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Group Actions, Cyclic Coverings and Families of K3-Surfaces

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Abstract. In this paper we describe six pencils of $K3$-surfaces which have large Picard number ($\rho = 19, 20$) and each contains precisely five special fibers: four have A-D-E singularities and one is non-reduced. In particular, we characterize these surfaces as cyclic coverings of some $K3$-surfaces described in a recent paper by Barth and the author. In many cases, using 3-divisible sets, resp., 2-divisible sets, of rational curves and lattice theory, we describe explicitly the Picard lattices.

Introduction

Recently, using various methods (toric geometry, mirror symmetry, etc.), many families of $K3$-surfaces with large Picard number and small number of special fibers have been constructed and studied (see [D, VY, Be]). Here we use group actions and cyclic coverings to describe six new families where the generic surface has Picard number 19 and we identify four surfaces with Picard number 20. These six pencils are related to three families of K3-surfaces studied by Barth and the author [BS]: the generic surface has Picard number 19 and the pencils contain four surfaces with singularities of $A-D-E$ type and $\rho = 20$ and one non-reduced fiber. The families arise as minimal resolutions of quotients X^n_λ/G_n where G_n is a special finite subgroup of $SO(4,\mathbb{R})$ containing the Heisenberg group and $\{X^n_\lambda\}_{\lambda\in\mathbb{P}_1}$ is a G_n-invariant pencil of surfaces in $\mathbb{P}_3$, the latter are described in [S1] (we recall some facts in Section 1). In Sections 1 and 2 we describe six normal subgroups H of G_n which contain the Heisenberg group, describe the fixed points of H on X^n_λ, and show that the minimal resolutions are pencils of K3-surfaces which contain five special surfaces. In Section 3 we show that the new families are certain cyclic coverings of the surfaces of [BS]. Then, by a classical result of Inose, [I, Cor. 1.2], they have the same Picard number, hence the general surface in each of the six pencils has Picard number 19 and we have four surfaces with Picard number 20. In Section 4 by using the rational curves on the minimal resolutions and 2-divisible and 3-divisible sets of rational curves, we describe completely the Picard lattice of many of the surfaces.

1 Notations and Preliminaries

There are two classical 2 : 1 coverings:

$$SU(2) \to SO(3,\mathbb{R}) \quad \text{and} \quad \sigma\colon SU(2)\times SU(2) \to SO(4,\mathbb{R}).$$

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Denote by $T, O \subset SO(3,\mathbb{R})$, the rotation groups of the tetrahedron and octahedron, by $\widetilde{T}, \widetilde{O}$ the corresponding binary subgroups of $SU(2)$, and let $G_6 := \sigma(\tilde{T} \times \tilde{T})$, $G_8 := \sigma(\tilde{O} \times \tilde{O})$. We denote an element of $SU(2) \times SU(2)$ and its image in $SO(4,\mathbb{R})$ by (p_1, p_2). Let $X_\lambda^6 = s_6 + \lambda q^3$ and $X_\lambda^8 = s_8 + \lambda q^4$ denote the pencils of G_6- and G_8-invariant surfaces in $\mathbb{P}_3$, which are described in [S1]; s_6 denotes a G_6-invariant homogeneous polynomial of degree six and s_8 denotes a G_8-invariant homogeneous polynomial of degree eight; and $q := x_0^2 + x_1^2 + x_2^2 + x_3^2$ is the equation of the quadric $\mathbb{P}_1 \times \mathbb{P}_1$ in $\mathbb{P}_3$. The base locus of the pencils X_λ^n are $2n$ lines on the quadric with n in each ruling, and each pencil contains exactly four nodal surfaces, *cf.* [S1].

Now recall the matrix:

$$C := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in O(4,\mathbb{R}),$$

which operates on an element $(p_1, p_2) \in G_1 \times G_2$ by:

$$C^{-1}(p_1, p_2)C = (p_2, p_1).$$

Moreover we specify the following matrices of $SO(4,\mathbb{R})$:

$$(q_1, 1) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (q_2, 1) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$(p_3, 1) = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad (p_4, 1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Using these matrices the groups have the following generators:

Group	Generators
G_6	$(q_2, 1), (1, q_2), (p_3, 1), (1, p_3)$
G_8	$(q_2, 1), (1, q_2), (p_3, 1), (1, p_3), (p_4, 1), (1, p_4)$

Denote by PG the image of a subgroup $G \subset SO(4,\mathbb{R})$ in $\mathbb{P}\,GL(4,\mathbb{R})$. We define the *types* of lines in $\mathbb{P}_3$ which are fixed by elements $(p_1, p_2) \in PG$ of order 2, 3 or 4 in the following way:

order	2	3	4
type	M	N	R

1.1 Normal Subgroups

In [S2] the author classifies all the subgroups of $SO(4,\mathbb{R})$ which contain the Heisenberg group $V \times V$. Here we consider all the normal subgroups of G_6 and of G_8 which contain the subgroup $V \times V$, resp. G_6. We denote by H such a normal subgroup, by $o(H)$ its order and by $i(H) = [G_n : H]$ the index of H in G_n. We list below all the groups H and their generators, following the notation of [S2]. Moreover, we do not consider separately the groups H and $C^{-1}HC$ or, in general, groups which are conjugate in $O(4,\mathbb{R})$. The group $T \times T$ is in fact the same as G_6, but to avoid confusion we use this notation when we consider it as subgroup of G_8.

