A Matrix Representation of Ascending and Descending Continued Fractions[1]

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The present paper describes briefly a notation for representing continued fractions in many dimensions, which has the advantage of providing a direct method of attack and of rendering intuitive, results which are usually proved by induction. The notation is the outcome of a generalisation which I previously made [1] in connection with the solution of certain difference equations. Only formal theorems are considered here. For a discussion of convergence reference may be made to the works [2, 3, 4, 5] cited at the end. The paper by Paley and Ursell is particularly important since these authors discuss very fully the non-cyclic simple continued fraction.

Let J_m denote the square matrix of k rows and k columns

$$\begin{bmatrix} a_m & 1 & 0 & 0 & \dots & 0 \\ b_m & 0 & 1 & 0 & \dots & 0 \\ c_m & 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ i_m & 0 & 0 & 0 & \dots & 1 \\ j_m & 0 & 0 & 0 & \dots & 0 \end{bmatrix}. \tag{1}$$

Let $J_{m,0}$ denote the first column of J_m. We shall suppose that $j_1 = 1$.

Now consider the product

$$M_m = J_1 J_2 J_3 \dots J_m = \begin{bmatrix} p_{1,m} & p_{2,m} & p_{3,m} & \dots \\ q_{1,m} & q_{2,m} & q_{3,m} & \dots \\ r_{1,m} & r_{2,m} & r_{3,m} & \dots \\ \cdot & \cdot & \cdot & \dots \end{bmatrix}.$$

Writing $m + 1$ for m we have

$$M_{m+1} = M_m J_{m+1},$$

[1] Also read at the International Congress of Mathematicians, Zürich, 1932.

which gives

$$p_{1,\,m+1} = a_{m+1}\, p_{1,\,m} + b_{m+1}\, p_{2,\,m} + \dots + j_{m+1}\, p_{k,\,m}\,,$$
$$p_{2,\,m+1} = p_{1,\,m}\,,\quad p_{3,\,m+1} = p_{2,\,m}\,,\ \dots,\quad p_{k,\,m+1} = p_{k-1,\,m}\,.$$

If then, we write p_m for $p_{1,m}$ it follows at once that the first row of M_m can be written

$$[p_m \quad p_{m-1} \quad p_{m-2} \quad \dots \quad p_{m-k+1}]$$

and similar results hold for every row. Thus we can write

$$(2) \qquad J_1\, J_2\, J_3 \dots J_n = \begin{bmatrix} p_n & p_{n-1} & p_{n-2} & \dots \\ q_n & q_{n-1} & q_{n-2} & \dots \\ \cdot & \cdot & \cdot & \dots \\ w_n & w_{n-1} & w_{n-2} & \dots \end{bmatrix},$$

where

$$(3) \qquad \begin{aligned} p_n &= a_n\, p_{n-1} + b_n\, p_{n-2} + \dots + j_n\, p_{n-k}, \\ q_n &= a_n\, q_{n-1} + b_n\, q_{n-2} + \dots + j_n\, q_{n-k}, \\ &\cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \\ w_n &= a_n\, w_{n-1} + b_n\, w_{n-2} + \dots + j_n\, w_{n-k}. \end{aligned}$$

These considerations suggest the following definitions:

DEFINITION I. *The matrix J_m of k rows and columns being defined by* (1), *the matrix product*

$$J_1\, J_2\, J_3 \ \dots \ J_n \ \dots$$

is a cyclic descending continued fraction in k dimensions.

If $j_m = 1$, $m = 1, 2, 3, \dots$ we shall call the continued fraction *simple.*

DEFINITION II. *The matrix product*

$$J_1\, J_2\, J_3 \ \dots \ J_{n-1}\, J_{n,\,0}$$

is the nth convergent of the continued fraction.

Thus we can write

$$(4) \qquad \begin{bmatrix} p_n \\ q_n \\ \cdot \\ w_n \end{bmatrix} = J_1\, J_2 \ \dots \ J_{n-1} \begin{bmatrix} a_n \\ b_n \\ \cdot \\ j_n \end{bmatrix}.$$

The numbers p_n, q_n, r_n, $\dots$, w_n are called the components of the nth convergent. Their successive values are related by (3).

DEFINITION III. *The value of a non-terminating continued fraction is the limit when $n \to \infty$ of the ratio*

$$p_n : q_n : r_n : \ldots : w_n$$

provided that this limit exists. If the fraction terminates, the value is the ratio of the components of the last convergent.

