Canad. J. Math. Vol. 73 (1), 2021 pp. 293–295
http://dx.doi.org/10.4153/S0008414X20000310


# Erratum: The Jiang–Su Absorption for Inclusions of Unital C*-algebras

http://dx.doi.org/10.4153/CJM-2017-033-7

Hiroyuki Osaka and Tamotsu Teruya

*Abstract.* We correct an error in the statement in a proposition and a theorem in *Jiang–Su absorption for inclusions of unital C*-algebras*. Canad. J. Math. **70**(2018), 400–425. This error was found by Dr. M. Ali Asadi-Vasfi and communicated to the authors by N. Christopher Phillips of the University of Oregon, who also suggested the outline for the following correct proofs.

We correct an error in the statement in a proposition and a theorem in *Jiang–Su absorption for inclusions of unital C*-algebras*. Canad. J. Math **70**(2018), 400–425. This error was found by Dr. M. Ali Asadi-Vasfi, and communicated to the authors by N. Christopher Phillips of University of Oregon, who also suggested the outline for the following correct proofs.

The proposition in question reads as follows. We put one extra condition, namely that $b$ is purely positive; that is, 0 is not an isolated point in $\sigma(b)$ by [5, Lemma 3.2].

***Proposition 6.2*** ([4]) *Let $P \subset A$ be an inclusion of unital C*-algebras with index-finite type. Suppose that $A$ is simple and $E: A \to P$ has the tracial Rokhlin property. If two positive elements $a, b \in P$ satisfy $a \precsim b$ in $A$ with $b$ being purely positive, then $a \precsim b$ in $P$.*

**Proof** Recall that $f_\epsilon(t) = \max\{t - \epsilon, 0\}$ and that $(a-\epsilon)_+$ is equal to $f_\epsilon(a)$. Then we have only to show that $(a-\epsilon)_+ \precsim b$ for every $\epsilon > 0$.

Since $a \precsim b$ in $A$, there exists $\delta > 0$ and $r \in A$ such that $(a-\epsilon)_+ = r^*(b-\delta)_+ r \sim (b-\delta)_+^{1/2} r r^* (b-\delta)_+^{1/2} = b_0$. Note that $b_0 \in \overline{bAb}$.

Since $\sigma(b) \cap (0, \delta) \neq \varnothing$, we can take a nonzero element $c \in \overline{(bPb)}_+$ such that $(b-\delta)_+ \perp c$. Take a Rokhlin proposition $e \in A' \cap A^\infty$ for $E$. Then there is a projection $g \in P' \cap P^\infty$ such that $1 - g \precsim ec$ in $A^\infty$. Hence,

$$\begin{aligned}(a-\epsilon)_+(1-g) &= (1-g)(a-\epsilon)_+(1-g)\\ &\precsim (1-g) \precsim ec\end{aligned}$$

in $A^\infty$. Note that since $c \in P$, we have $ec = ce$. By [4, Lemma 6.1], $(a-\epsilon)_+(1-g) \precsim c$ in $P^\infty$.

Received by the editors August 28, 2019; revised May 5, 2020.
Published online on Cambridge Core June 11, 2020.
Research of the first author partially supported by the JSPS grant for Scientific Research No. 23540256.
AMS subject classification: **46L55**, 46L35.
Keywords: Jiang–Su absorption, inclusion of C*-algebras, strictly comparison.

Then we have in $P^\infty$:

$$\begin{aligned}(a-\varepsilon)_+ &= (a-\varepsilon)_+ g + (a-\varepsilon)_+(1-g)\\ &= \beta((a-\varepsilon)_+) + (a-\varepsilon)_+(1-g)\\ &\lesssim \beta(b_0) + (a-\varepsilon)_+(1-g) && ([2, \text{Proposition 1.1}])\\ &\lesssim \beta(b_0) + c \in \overline{bP^\infty b}, && ([2, \text{Proposition 1.1}])\end{aligned}$$

where $\beta\colon A \to gP^\infty g$ is defined as in [4, Lemma 4.9]. Hence, $(a-\varepsilon)_+ \lesssim b$ in $P^\infty$.