$H \subset G_6$	Generators	$o(H)$	$i(H)$	$H \subset G_8$	Generators	$o(H)$	$i(H)$
$T \times V$	$(q_1,1),(1,q_1)$ $(p_3,1)$	96	3	$O \times T$	$(q_1,1),(1,q_1)$ $(p_3,1),(1,p_3)$ $(p_4,1)$	576	2
$(TT)'$	$(q_1,1),(1,q_1)$ $(q_2,1),(1,q_2)$ (p_3,p_3)	96	3	$(OO)''$	$(q_1,1),(1,q_1)$ $(p_3,1),(1,p_3)$ (p_4q_2,p_4q_2)	576	2
$V \times V$	$(q_1,1),(1,q_1)$ $(q_2,1),(1,q_2)$	32	9	$T \times T$	$(q_1,1),(1,q_1)$ $(p_3,1),(1,p_3)$	288	4

1.2 Fixed Points

We analyze the different kind of fixed points for elements of the subgroups $PH \subset PG$ in the same way as in [BS]. Recall that the elements of the form $(p,1)$ or $(1,p)$ have each two disjoint lines of fixed points contained in one ruling, respectively, in the other ruling of the quadric (*cf.* [S1, 5.4, p. 439]).

Fixed Points on the Quadric: The subgroups $G_1 \times 1$ and $1 \times G_2$ of PH operate on the two rulings of the quadric and determine orbits of lines. We give the lengths of the orbits in the following tables. In the first row we write the order of the element which fixes two lines of the orbit:

Order of $(p,1)$	2	3	4
$T \times V$	6	4, 4	–
$O \times T$	12	8	6
$(TT)'$	6	–	–
$(OO)''$	6	8	–
$V \times V$	2, 2, 2	–	–
$T \times T$	6	4, 4	–

Order of $(1,p)$	2	3	4
$T \times V$	2, 2, 2	–	–
$O \times T$	6	4, 4	–
$(TT)'$	6	–	–
$(OO)''$	6	8	–
$V \times V$	2, 2, 2	–	–
$T \times T$	6	4, 4	–

In particular observe that in the case of the groups $(TT)'$ and $(OO)''$ the meeting points of the fixed lines of the two rulings of $\mathbb{P}_1 \times \mathbb{P}_1$ split into three orbits of length 12 and two orbits of length 32, in the other cases these meeting-points form just one orbit.

Fixed Points Off the Quadric: We denote by F_L the fixed group of a line L of $\mathbb{P}_3$ in PH and by H_L the stabilizer group of L in PH, *i.e.*,

$$F_L := \{h \in PH \ s.t.\ hx = x \ for\ all\ x \in L\}$$
$$H_L := \{h \in PH \ s.t.\ hL = L\}.$$

Moreover, denote by $\ell(L)$ the length of the H-orbit of the line L and by g a representative of a conjugacy class in H:

Group	$T \times V$			$(TT)'$				$V \times V$
g	(q_1, q_1)	(q_1, q_2)	(q_1, q_3)	(q_1, q_1)	(q_1, q_2)	(q_1, q_3)	(p_3, p_3)	(q_i, q_j)
F_L	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$
type	M_1	M_2	M_3	M_1	M_2	M_3	N	M_{ij}
$\ell(L)$	6	6	6	6	6	6	16	2
$\lvert H_L\rvert/\lvert F_L\rvert$	4	4	4	4	4	4	1	4

Here we denote by $q_3 \in SU(2)$ the product of q_1 and q_2. In the last column of the table the sum runs over $i, j = 1, 2, 3$. In this case we have nine distinct conjugacy classes with just one element.

Group	$O \times T$			$(OO)''$			
g	(q_1, q_1)	(p_3, p_3)	(p_4q_2, q_2)	(p_4, p_4)	(p_3, p_3)	(p_3^2, p_3)	(p_4q_2, p_4q_2)
F_L	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_2$
type	M	N	M'	R	N	N'	M
$\ell(L)$	18	32	36	18	16	16	72
$\lvert H_L\rvert/\lvert F_L\rvert$	8	3	3	4	8	8	2

Group	$T \times T$		
g	(q_2, q_2)	(p_3, p_3)	(p_3^2, p_3)
F_L	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_3$
Type	M	N	N'
$\ell(L)$	18	16	16
$\lvert H_L\rvert/\lvert F_L\rvert$	4	3	3

Remark 1.1 By taking the generator (p_3^2, p_3) for $(TT)'$ instead of (p_3, p_3) we find a group $(TT)''$ which is conjugate in $O(4, \mathbb{R})$ to $(TT)'$. The description of the fixed points is similar to that in the case of $(TT)'$.

2 Quotient Surfaces

2.1 Quotient Singularities

We now consider the projections:

$$\pi_H \colon X_\lambda^6 \longrightarrow X_\lambda^6/H, \qquad \pi_{H'} \colon X_\lambda^8 \longrightarrow X_\lambda^8/H'$$

with $H = T \times V$, $(TT)'$ or $V \times V$; $H' = O \times T$, $(OO)''$ or $T \times T$. In this section we run the same program as in [BS, §3] and describe the singularities of the quotients (for the details *cf.* [BS])

Fixed Lines on q: The image in the quotient of the lines of the base locus of the pencils X^6_λ and X^8_λ and of the intersection points of the lines of the base locus are smooth. Observe that the points of intersection of the lines of the base locus of the pencils form one orbit under the action of $T \times V$, $V \times V$, $O \times T$ and $T \times T$. In the case of the groups $(TT)'$ we have three orbits and in the case of the group $(OO)''$ we have two orbits, as described in §1.2, this means that the lines in the quotient will meet three times and two times. Now we consider the points of intersection of the lines of the base locus with the other fixed lines on q. In the table below we do not write the groups $(TT)'$ and $V \times V$ because they do not have fixed points on q other than the lines of the base locus. We denote by $\mathrm{Fix}(P)$ the fix-group in PG of a point P. In the next table we write the length and the number of orbits of fixed points, and we describe which kind of singularities we have in the quotient:

Group	$T \times V$	$O \times T$			$(OO)''$		$T \times T$	
$\mathrm{Fix}(P)$	$\mathbb{Z}_3 \times \mathbb{Z}_2$	$\mathbb{Z}_3 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_3$	$\mathbb{Z}_2 \times \mathbb{Z}_3$	$\mathbb{Z}_2 \times \mathbb{Z}_3$	$\mathbb{Z}_3 \times \mathbb{Z}_2$	$\mathbb{Z}_3 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_3$
Length	8	48	24	48	48	48	24	24
Number	6	1	2	2	1	1	2	2
Sing.	$6A_2$	$1A_1$	$2A_3$	$2A_1$	$1A_1$	$1A_1$	$2A_1$	$2A_1$