Since the determinant of the matrix J_m is $(-1)^{k-1} j_m$ it follows from (2) that

$$\begin{vmatrix} p_n & p_{n-1} & p_{n-2} & \ldots \\ q_n & q_{n-1} & q_{n-2} & \ldots \\ r_n & r_{n-1} & r_{n-2} & \ldots \\ . & . & . & \ldots \end{vmatrix} = (-1)^{n(k-1)} j_2 j_3 \ldots j_n. \tag{5}$$

In the case $k = 2$ we have the ordinary or two-dimensional continued fraction

$$\begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_2 & 1 \\ b_2 & 0 \end{bmatrix} \ldots \begin{bmatrix} a_n & 1 \\ b_n & 0 \end{bmatrix} \ldots$$

and (5) becomes

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^n b_2 b_3 \ldots b_n,$$

a well known result of which (5) is the generalisation.

Periodic Cyclic Continued Fractions.

A continued fraction of the form

$$J_1 J_2 J_3 \ldots J_i K_1 K_2 \ldots K_h K_1 K_2 \ldots K_h K_1 K_2 \ldots K_h \ldots$$

where K_s has been written for J_{i+s}, $s = 1, 2, \ldots, h$, is said to be periodic. The same set of matrices is continually repeated in the same order. Put

$$J = J_1 J_2 \ldots J_i, \qquad K = K_1 K_2 \ldots K_h, \qquad n = i + mh + j, \ j < h,$$

$$J K J^{-1} = \begin{bmatrix} \alpha_1 & \beta_1 & \ldots & \kappa_1 \\ \alpha_2 & \beta_2 & \ldots & \kappa_2 \\ . & . & . & \\ \alpha_k & \beta_k & \ldots & \kappa_k \end{bmatrix}.$$

We then have

$$\begin{bmatrix} p_n \\ q_n \\ . \end{bmatrix} = J K^m K_1 K_2 \ldots K_j,$$

$$\begin{bmatrix} p_{n+h} \\ q_{n+h} \\ . \end{bmatrix} = J K J^{-1} \begin{bmatrix} p_n \\ q_n \\ . \end{bmatrix}.$$

This gives k linear difference equations for the components of the convergents, the first equation being

$$p_{n+h} = a_1 p_n + \beta_1 q_n + \gamma_1 r_n + \ldots + \kappa_1 w_n.$$

Putting $p_n = px^n$, $q_n = qx^n$, $\ldots$, we have

$$p(a_1 - x^h) + q\beta_1 + \ldots + w\kappa_1 = 0,$$
$$p\,a_2 + q(\beta_2 - x^h) + \ldots + w\kappa_2 = 0,$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\,,$$

so that

$$(6)\qquad \begin{vmatrix} a_1 - x^h & \beta_1 & \ldots & \kappa_1 \\ a_2 & \beta_2 - x^h & \ldots & \kappa_2 \\ \cdot & \cdot & \cdot & \cdot \\ a_k & \beta_k & \ldots & \kappa_k - x^h \end{vmatrix} = 0,$$

and the values of x^h are the latent roots of the matrix $J K J^{-1}$, while corresponding to each value of x^h, the numbers p, q, $\ldots$, w are proportional to the cofactors of the elements of the top row in the determinant (6). If (6) has one root, x, which is greater in absolute value than every other root we have

$$p_n \sim px^n, \quad q_n \sim qx^n, \quad \ldots$$

so that the value of the fraction is $p : q : r : \ldots : w$.

As an illustration consider

$$\begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \ldots$$

Here

$$J K J^{-1} = \begin{bmatrix} -119 & 402 \\ -37 & 125 \end{bmatrix}$$

$$\begin{vmatrix} -119 - x^3 & 402 \\ -37 & 125 - x^3 \end{vmatrix} = 0,$$

whence $x^3 = 3 + \sqrt{10}$, and the value of the fraction is $(122 - \sqrt{10}) : 37$.

Generalised Continuants.