Therefore, we have $a \lesssim b$ in $P^\infty$ by [7, Proposition 2.4 (iv)], and $a \lesssim b$ in $P$. ∎

We note that Corollary 6.3 is wrong in general as pointed out by N. Christopher Phillips. (See [1, Example 4.7].)

Using Proposition 6.2, we have the following theorem.

***Theorem 7.1*** ([4]) *Let $P \subset A$ be an inclusion of unital $C^*$-algebras with index-finite type. Suppose that $A$ is simple, stably finite, and exact, which is not type I, and $A$ has strict comparison and $E\colon A \to P$ has the tracial Rokhlin property. Then $P$ has strict comparison.*

**Proof** Since $E\colon A \to P$ is of index-finite type and $A$ is simple and exact, $P$ is exact and simple by [4, Proposition 4.7(i)]. Note that the strict comparison property is equivalent to the the strict comparison property given by traces; *i.e.*, for all $x, y \in W(A)$, one has that $x \le y$ if $d_\tau(x) < d_\tau(y)$ for all tracial states $\tau$ on $A$ (see [9, Remark 6.2] and [8, Corollary 4.6]).

Since $E \otimes \mathrm{id}\colon A \otimes M_n \to P \otimes M_n$ is of index-finite type and has the tracial Rokhlin property, it suffices to verify the condition that whenever $a, b \in P$ are positive elements such that $d_\tau(a) < d_\tau(b)$ for all $\tau \in \mathrm{T}(P)$, then $a \lesssim b$.

Let $a, b \in P$ be positive elements such that $d_\tau(a) < d_\tau(b)$ for all $\tau \in \mathrm{T}(P)$. Then for any tracial state $\tau \in \mathrm{T}(A)$ the restriction $\tau_{|P}$ belongs to $\mathrm{T}(P)$. Hence, we have $d_\tau(a) < d_\tau(b)$ for all tracial states $\tau \in \mathrm{T}(A)$. Since $A$ has strict comparison, $a \lesssim b$ in $A$. Then we can split into two cases in which $b$ is purely positive or not.

When $b$ is purely positive, by Proposition 6.2, $a \lesssim b$ in $P$. When $b$ is not purely positive, we can assume that $b$ is a nonzero projection in $P$.

We consider two cases for $\langle a\rangle$; that is, Case 1: $\langle a\rangle$ is the class of a projection; Case 2: $\langle a\rangle$ is not the class of a projection.

Case 1: We can assume that $a$ is a projection. Since the function $\mathrm{T}(A) \ni \tau \to d_\tau(b) - d_\tau(a) = \tau(b) - \tau(a)$ is continuous, positive, and $\mathrm{T}(A)$ is compact, it follows that $\rho = \inf_{\tau\in\mathrm{T}(A)}(d_\tau(b) - d_\tau(a)) > 0$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \rho$. Use [5, Lemma 3.6] to find $c, d \in (bPb \otimes \mathbb{K})_+\setminus\{0\}$ such that $\langle c\rangle \le \langle b\rangle \le \langle c\rangle + \langle d\rangle$, and $n\langle d\rangle \le \langle 1\rangle$ in $\mathrm{Cu}(P)$, where $c$ is purely positive and $\mathrm{Cu}(P) = (P \otimes \mathbb{K})/\sim$, the Cuntz semigroup of $P$. Since for $\tau \in \mathrm{T}(A)$ $d_\tau(\langle d\rangle) \le \frac{1}{n} < \rho$, $d_\tau(\langle c\rangle) > d_\tau(\langle b\rangle) - \rho > d_\tau(\langle a\rangle)$. Since $A$ has strictly comparison, $a \lesssim c$ in $A$. Since $c$ is purely positive, $a \lesssim c$ in $P$. Therefore, since $\langle c\rangle \le \langle b\rangle$ in $\mathrm{Cu}(P)$, $a \lesssim b$ in $P$.