Fixed Lines Off q: Denote by $o(L)$ the order of the fix-group F_L of L. The number of points not on q cut out on X^n_λ by L is:

Group	$T \times V$	$(TT)'$		$V \times V$	$O \times T$		$(OO)''$		$T \times T$	
$o(L)$	2	2	3	2	2	3	4	3	2	3
Number	4	4	6	4	8	6	8	6	8	6

In the next table we show in each case length and number of H_L-orbits, the number and type(s) of the quotient singularity(ies):

Group	$T \times V$			$(TT)'$				$V \times V$
$o(L)$	2	2	2	2	2	2	3	2
Type	M_1	M_2	M_3	M_1	M_2	M_3	N	M_{ij}
Length	4	4	4	4	4	4	1	4
Number	1	1	1	1	1	1	6	1
Singularities	A_1	A_1	A_1	A_1	A_1	A_1	$6A_2$	A_1

Group	$O \times T$			$(OO)''$				$T \times T$		
$o(L)$	2	3	2	4	3	3	2	2	3	3
Type	M	N	M'	R	N	N'	M	M	N	N'
Length	8	3	3	4	6	6	2	4	3	3
Number	1	2	2	2	1	1	4	2	2	2
Singularities	A_1	$2A_2$	$2A_1$	$2A_3$	A_2	A_2	$4A_1$	$2A_1$	$2A_2$	$2A_2$

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The Singular Surfaces: We denote by ns the number of nodes on the surfaces and by F the fix-group of a node in H. In the table below, we give the number of orbits of nodes and their fix-groups in PH, PH' and we describe the singularities in the quotient. We recall [BS, Proposition 3.1]:

Proposition 2.1 *Let X be a nodal surface with $F \subset SO(3)$ the fix-group of the node. Then the image of this node on X/H is a quotient singularity locally isomorphic with $\mathbb{C}^2/\tilde{F}$, where $\tilde{F} \subset SU(2)$ is the binary group which corresponds to F.*

Group	$T \times V$				$(TT)'$				$V \times V$			
λ	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
ns	12	48	48	12	12	48	48	12	12	48	48	12
Orbit	1	1	1	1	3	3	1	1	3	3	3	3
F	$\mathbb{Z}_2 \times \mathbb{Z}_2$	id	id	$\mathbb{Z}_2 \times \mathbb{Z}_2$	T	$\mathbb{Z}_3$	id	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	id	id	$\mathbb{Z}_2 \times \mathbb{Z}_2$
Lines	$1M_1$	–	–	$1M_1$	$3M_i$	$1N$	–	$1M_1$	$3M_{ij}$	–	–	$3M_{ij}$
Meeting	$1M_2$			$1M_2$	$4N$			$1M_2$				
	$1M_3$			$1M_3$				$1M_3$				
Sing.	D_4	A_1	A_1	D_4	$3E_6$	$3A_5$	A_1	D_4	$3D_4$	$3A_1$	$3A_1$	$3D_4$

Group	$O \times T$				$(OO)''$				$T \times T$			
λ	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
ns	24	72	144	96	24	72	144	96	24	72	144	96
Orbit	1	1	1	1	2	1	1	2	2	1	1	2
F	T	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	O	$\mathbb{Z}_4$	$\mathbb{Z}_2$	D_3	T	$\mathbb{Z}_2$	id	$\mathbb{Z}_3$
Lines	$3M$	$1M$	$1M'$	$1N$	$3R$	$1R$	$1M$	$1N(N')$	$3M$	$1M$	–	$1N(N')$
Meeting	$4N$	$2M'$			$4N(N')$			$3M$	$4N(N')$			
					$6M$							
Sing.	E_6	D_4	A_3	A_5	$2E_7$	A_7	A_3	$2D_5$	$2E_6$	A_3	A_1	$2A_5$

2.2 Rational Curves

Let

$$\mu \colon Y_{\lambda,H} \longrightarrow X^n_\lambda/H$$

be the minimal resolution of the singularities of X^n_λ/H. In the following table we give the number of rational curves coming from the curves of the base locus of X^n_λ (denote it by ν_1) and from the resolution of the singularities. The latter are of three kinds: those coming from the intersection points of the lines of the base locus with other fixed lines on q, those coming from fixed points which are off q and do not come from nodes of X^n_λ, and those coming from the nodes. We denote their numbers by ν_2, ν_3 and ν_4; then the total number of rational curves is $\nu := \nu_1 + \nu_2 + \nu_3 + \nu_4$. The configurations of some of the curves are then given in the figures in Section 5. In the table we write the discriminant d of the intersection matrix, too. This is easy to compute because we know the configurations of the rational curves. Since in each case $d \neq 0$, the classes of the curves are independent in $NS(Y_{\lambda,H})$.

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The Smooth X^n_λ***:***

Group	$T\times V$	$(TT)'$	$V\times V$	$O\times T$	$(OO)''$	$T\times T$
ν_1	4	2	6	3	2	4
ν_2	12	–	–	9	2	4
ν_3	3	15	9	7	14	10
ν	19	17	15	19	18	18
d	$2^5\cdot 3^3\cdot 5$	$2^3\cdot 3^6\cdot 5$	$2^{13}\cdot 5$	$2^5\cdot 3^3\cdot 7$	$-2^8\cdot 3^2\cdot 7$	$-2^2\cdot 3^6\cdot 7$

The Singular X^n_λ***:*** In this case the surfaces X^n_λ do not have extra singularities on q, hence the number ν_1 and ν_2 remain the same as above and we do not write them again.