For brevity we consider three-dimensional continued fractions. Let

$$(7)\qquad \begin{bmatrix} K(i, n) & K(i, n-1) & K(i, n-2) \\ q(i, n) & q(i, n-1) & q(i, n-2) \\ r(i, n) & r(i, n-1) & r(i, n-2) \end{bmatrix} = \begin{bmatrix} a_i & 1 & 0 \\ b_i & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} C_{i+1} \ldots C_n$$

where

$$C_s = \begin{bmatrix} a_s & 1 & 0 \\ b_s & 0 & 1 \\ c_s & 0 & 0 \end{bmatrix}.$$

It follows that

$$(8) \quad P(i, n) = C_i C_{i+1} \dots C_n = \begin{bmatrix} K(i, n) & K(i, n-1) & K(i, n-2) \\ q(i, n) & q(i, n-1) & q(i, n-2) \\ c_i r(i, n) & c_i r(i, n-1) & c_i r(i, n-2) \end{bmatrix}.$$

Hence we have $C_i P(i+1, n) = P(i, n)$, which gives

$$(9) \quad \begin{aligned} K(i, n) &= a_i K(i+1, n) + q(i+1, n), \\ q(i, n) &= b_i K(i+1, n) + c_{i+1} r(i+1, n), \\ r(i, n) &= \quad K(i+1, n), \end{aligned}$$

whence we obtain the recurrence relation

$$(10) \qquad K(i, n) = a_i K(i+1, n) + b_{i+1} K(i+2, n) + c_{i+2} K(i+3, n).$$

We call the function $K(i, n)$ a generalised continuant, since it is an obvious extension of the continuant defined by a two-dimensional fraction. The definition and method of inference by which (10) was established are clearly general. We also note that

$$K(1, n) = p_n.$$

Also, as in (3), we have at once

$$(11) \qquad K(i, n) = a_n K(i, n-1) + b_n K(i, n-2) + c_n K(i, n-3).$$

From (7) we see that

$$K(i, i) = a_i, \quad K(i, i-1) = 1, \quad K(i, i-2) = 0$$

are appropriate interpretations.

From (8) we see that $P(1, n) = P(1, i) \ P(i+1, n)$; whence using (9) we have the theorem

$$(12) \quad K(1, n) = K(1, i) \ K(i+1, n) + K(1, i-1) \{b_{i+1} K(i+2, n) + c_{i+2} K(i+3, n)\} \\ + c_{i+1} K(1, i-2) \ K(i+2, n).$$

In the two-dimensional case this reduces to

$$K(1, n) = K(1, i) \ K(i+1, n) + b_{i+1} K(1, i-1) \ K(i+2, n).$$

Euler's Theorem on Continuants.

Euler gave for two-dimensional simple continuants a theorem which is equivalent to

$$K(1, n) \ K(i, m) - K(1, m) \ K(i, n) \\ = (-1)^{m-i+1} b_i b_{i+1} \dots b_{m+i} K(1, i-2) \ K(m+2, n), \ 1 < i < m < n.$$

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This theorem can easily be generalised for our continued fractions. We give the method applied to the three-dimensional case. If $k < i$ we have by repeated use of (11)

$$(13)\quad \begin{bmatrix} K(1,n) \\ K(i,n) \\ K(k,n) \end{bmatrix} = \begin{bmatrix} K(1,i-1) & K(1,i-2) & K(1,i-3) \\ K(\ldots.) & 0 & 0 \\ K(k,i-1) & K(k,i-2) & K(k,i-3) \end{bmatrix} C_i \ldots C_{n-1} \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix}.$$

If, then, $1 < k < i < t < m < n$, we have

$$(14)\quad \begin{bmatrix} K(1,n) & K(1,m) & K(1,t) \\ K(i,n) & K(i,m) & K(i,t) \\ K(k,n) & K(k,m) & K(k,t) \end{bmatrix}$$

$$= \begin{bmatrix} K(1,i-1) & K(1,i-2) & K(1,i-3) \\ K(\ldots.) & 0 & 0 \\ K(k,i-1) & K(k,i-2) & K(k,i-3) \end{bmatrix} C_i \ldots C_{t-1} N$$

where

$$N = C_t \ldots C_{n-1} \begin{bmatrix} a_n & 0 & 0 \\ b_n & 0 & 0 \\ c_n & 0 & 0 \end{bmatrix} + C_t \ldots C_{m-1} \begin{bmatrix} 0 & a_m & 0 \\ 0 & b_m & 0 \\ 0 & c_m & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & a_t \\ 0 & 0 & b_t \\ 0 & 0 & c_t \end{bmatrix}$$

$$= \begin{bmatrix} K(t,n) & K(t,m) & a_t \\ q\,(t,n) & q\,(t,m) & b_t \\ c_t\, r(t,n) & c_t\, r(t,m) & c_t \end{bmatrix}.$$