Case 2: Since $\langle a\rangle$ is not the class of a projection, $a$ is purely positive by [5, Lemma 5]. For any $\epsilon > 0$ take a continuous function $f\colon [0,\infty) \to [0,1]$ such that $f(\lambda) > 0$

https://doi.org/10.4153/S0008414X20000310 Published online by Cambridge University Press

for $\lambda \in (0, \epsilon)$ and $f(\lambda) = 0$ for $\lambda \in \{0\} \cup [\epsilon, \infty)$. Then, $f(a) \neq 0$. Therefore, $\rho = \inf\{d_\tau(f(a)) : \tau \in \mathrm{T}(A)\}$ satisfies $\rho > 0$.

Claim that for any $\tau \in \mathrm{T}(A)$, $d_\tau((a-\epsilon)_+) + \rho < d_\tau(b)$. Indeed,

$$\begin{aligned} d_\tau\big((a-\epsilon)_+\big) + \rho &\le d_\tau\big((a-\epsilon)_+\big) + d_\tau\big(f(a)\big) \\ &= d_\tau\big((a-\epsilon)_+ + f(a)\big) \le d_\tau(a) < d_\tau(b). \end{aligned}$$

As in Case 1, choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \rho$. Use [5, Lemma 3.6] to find $c$, $d \in (bPb \otimes \mathbb{K})_+ \setminus \{0\}$ such that $\langle c \rangle \le \langle b \rangle \le \langle c \rangle + \langle d \rangle$, and $n\langle d \rangle \le \langle 1 \rangle$ in $\mathrm{Cu}(P)$, where $c$ is purely positive and $\mathrm{Cu}(P) = (P \otimes \mathbb{K})/\sim$, the Cuntz semigroup of $P$. Since for $\tau \in \mathrm{T}(A)$ $d_\tau(\langle d \rangle) \le \frac{1}{n} < \rho$, $d_\tau(\langle c \rangle) > d_\tau(\langle b \rangle) - \rho > d_\tau(\langle (a-\epsilon)_+ \rangle)$. Since $A$ has strictly comparison, $(a-\epsilon)_+ \precsim c$ in $A$. Since $c$ is purely positive, $(a-\epsilon)_+ \precsim c$ in $P$.

Therefore, since $\langle c \rangle \le \langle b \rangle$ in $\mathrm{Cu}(P)$, $(a-\epsilon)_+ \precsim b$ in $P$. Moreover, $\epsilon > 0$ is arbitrary, so we conclude that $a \precsim b$ in $P$. ■

## References


[1] M. A. Asadi-Vasfi, N. Golestani, and N. C. Phillips, *The Cuntz semigroup and the radius of comparison of the crossed product by a finite group*. arxiv:1908.06343

[2] J. Cuntz, *Dimension functions on simple C*-algebras*. Math. Ann. 233(1978), 145–153. https://doi.org/10.1007/BF01421922

[3] H. H. Lee and H. Osaka, *Tracially sequentially-split*-homomorphisms between C*-algebras II*. arxiv:1906.06950

[4] H. Osaka and T. Teruya, *Jiang–Su absorption for inclusions of C*-algebras*. Canad. J. Math. 70(2018), 400–425. https://doi.org/10.4153/CJM-2017-033-7

[5] N. C. Phillips, *Large subalgebras*. arxiv:1408.5546v1

[6] N. C. Phillips, *Private communication*. 2019.

[7] M. Rørdam, *On the structure of simple C*-algebras tensored with a UHF-algebra II*. J. Funct. Anal. 107(1992), 255–269. https://doi.org/10.1016/0022-1236(92)90106-S

[8] M. Rørdam, *The stable and the real rank of $\mathcal{Z}$-absorbing C*-algebras*. Inter. J. Math. 10(2004), 1065–1084. https://doi.org/10.1142/S0129167X04002661

[9] M. Rørdam and W. Winter, *The Jiang–Su algebra revisited*. J. Reine Angew. Math. 642(2010), 129–155. https://doi.org/10.1515/CRELLE.2010.039



*Department of Mathematical Sciences, Ritsumeikan University, Kusatsu, Shiga, 525-8577, Japan*
*e-mail:* osaka@se.ritsumei.ac.jp

*Faculty of Education, Gunma University, 4-2 Aramaki-machi, Maebashi City, Gunma, 371-8510, Japan*
*e-mail:* teruya@gunma-u.ac.jp