Group	$T\times V$				$(TT)'$			
λ	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
ν_3	–	3	3	–	–	3	15	12
ν_4	4	1	1	4	18	15	1	4
ν	20	20	20	20	20	20	18	18
d	$-2^4\cdot 3^3\cdot 5$	$-2^6\cdot 3^3\cdot 5$	$-2^6\cdot 3^3\cdot 5$	$-2^4\cdot 3^3\cdot 5$	$-3^3\cdot 5$	$-2^6\cdot 3^3\cdot 5$	$-2^4\cdot 3^6\cdot 5$	$-2^2\cdot 3^6\cdot 5$

Group	$V\times V$				$O\times T$			
λ	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
ν_3	–	9	9	–	2	4	5	3
ν_4	12	3	3	12	6	4	3	5
ν	18	18	18	18	20	20	20	20
d	$-2^{10}\cdot 5$	$-2^{16}\cdot 5$	$-2^{16}\cdot 5$	$-2^{10}\cdot 5$	$-2^4\cdot 3^2\cdot 7$	$-2^4\cdot 3^3\cdot 7$	$-2^5\cdot 3^3\cdot 7$	$-2^6\cdot 3^2\cdot 7$

Group	$(OO)''$				$T\times T$			
λ	λ_1	λ_2	λ_3	λ_4	λ_1	λ_2	λ_3	λ_4
ν_3	–	8	12	6	–	8	10	2
ν_4	16	7	3	10	12	3	1	10
ν	20	19	19	20	20	19	19	20
d	$-2^4\cdot 7$	$2^7\cdot 3^2\cdot 7$	$2^8\cdot 3^2\cdot 7$	$-2^8\cdot 7$	$-3^4\cdot 7$	$2^2\cdot 3^6\cdot 7$	$2^3\cdot 3^6\cdot 7$	$-2^4\cdot 3^4\cdot 7$

2.3 K3-Surfaces

Since the groups H and H' contain the subgroups $V\times V$ of G_6, resp., $T\times T$ of G_8, the projections π_H and $\pi_{H'}$ are ramified on the lines of the base locus of the families X^n_λ with ramification index two and three. By using Hurwitz' formula and the fact that in each case the previous rational curves are independent in the Neron–Severi group, the same computation as in [BS, section 5] shows that the minimal resolutions of the quotients are $K3$-surfaces, a direct proof of this fact is given in the next section.

3 Cyclic Coverings

We give another description of the pencils of K3-surfaces by using cyclic coverings. We consider the pairs G_n and H so that G_n/H is cyclic, in our cases either $|G_n/H| = 3$ or $|G_n/H| = 2$, and we consider the map:

$$\pi\colon X_\lambda^n/H \longrightarrow X_\lambda^n/G_n$$

3.1 The General Case

For the moment assume that X_λ^n is smooth. The group G_n/H acts on the points of the fiber $\pi^{-1}(P)$. If the point P is not fixed by G_n/H, then the map is 3:1 or 2:1 there. If P is fixed by G_n/H then we have a singularity on X_λ^n/H, more precisely an A_2 or an A_1, now the fiber $\pi^{-1}(P)$ is one point and the map has multiplicity 2 or 3 there (*cf.* [M2, Lemma 3.6 p. 80]). We have a rational map between the minimal resolutions of X_λ^n/H and X_λ^n/G_n:

$$\gamma\colon Y_{\lambda,H} \dashrightarrow Y_{\lambda,G_n}$$

which is 3:1 or 2:1. Observe that this map is not defined over the (-2)-rational curves in the blow up of singular points of X_λ^n/G_n that come from fixed points of G_n/H on X_λ^n/H. The surfaces $Y_{\lambda,H}$ are $K3$-surfaces as well and by [I, Cor. 1.2] these have the same Picard number $\rho(Y_{\lambda,H}) = \rho(Y_{\lambda,G_n}) = 19$.

In this section we describe the map γ by using cyclic coverings. For the general theory about 2-cyclic coverings and 3-cyclic coverings we return to the article by Nikulin [N] and to articles by Miranda [M1] and Tan [T]. For the convenience of the reader, in Section 5, Figure 1, we recall the configurations of (-2)-rational curves on the smooth surfaces Y_{λ,G_6} and on Y_{λ,G_8} given in [BS]. By [BS, Proposition 6.1] the following classes are 3-divisible in $NS(Y_{\lambda,G_6})$:

$$\mathcal{L} := L_1 - L_2 + L_4 - L_5 + N_1 - N_2 + N_3 - N_4 + N_5 - N_6 + N_7 - N_8,$$
$$\mathcal{L}' := L_1' - L_2' + L_4' - L_5' + N_1 - N_2 + N_3 - N_4 - N_5 + N_6 - N_7 + N_8,$$

and also:

$$\mathcal{L} - \mathcal{L}' = L_1 - L_2 + L_4 - L_5 - L_1' + L_2' - L_4' + L_5' + 2(N_5 - N_6 + N_7 - N_8),$$
$$\mathcal{L} + \mathcal{L}' = L_1 - L_2 + L_4 - L_5 + L_1' - L_2' + L_4' - L_5' + 2(N_1 - N_2 + N_3 - N_4).$$

Making reduction modulo three we find the classes:

$$\mathcal{M} := L_1 - L_2 + L_4 - L_5 - L_1' + L_2' - L_4' + L_5' - (N_5 - N_6 + N_7 - N_8),$$
$$\mathcal{M}' := L_1 - L_2 + L_4 - L_5 + L_1' - L_2' + L_4' - L_5' - (N_1 - N_2 + N_3 - N_4).$$

In $NS(Y_{\lambda,G_8})$ the following classes are 2-divisible:

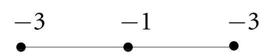
$$\mathcal{L} := L_1 + L_3 + L_5 + M_1 + M_3 + M_4 + R_1 + R_3,$$
$$\mathcal{L}' := L_1' + L_3' + L_5' + M_2 + M_3 + M_4 + R_1 + R_3.$$

Consider also the classes $\mathcal{L}+\mathcal{L}'$ and $\mathcal{L}-\mathcal{L}'$, which after reduction modulo two are the same as:

