12}{6}z^3 = 0,$$

respectively. It suffices to show that (3.14) and $2z^2 - 17z + 2 = 0$ have no common root, which is equivalent to showing that $2z^2 - 17z + 2$ does not divide the polynomial in (3.13). This can be done by a direct computation, which we omit.

Case III. $J_2(y) = J_3(y) = 0$. Since $J_2(y) = 0, y_j = 0$ for some j $(1 \leqq j \leqq 3)$. Assume that $y_3 = 0$. Then the two equations $J_2(y) = 0$ and $J_3(y) = 0$ can be solved by setting $z = y_2^2/y_1^2$. We find that $y_2/y_1 = \pm\tau^{-1}$. Similar calculations may be carried out for $y_1 = 0$ and $y_2 = 0$. The common solutions to $J_2(y) = 0$, $J_3(y) = 0$ are then given by the 24 rays $y_{\sigma(1)} = t$, $y_{\sigma(2)} = \pm\tau^{\pm 1}t$, $y_{\sigma(3)} = 0$, where $t > 0$ and $\sigma(j)$ denotes any permutation of 1, 2, 3. Let L_2 denote the union of these rays, so that $y \in L_2$.

Since $J_2(y) = 0$, Lemma 3.2 implies that $P_m(x, y) = 0$ for m odd. Equations (3.2) show that $P_4(x, y) \in (I_2)$. Now $P_6(x, y) = R(x, y) + b_1(y)I_2^2(x)$. It is seen by inspection that the rays in L_1 are distinct from those in L_2. Since L_1 is the solution to $J_2(y) = b_1(y) = 0$, we must have $b_1(y) \neq 0$ for $y \in L_2$. Thus $(P_2, P_6) = (I_1, I_2^2)$. Using (3.4) and (3.5) we have $P_8 \in (I_1) \subset (I_1, I_2^2)$ since $J_2(y) = 0$ and $J_3(y) = 0$ imply that $b_0(y) = 0$. Writing out P_{10} and P_{12} as polynomials in $I_1(x)$, $I_2(x)$, $I_3(x)$, we find that $P_{10} \in (I_1, I_2^2)$, $P_{12}(x, y) = T(x, y) + b_3(y)I_3^3(x)$, where $T \in (I_1, I_2^2)$. We show that $b_3(y) \neq 0$ so that

$(P_2, P_6, P_{12}) = (I_1, I_2^2, I_3^3)$. Writing P_m as a polynomial in I_1, I_2, I_3, we find that for $m \geqq 12$ we have $P_m \in (I_1, I_2^2, I_3^3)$. Thus $\mathscr{P}_y = (I_1, I_2^2, I_3^3)$ for $y \in L_2$.

Since $I_1(1, i, 0) = I_2(1, i, 0) = 0$ and $I_3(1, i, 0) = -1$, we have

$$P_{12}(1, i, 0, y) = -b_3(y).$$

Let $(y_1, y_2, 0) \in L_2$. Using (3.12) we have

$$(3.15)\quad \tfrac{1}{4}P_{12}(1, i, 0, y_1, y_2, 0) = 4(y_1^{12} + y_2^{12}) - 2\binom{12}{2}(y_1^{10}y_2^{2} + y_2^{10}y_1^{2}) + 2\binom{12}{4}(y_1^{8}y_2^{4} + y_1^{4}y_2^{8}) - 2\binom{12}{6}y_1^{6}y_2^{6}.$$

Let $z = y_2^2/y_1^2$. $J_3(y_1, y_2, 0) = 0$ and $b_3(y) = 0$ become

$$(3.16)\qquad z^2 - 3z + 1 = 0$$

and

$$(3.17)\qquad 2(z^6 + 1) - \binom{12}{2}(z^5 + z) + \binom{12}{4}(z^4 + z^2) - \binom{12}{6}z^3 = 0,$$

respectively.

$J_3(y_1, y_2, 0) = 0$ and $b_3(y) = 0$ will be incompatible provided that (3.16) and (3.17) have no common root. However, (3.16) and (3.17) have a common root if and only if $z^2 - 3z + 1$ divides the polynomial of (3.17). A direct computation, which we omit here, shows that this is not so. Hence, if $(y_1, y_2, 0) \in L_2$, then $b_3(y) \neq 0$. Reasoning in a similar fashion for $y_1 = 0$ and $y_2 = 0$, we see that $b_3(y) \neq 0$ for $y \in L_2$. Thus for $y \in L_2$, we have $\mathscr{P}_y = (I_1, I_2^2, I_3^3)$.

We observe that in all cases $\mathscr{P}_y = (Q_1, Q_2, Q_3)$, where Q_j $(1 \leqq j \leqq 3)$ is homogeneous and $Q_1(x) = Q_2(x) = Q_3(x) = 0$ if and only if $x = 0$. Furthermore, S is the solution space to the system $Q_j(\partial/\partial x)f = 0$ $(1 \leqq j \leqq 3)$. It follows from a result in (**6**, the corollary to Theorem 2.2) that S is a finite-dimensional space spanned by homogeneous polynomials and $\dim S = k_1k_2k_3$, $\deg S = \sum_{j=1}^{3}(k_j - 1)$, where $\deg Q_j = k_j$. $\dim S$ denotes the dimension of S while $\deg S$ denotes the maximum degree of the polynomials in S. We summarize our results in the following table:

Case	ideal $\mathscr{P}_y$	dim S	deg S
$J_2(y) \neq 0$, $J_3(y) \neq 0$	(I_1, I_2, I_3)	24	6
$J_2(y) \neq 0$, $J_3(y) = 0$	(I_1, I_2, I_3^2)	48	10
$J_2(y) = 0$, $J_3(y) \neq 0$, $y \notin L_1$	(I_1, I_2^2, I_3)	48	9
$J_2(y) = 0$, $J_3(y) \neq 0$, $y \in L_1$	(I_1, I_2^4, I_3)	96	15
$J_2(y) = 0$, $J_3(y) = 0$	(I_1, I_2^2, I_3^3)	144	17

We note that if $J_2(y) \neq 0$, $J_3(y) \neq 0$, then $S = D\Pi$, where

$$\Pi = \prod_{1 \leqq j < k \leqq 3} (x_j^2 - x_k^2).$$

It is easily checked that $J_2(1, 0, 0) = 0$, $J_3(1, 0, 0) \neq 0$, $(1, 0, 0) \notin L_1$. Thus $\mathscr{P}_{(1,0,0)} = \mathscr{P}_y$ for $J_2(y) = 0$, $J_3(y) \neq 0$, $y \notin L_1$. The orbit of $(1, 0, 0)$ under G is given by the permutations of $(\pm 1, 0, 0)$, these being the vertices of the octahedron β_3. It follows from the vertex problem that in this case $S = D\Pi$, where

$$\Pi = x_1 x_2 x_3 \prod_{1 \leqq j < k \leqq 3} (x_j^2 - x_k^2).$$

It is not known whether a similar characterization of S can be given in the remaining cases.

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