Using (9) and (10) we easily obtain

$$\begin{vmatrix} K(t,n) & K(t,m) & a_t \\ q\,(t,n) & q\,(t,m) & b_t \\ c_t\, r(t,n) & c_t\, r(t,m) & c_t \end{vmatrix} = c_t\, c_{t+1}\, c_{t+2} \begin{vmatrix} K(t+3,n) & K(t+3,m) \\ K(t+2,n) & K(t+2,m) \end{vmatrix}.$$

Now the determinant of C_s is c_s. If then, in (14), we replace the matrices by determinants, we have the required generalisation to three dimensions of Euler's theorem, namely

$$\begin{vmatrix} K(1,n) & K(1,m) & K(1,t) \\ K(i,n) & K(i,m) & K(i,t) \\ K(k,n) & K(k,m) & K(k,t) \end{vmatrix}$$

$$= c_i\, c_{i+1} \ldots c_{t+2} \begin{vmatrix} K(1,i-2) & K(1,i-3) \\ K(k,i-2) & K(k,i-3) \end{vmatrix} \times \begin{vmatrix} K(t+2,n) & K(t+2,m) \\ K(t+3,n) & K(t+3,m) \end{vmatrix}.$$

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Non-Cyclic Continued Fractions.

The non-cyclic continued fraction, which was introduced by Paley and Ursell [3] for simple continued fractions, is equivalent to the following type of matrix product (in the general case for three dimensions):

$$\begin{bmatrix} a_1 & 1 & 0 \\ b_1 & 0 & 1 \\ c_1 & 0 & 0 \end{bmatrix}\begin{bmatrix} c_2 & 0 & 1 \\ a_2 & 0 & 0 \\ b_2 & 1 & 0 \end{bmatrix}\begin{bmatrix} a_3 & 1 & 0 \\ b_3 & 0 & 1 \\ c_3 & 0 & 0 \end{bmatrix}\begin{bmatrix} c_4 & 1 & 0 \\ a_4 & 0 & 1 \\ b_4 & 0 & 0 \end{bmatrix}\begin{bmatrix} b_5 & 0 & 1 \\ c_5 & 0 & 0 \\ a_5 & 1 & 0 \end{bmatrix}\begin{bmatrix} c_6 & 0 & 1 \\ a_6 & 0 & 0 \\ b_6 & 1 & 0 \end{bmatrix}\cdots$$

If we refer to those rows which contain at most one element different from zero, as *zero rows*, and to the other rows, as *unit rows*, the simple continued fraction corresponds to the case in which the letter in each zero row is replaced by unity. The characteristic formal features of these fractions are:

(i) The first row of each matrix, except possibly in the first matrix, is a unit row.

(ii) The letter in the first row of each matrix is the same as the letter in the zero row of the immediately preceding matrix.

(iii) In any given matrix the letters and units are arranged cyclically from the zero row;

for example $c\ 1\ 0$ follows $b\ 0\ 0$ and $a\ 0\ 1$ follows $c\ 1\ 0$.

By an obvious adaptation of our matrix notation we write the above product in the form

$$C_1\ A_2\ C_3\ B_4\ C_5\ A_6\ \ldots$$

where the capital letter indicates the letter of the zero row, and the letter in the first row of each matrix is indicated by the capital letter of the matrix which immediately precedes.

The nth convergent is still given by Definition II, but the law of recurrence is different, namely

$$\begin{aligned} p_n &= a_n\, p_f + b_n\, p_g + c_n\, p_h, \\ q_n &= a_n\, q_f + b_n\, q_g + c_n\, q_h, \\ r_n &= a_n\, r_f + b_n\, r_g + c_n\, r_h, \end{aligned}$$

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where f is the suffix of the latest matrix before the nth which has a in the zero row, and g and h are similarly defined. This result follows at once from the form of the matrices which shows that p_f say, cannot be annihilated until the letter a has again occurred in a zero row.

Thus for example

$$p_6 = a_6\, p_2 + b_6\, p_4 + c_6\, p_5$$

and

$$\begin{bmatrix} p_5 & p_2 & p_4 \\ q_5 & q_2 & q_4 \\ r_5 & r_2 & r_4 \end{bmatrix} = C_1\, A_2\, C_3\, B_4\, C_5;$$

whence, replacing the matrices by determinants, we have

$$\begin{vmatrix} p_5 & p_2 & p_4 \\ q_5 & q_2 & q_4 \\ r_5 & r_2 & r_4 \end{vmatrix} = a_2\, c_3\, b_4\, c_5$$

which indicates the form taken by (5) when applied to fractions of this type. It may be noted that the determinant of each of the matrices A, B, C is represented by the corresponding small letter.