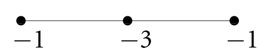
$$\mathcal{M} := L_1 + L_3 + L_5 + L_1' + L_3' + L_5' + M_1 + M_2.$$

These classes consist of six disjoint A_2-configurations of curves and of eight disjoint A_1-configurations of curves (according to [T, N]). These are the resolutions of A_2 and A_1 singularities of X^n_λ/G_n which arise by doing the quotient of X^n_λ/H by G_n/H. We construct the 3-cyclic coverings and the 2-cyclic coverings by using the divisors $\mathcal{L}, \mathcal{L}', \mathcal{M}, \mathcal{M}'$. We recall briefly the construction in the case of 3-cyclic coverings. Then in the case of 2-cyclic coverings it is similar. First to avoid producing singularities, we blow up the meeting points of the A_2-configurations. Call Y^0_{λ,G_6} the surface which we obtain after these blow-ups. The meeting points are replaced by (-1)-curves and the two (-2)-curves become now (-3)-curves. Denote by $\phi\colon Y^1_{\lambda,G_6} \to Y^0_{\lambda,G_6}$ the 3-cyclic covering with branching divisor $\mathcal{L}$, $\mathcal{L}'$ or $\mathcal{M}$, $\mathcal{M}'$ then

Proposition 3.1 *A configuration of curves on Y^0_{λ,G_6}:*



becomes a configuration:



on Y^1_{λ,G_6}.

Proof We do the computation for one configuration of curves $L_1 - L_2$; this is the same in the other cases. Denote again by L_1 and L_2 the curves on Y^0_{λ,G_6} which now are (-3)-curves and denote by E the exceptional (-1)-curve. By the properties of cyclic coverings we have $\phi^* L_i = 3\tilde{L}_i$, where $\tilde{L}_i$ is the strict transform of L_i. Then:

$$9(\tilde{L}_i)^2 = (\phi^* L_i)^2 = (\deg \phi) L_i^2 = -9.$$

Hence $(\tilde{L}_i)^2 = -1$. Since $E \cdot (L_1 - L_2) = 0$ the map ϕ is not ramified on E and the restriction $\phi_{|\tilde{E}}$ is 3:1 onto E. Hence we have $\phi^* E = \tilde{E}$ and $\tilde{E}^2 = (\phi^* E)^2 = (\deg \phi) E^2 = 3E^2 = -3$. ∎

Our surface Y^1_{λ,G_6} is now no longer minimal. By blowing down the (-1)-curves, the curve $\tilde{E}$ also becomes a (-1)-curve, so we blow it down, too. By construction, the surfaces which we obtain are minimal $K3$-surfaces and are exactly the surfaces $Y_{\lambda,H}$

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which are obtained as the minimal resolutions of X_λ^6/H, in fact we have a commutative diagram:

$$\begin{tikzcd}
 & Y^1_{\lambda,G_6} \arrow[r, "\phi"] \arrow[dl] & Y^0_{\lambda,G_6} \arrow[dr] & \\
Y_{\lambda,H} \arrow[rrr, dashed, "\gamma"] \arrow[dr] & & & Y_{\lambda,G_6} \arrow[dl] \\
 & X^6_\lambda/H \arrow[r, "\pi"] & X^6_\lambda/G_6 &
\end{tikzcd}$$

The construction is similar in the case of 2-cyclic coverings of the surfaces Y_{λ,G_8}. This gives another description of the families of $K3$-surfaces $Y_{\lambda,T\times V}$, $Y_{\lambda,(TT)'}$ and $Y_{\lambda,O\times T}$, $Y_{\lambda,(OO)''}$ as finite coverings of the families Y_{λ,G_6} and Y_{λ,G_8}.

Remark By using the divisors $\mathcal{L}'$ and $\mathcal{M}'$ on Y_{λ,G_6} for the coverings we obtain the surfaces $Y_{\lambda,V\times T}$ and $Y_{\lambda,(TT)''}$ and by taking the divisor $\mathcal{L}'$ on Y_{λ,G_8} we obtain the surface $Y_{\lambda,T\times O}$. We do not discuss these surfaces separately in the sequel.

3.2 The Special Cases

In these cases, the situation is a little more complicated. Now in the counterimage $\pi^{-1}(P)$ of some singular point P of X_λ^n/G_n coming from the A_1-singularities of X_λ^n we have singularities on X_λ^n/H, too. In the following table we give the singularities in the quotient X_λ^n/G_n, $n = 6, 8$ and the type and the number of singularities in the counterimage on X_λ^n/H. As in [BS] we denote by $6, 1, \dots, 8, 4$ the special surfaces in the families.

	6, 1	6, 2	6, 3	6, 4		8, 1	8, 2	8, 3	8, 4
G_6	E_6	A_5	A_5	E_6	G_8	E_7	D_6	D_4	D_5
$T\times V$	D_4	A_1	A_1	D_4	$O\times T$	E_6	D_4	A_3	A_5
$(TT)'$	$3E_6$	$3A_5$	A_1	D_4	$(OO)''$	$2E_7$	A_7	A_3	$2D_5$
$V\times V$	$3D_4$	$3A_1$	$3A_1$	$3D_4$	$T\times T$	$2E_6$	A_3	A_1	$2A_5$

By resolving the quotients we get a map like γ as before and so again by the result of Inose the minimal resolutions of the special $K3$-surfaces are $K3$-surfaces too with Picard number 20. We can describe this map as before by using cyclic coverings. In the case of the special surfaces in the family Y_{λ,G_6} we construct 3-cyclic covering as in the general case by using the divisors $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{M}$, $\mathcal{M}'$, which are in the case of the special surfaces 3-divisible, too, *cf.* [BS, 6.2]. By taking $\mathcal{L}$ and $\mathcal{L}'$ we obtain the special $K3$-surfaces in the families $Y_{\lambda,T\times V}$, resp., $Y_{\lambda,V\times T}$, by taking $\mathcal{M}$ and $\mathcal{M}'$ we obtain the special $K3$-surfaces in the covering $Y_{\lambda,(TT)'}$ and $Y_{\lambda,(TT)''}$. In the case of the

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special surfaces in the family Y_{λ,G_8} we take the divisors $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{M}$ and we do 2-cyclic coverings. By taking $\mathcal{L}$ or $\mathcal{L}'$ we find the singular surfaces in the family $Y_{\lambda,O\times T}$, resp., $Y_{\lambda,T\times O}$, and by taking $\mathcal{M}$ we find the singular surfaces in the family $Y_{\lambda,(OO)''}$.

4 Picard Lattices

We compute the Picard lattices of the general $K3$-surface in the families $Y_{\lambda,T\times V}$, $Y_{\lambda,O\times V}$ and of the special surfaces with $\rho = 20$ in each pencil. First we recall some facts. Denote by W the lattice spanned by the curves of Section 2, 2.2. If W is not the total Picard lattice, which we call NS, there is an integral lattice W' such that $W \subset W' \subset NS$ with $p := [W' : W]$, a prime number. Denote by $d(W)$, $d(W')$ the discriminant of the lattices W, W'. Since $[W' : W]^2 = d(W) \cdot d(W')^{-1}$ (*cf.* [BPV, Lemma 2.1, p. 12]), we find that p^2 divides the discriminant of W. Denote by $(W^\vee/W)^p$ the p-subgroup of $(W^\vee/W) \subset NS^\vee/NS$ and denote by T the transcendental lattice orthogonal to the Picard lattice. Since the discriminant groups $T^\vee/T$ and $NS^\vee/NS$ are isomorphic (*cf.* [BPV, p. 13, Lemma 2.5]), they have the same rank which is $\leq \mathrm{rk}(T)$. It follows that also $\mathrm{rk}(W^\vee/W)^p \leq \mathrm{rk}(T)$.

Proposition 4.1 *The Picard lattices of the generic surface $Y_{\lambda,T\times V}$ and $Y_{\lambda,O\times T}$ are generated by the* 19 *rational curves of Section* 2.2 *and the classes:*

$$\frac{\bar{L}'}{3} := \frac{L_1 - L_2 + L_4 - L_5 + L_1' - L_2' + L_4' - L_5' + L_1'' - L_2'' + L_4'' - L_5''}{3},$$
$$\frac{h_1}{2} := \frac{L_1 + L_3 + L_5 + L_1' + L_3' + L_5' + M_1 + M_2}{2},$$
$$\frac{h_2}{2} := \frac{L_1 + L_3 + L_5 + L_1'' + L_3'' + L_5'' + M_1 + M_3}{2},$$

of $NS(Y_{\lambda,T\times V})$, then the lattice has discriminant $2 \cdot 3 \cdot 5$, *resp., the classes*

$$\frac{\bar{L}''}{2} := \frac{L_1 + L_3 + L_5 + L_1' + L_3' + L_5' + M_1 + M_2}{2},$$
$$\frac{k_1}{3} := \frac{L_1 - L_2 + L_4 - L_5 - L_1' + L_2' - L_4' + L_5' + N_1 - N_2 + N_3 - N_4}{3},$$

of $NS(Y_{\lambda,O\times T})$, then the lattice has discriminant $2^3 \cdot 3 \cdot 7$.

Proof

Step 1. The discriminant of the lattice generated by the 19 curves is $2^5 \cdot 3^3 \cdot 5$ hence we can have 2-divisible classes or 3-divisible classes. The divisor $\bar{L}'$ is 3-divisible since it is the pull back of the divisor $\mathcal{L}'$ on Y_{λ,G_6} which is 3-divisible, too. And we cannot have more 3-divisible classes. If there are no 2-divisible classes, then the group

$(W^\vee/W)^2$ would contain the classes $M_1/2, M_2/2, M_3/2, (L_1+L_3+L_5+L'_1+L'_3+L'_5)/2$, $(L_1+L_3+L_5+L''_1+L''_3+L''_5)/2$, $(L'_1+L'_3+L'_5+L''_1+L''_3+L''_5)/2$ which are independent classes with respect to the intersection form. Since the rank of $(W^\vee/W)^2$ is less than or equal to the rank of $T^\vee/T$ which is at most three, it cannot happen that we find five classes as before. Hence some combination of them must be contained in the Neron–Severi group. So we have

$$\frac{1}{2}\Big(\lambda(L_1+L_3+L_5)+\lambda'(L'_1+L'_3+L'_5)+\lambda''(L''_1+L''_2+L''_3) \\ +\mu_1M_1+\mu_2M_2+\mu_3M_3\Big) \in NS$$

for some parameters $\lambda, \lambda', \lambda'', \mu_1, \mu_2, \mu_3 \in \mathbb{Z}_2$.

By Nikulin [N] such a 2-divisible set contains 8 curves. So putting $\lambda''=0$ and $\mu_3=0$ we get the divisor $h_1/2$, putting $\lambda'=0$ and $\mu_2=0$ we get the divisor $h_2/2$. The discriminant of the lattice W together with these three classes now become $2\cdot3\cdot5$, hence we cannot have more torsion classes.

Step 2. Again the class $\overline{L''}$ is the pull back of the class $\mathcal{M}'$ on Y_{λ,G_8} hence it is 2-divisible. If there are no 3-divisible classes then the group $(W^\vee/W)^3$ would contain the classes $N_1-N_2/3$, $N_3-N_4/3$ and $(L_1-L_2+L_4-L_5+L'_1-L'_2+L'_4-L'_5)/3$ which are independent. By specializing to the surfaces $Y^{(8,2)}_{\lambda,O\times O}$ and $Y^{(8,3)}_{\lambda,O\times O}$, we also find here these three independent classes and so $\mathrm{rk}(W^\vee/W)^3 \geq 3$. This is not possible. In fact, on these surfaces we have $\mathrm{rk}(W)$=20 which implies $\mathrm{rk}(W^\vee/W)^3 \leq 2$. This means that the three classes fit together giving a 3-divisible class in $NS(Y^{(8,2)}_{\lambda,O\times O})$ and $NS(Y^{(8,3)}_{\lambda,O\times T})$ and so in $NS(Y_{\lambda,O\times T})$ (*cf.* [vGT, Lemma 2.3]). ■

In the same way as before we can compute the Picard lattices of the special surfaces in the families. We state the results leaving the proofs to the reader.