The matrix method of generation of general non-cyclic continued fractions from numbers is illustrated by the following numerical case.

$$\begin{bmatrix} 150 \\ 103 \\ 116 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 58 \\ 34 \\ 45 \end{bmatrix}, \quad \begin{bmatrix} 58 \\ 34 \\ 45 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 17 \\ 11 \\ 7 \end{bmatrix}.$$

The first of these results is obtained by dividing each member by 58, the second by dividing by 17.

Proceeding in this way we obtain

$$\begin{bmatrix} 150 \\ 103 \\ 116 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 2 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The result is of course not unique as it depends on the choice of divisor at each stage.

For further developments of non-cyclic fractions the paper [3] cited at the end should be consulted.

Ascending Continued Fractions.

If the two-dimensional continued fraction

$$a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \dots}} \tag{15}$$

be reflected in a mirror placed above the fraction, with its plane perpendicular to the plane of the paper, we obtain the ascending continued fraction

$$a_1 + \frac{a_2 + \frac{a_3 + \dots}{b_3}}{b_2} \tag{16}$$

which is at once seen to be equivalent to the series

$$a_1 + \frac{a_2}{b_2} + \frac{a_3}{b_2 b_3} + \frac{a_4}{b_2 b_3 b_4} + \dots \tag{17}$$

Continued fractions of this type arise naturally from the non-homogeneous linear difference equation of the first order. For consider the equation

$$b(x)\, u(x) = u(x+1) + a(x).$$

We have by repeated substitution

$$b(x)\, u(x) = a(x) + \frac{a(x+1) + \frac{a(x+2) + \dots}{b(x+2)}}{b(x+1)},$$

which furnishes a particular solution if the ascending continued fraction converges. Thus for instance the equation

$$u(x+1) - x\, u(x) = -e^{-r} r^x$$

yields the particular solution

$$\begin{aligned} u(x) &= e^{-r} r^x \left\{ \frac{1}{x} + \frac{r}{x(x+1)} + \frac{r^2}{x(x+1)(x+2)} + \dots \right\} \\ &= \int_0^r t^{x-1} e^{-t}\, dt, \end{aligned}$$

when the real part of x is positive: this is an Incomplete Gamma function, which reduces to Prym's function when $r = 1$.

If we denote the sum of the first n terms of the series (17) by $p(n)/q(n)$ we have

$$\frac{p(n)}{q(n)} = \frac{p(n-1)}{q(n-1)} + \frac{a_n}{b_2 b_3 \dots b_n}$$

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which gives, on the supposition that no common factors are cancelled,

$$(18)\qquad \begin{aligned} q(n) &= b_2 b_3 \ldots b_n, \\ p(n) &= b_n\, p(n-1) + a_n. \end{aligned}$$

In the matrix notation this becomes

$$(19)\qquad \begin{bmatrix} q(n) & 0 \\ p(n) & 1 \end{bmatrix} = \begin{bmatrix} q(n-1) & 0 \\ p(n-1) & 1 \end{bmatrix} \begin{bmatrix} b_n & 0 \\ a_n & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ a_1 & 1 \end{bmatrix} \begin{bmatrix} b_2 & 0 \\ a_2 & 1 \end{bmatrix} \begin{bmatrix} b_3 & 0 \\ a_3 & 1 \end{bmatrix} \ldots \begin{bmatrix} b_n & 0 \\ a_n & 1 \end{bmatrix}$$

which is the matrix representation of an ascending continued fraction; and is seen to be the reflection, in the manner already described, of the matrix representation of a two-dimensional descending fraction.

To generalise this result we denote by B_m the matrix of k rows and columns

$$(20)\qquad \begin{bmatrix} b_m & 0 & 0 & 0 & \ldots & 0 \\ a_m & 1 & 0 & 0 & \ldots & 0 \\ j_m & 0 & 1 & 0 & \ldots & 0 \\ i_m & 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \ldots & 0 \\ c_m & 0 & 0 & 0 & \ldots & 1 \end{bmatrix},$$

and by $B_{m,0}$ the matrix consisting of the first column of B_m. We shall suppose that $b_1 = 1$.

DEFINITION IV. *The matrix B_m of k rows and columns being defined by* (20), *the matrix product*

$$B_1\, B_2\, B_3 \ldots B_n \ldots$$

is a cyclic ascending continued fraction in k dimensions.