Proposition 4.2

(i) *The Picard lattice of the special surfaces in $Y_{\lambda,T\times V}$ and $Y_{\lambda,O\times T}$ is generated in all the cases but $Y^{(8,4)}_{\lambda,O\times T}$ by the curves of Section* 2.2 *and by the classes $\bar{L}'/3$, $h_1/2$, $h_2/2$, resp. $\bar{L}''/2$, $k_1/3$ of proposition* 4.1. *In the case of $Y^{(8,4)}_{\lambda,O\times T}$ the class*

$$\frac{L_1+L_3+L_5+N_1+C+N_4+R_2+M_1}{2}$$

is a generator too, and they span the 20*-dimensional Picard lattice.*

(ii) *In the case of $Y^{(6,1)}_{\lambda,(TT)'}$ and of $Y^{(6,2)}_{\lambda,(TT)'}$ the class*

$$\frac{\bar{L}}{3} := \frac{N_1-N_2+N_3-N_4+N_5-N_6+N_7-N_8+N_9-N_{10}+N_{11}-N_{12}}{3}$$

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is in the Neron–Severi group and in the case of $Y^{(6,2)}_{\lambda,(TT)'}$*, the classes*

$$\frac{N_1 + C_1 + N_4 + N_5 + C_2 + N_8 + M_1 + M_2}{2},$$
$$\frac{N_1 + C_1 + N_4 + N_9 + C_3 + N_{12} + M_1 + M_3}{2}$$

are in the Neron–Severi group, too. These together with the 20 *curves of Section* 2.2 *span the* 20*-dimensional Picard lattice.*

(iii) *In the case of* $Y^{(8,1)}_{\lambda,(OO)''}$ *and* $Y^{(8,4)}_{\lambda,(OO)''}$*, the class*

$$\frac{\bar{L}}{2} := \frac{M_1 + M_2 + M_3 + M_4 + R_1 + R_3 + R'_1 + R'_3}{2}$$

is in the Neron–Severi group and in the case of $Y^{(8,4)}_{\lambda,(OO)''}$*, the class*

$$\frac{W}{4} := \frac{R_1 + 2R_2 + 3R_3 + R'_1 + 2R'_2 + 3R'_3}{4} + \frac{2N_1 + 2C_1 + 3M_1 + M_2 + 2N_3 + 2C_2 + 3M_3 + M_4}{4}$$

is in the Neron–Severi group, too. Again these classes together with the 20 *curves of Section* 2.2 *span the* 20*-dimensional Picard lattice. The discriminants of the Picard lattices then are:*

	$Y_{\lambda,T\times V}$				$Y_{\lambda,(TT)'}$	
	6, 1	6, 2	6, 3	6, 4	6, 1	6, 2
d	$-3\cdot 5$	$-2^2\cdot 3\cdot 5$	$-2^2\cdot 3\cdot 5$	$-3\cdot 5$	$-3\cdot 5$	$-2^2\cdot 3\cdot 5$

	$Y_{\lambda,O\times T}$				$Y_{\lambda,(OO)''}$	
	8, 1	8, 2	8, 3	8, 4	8, 1	8, 4
d	$-2^2\cdot 7$	$-2^2\cdot 3\cdot 7$	$-2^3\cdot 3\cdot 7$	$-2^2\cdot 7$	$-2^2\cdot 7$	$-2^4\cdot 7$

4.1 More Cyclic Coverings

Now we can construct the 3-cyclic covering of $Y_{\lambda,T\times V}$, by using the 3-divisible classes $\overline{L'}$ and the 2-cyclic coverings of $Y_{\lambda,O\times T}$, by using the 2-divisible class $\overline{L''}$. We can do this for the general surface in the pencil and for the special surfaces, too. In this case we obtain another description of the families $Y_{\lambda,V\times V}$ and $Y_{\lambda,T\times T}$. In particular, in these cases the general surface in the family also has Picard number 19 and we have four surfaces with Picard number 20. The description of the Picard lattices of the surfaces with $\rho = 20$ is given in the following proposition (again, we leave the proof to the reader).

Proposition 4.3 *The classes*

$$\frac{L_2 - L_4 + L_3' - L_1' + N_1 - N_2 + N_3 - N_4 + N_5 - N_6 + N_7 - N_8}{3},$$
$$\frac{L_2' - L_4' + L_3 - L_1 + N_1 - N_2 - N_3 + N_4 + N_5 - N_6 - N_7 + N_8}{3}$$

are in $NS(Y^{(8,1)}_{\lambda,T\times T})$ and in $NS(Y^{(8,4)}_{\lambda,T\times T})$. Moreover, the class

$$\frac{N_1 + C_1 + N_4 + N_5 + C_2 + N_8 + M_1 + M_2}{2}$$

is in $NS(Y^{(8,4)}_{\lambda,T\times T})$ too. These classes together with the rational curves of Section 2, 2.2 span the 20-dimensional Picard lattices of the surfaces $Y^{(8,1)}_{\lambda,T\times T}$ and $Y^{(8,4)}_{\lambda,T\times T}$. Then the lattices have discriminant -7, resp., $-2^2 \cdot 7$.

5 Configurations of Rational Curves

In Figures 1, 2, 3, and 4 we give the configurations of rational curves on the surfaces with Picard number 19 and 20. In the case of the singular surfaces of the families $Y_{\lambda,T\times V}$ and $Y_{\lambda,O\times T}$ also the curves L_i, L_i' and L_i'' are contained on the surfaces, but we do not draw the picture again. Moreover the configurations of curves on the surfaces $Y^{(6,4)}_{\lambda,T\times V}$ and $Y^{(6,3)}_{\lambda,T\times V}$ are the same as on the surfaces $Y^{(6,1)}_{\lambda,T\times V}$, resp., $Y^{(6,2)}_{\lambda,T\times V}$, so again we draw only one picture.

6 Final Remarks

In Section 4 we identify explicitly the Picard lattice of some K3-surfaces. It is our next aim to compute the transcendental lattices orthogonal to the Picard lattices to classify the $K3$-surfaces. In particular by a result of Shioda and Inose [SI] $K3$-surfaces with $\rho = 20$ are classified by means of their transcendental lattice.

By a result of Morrison [Mo], each $K3$-surface with $\rho = 19$ or 20 admits a so called Shioda–Inose structure. This means that there is a Nikulin involution, an involution with eight isolated fixed points and the quotient is birational to a Kummer surface. It would be desirable to have an explicit description of this structure for our surfaces.

We do not describe the quotients 3-folds $\mathbb{P}_3/G_n$, $\mathbb{P}_3/H$, H a normal subgroup of G_n. It would be interesting to have a global resolution of these spaces and to see our $K3$-surfaces as smooth pencils on the smooth 3-folds.

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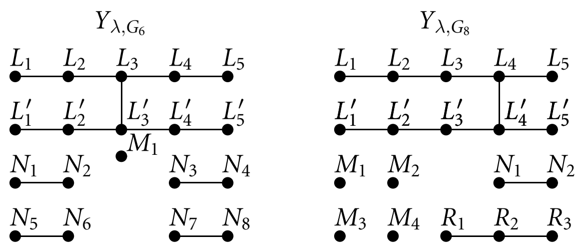

Figure 1

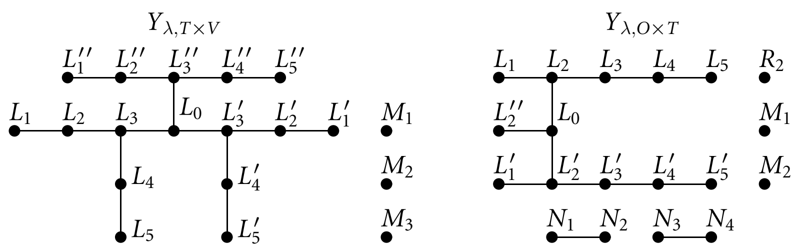

Figure 2

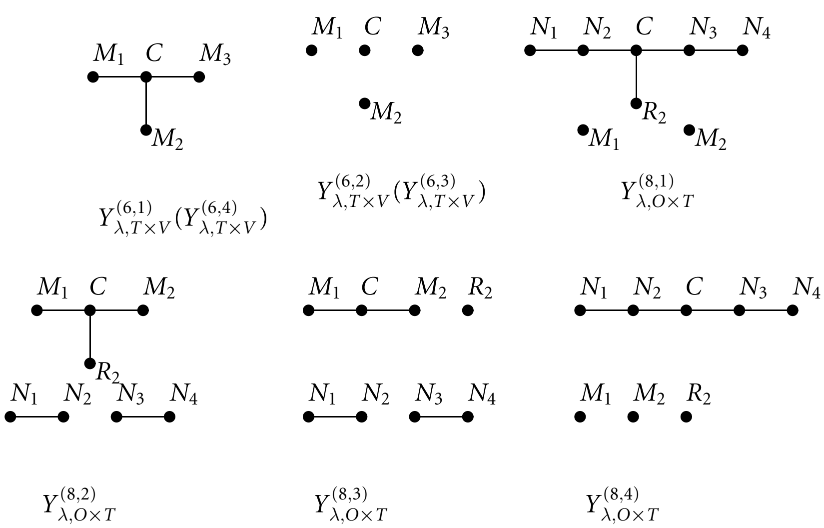

Figure 3

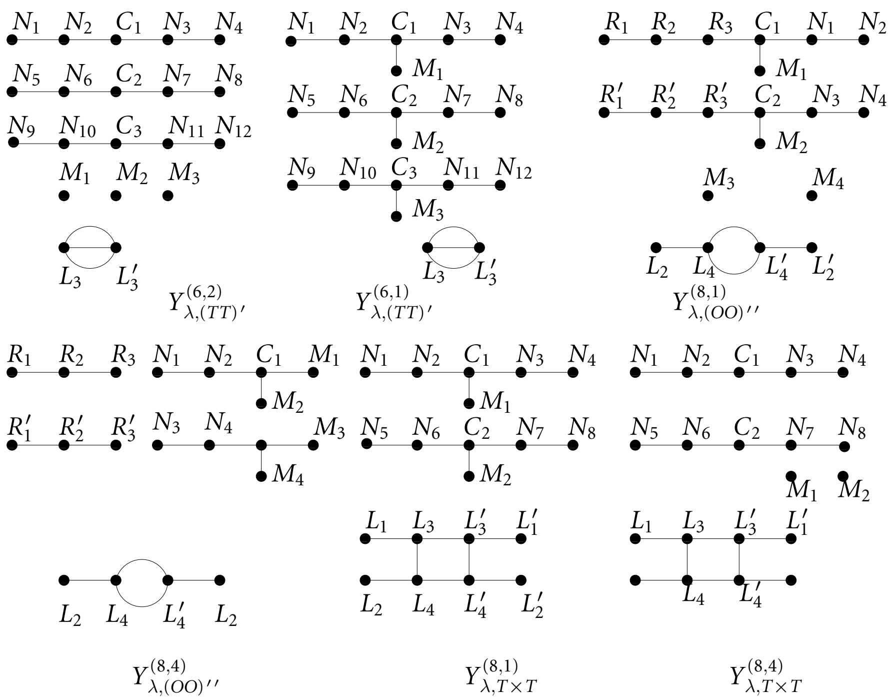

Figure 4

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