It may be observed that a matrix of type B_m arises from the matrix J_m of (1) if the last row of the latter be moved to the top. This operation may be considered as the analogue of reflection in a mirror of the two-dimensional descending fraction.

The above definition does not exhaust the possibilities of generalising (19). In fact it would appear that we could define a generalised ascending fraction as a product of matrices obtained by arbitrary rearrangements of the last $k-1$ rows of B_m. This matter is reserved for future consideration.

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DEFINITION V. *The matrix product*

$$B_1 B_2 \ldots B_{n-1} B_{n,0}$$

is the nth convergent of the ascending continued fraction.

Thus we write

$$(21) \qquad \begin{bmatrix} q(n) \\ p(n) \\ w(n) \\ \cdot \\ r(n) \end{bmatrix} = B_1 B_2 \ldots B_{n-1} \begin{bmatrix} b_n \\ a_n \\ j_n \\ \cdot \\ c_n \end{bmatrix},$$

and we call the numbers $p(n)$, $q(n)$, $\ldots$, the components of the nth convergent.

DEFINITION VI. *The value of a non-terminating ascending continued fraction is the limit when $n \to \infty$ of the ratio $p(n) : q(n) : \ldots : w(n)$, provided that this limit exists. If the fraction terminates, the value is the ratio of the components of the last convergent.*

From Definition V we have

$$(22) \quad \begin{bmatrix} q(n) & 0 & \ldots & 0 \\ p(n) & 1 & \ldots & 0 \\ \cdot & \cdot & \ldots & \cdot \\ r(n) & 0 & \ldots & 1 \end{bmatrix} = \begin{bmatrix} q(n-1) & 0 & \ldots & 0 \\ p(n-1) & 1 & \ldots & 0 \\ \cdot & \cdot & \ldots & \cdot \\ r(n-1) & 0 & \ldots & 1 \end{bmatrix} \begin{bmatrix} b_n & 0 & \ldots & 0 \\ a_n & 1 & \ldots & 0 \\ \cdot & \cdot & \ldots & \cdot \\ c_n & 0 & \ldots & 1 \end{bmatrix}$$

which gives the recurrence relations

$$(23) \qquad \begin{aligned} p(n) &= b_n\, p(n-1) + a_n, \\ q(n) &= b_n\, q(n-1), \\ r(n) &= b_n\, r(n-1) + c_n, \\ &\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

If in (22) we replace the matrices by determinants we obtain

$$q(n) = b_2 b_3 \ldots b_n,$$

which is indeed obvious from the second of relations (23).

Periodic Ascending Continued Fractions.

A periodic fraction will be of the type

$$B_1 B_2 \ldots B_i C_1 C_2 \ldots C_h C_1 C_2 \ldots C_h C_1 C_2 \ldots C_h \ldots$$

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where C_s denotes a matrix of the type B_s. Writing

$$B = B_1 B_2 \ldots B_i, \quad C = C_1 C_2 \ldots C_h, \quad n = i + mh + j, \quad j < h,$$

$$BCB^{-1} = \begin{bmatrix} \beta & 0 & 0 & \ldots & 0 \\ \alpha & 1 & 0 & \ldots & 0 \\ \kappa & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \\ \gamma & 0 & 0 & \ldots & 1 \end{bmatrix},$$

we have

$$\begin{bmatrix} q(n+h) \\ p(n+h) \\ \cdot \\ r(n+h) \end{bmatrix} = BCB^{-1} \begin{bmatrix} q(n) \\ p(n) \\ \cdot \\ r(n) \end{bmatrix},$$

which gives the difference equations

$$\begin{aligned} p(n+h) &= \alpha q(n) + p(n), \\ q(n+h) &= \beta q(n), \\ r(n+h) &= \gamma q(n) + r(n). \end{aligned}$$

Putting $p(n) = px^n$, $q(n) = qx^n$, $\ldots$, the second equation gives $x^h = \beta$ so that the value of the continued fraction is

$$p : q : \ldots : w = \alpha : (\beta - 1) : \gamma : \ldots : \kappa$$

and the value exists if $\beta \neq 1$.

Thus for example in the case of the ascending fraction

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 3 & 1 \end{bmatrix} \ldots, \; BCB^{-1} = \begin{bmatrix} 10 & 0 \\ 17 & 1 \end{bmatrix},$$

the value is $17 : 9$ which is readily verified from the series (17).

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