ON A NON-ANALYTIC PERTURBATION PROBLEM

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(received April 19, 1961)

1. Introduction. The purpose here is to study a type of perturbation problem, arising from a differential equation, which is not included in the realm of analytic or asymptotic perturbation theory [4], [6]. Such a problem arises when the domain of the differential operator has been subjected to a variation (rather than the formal operator). We propose to outline one simple problem of this type, concerned with a second order ordinary differential operator. Our purpose is to obtain asymptotic estimates for the characteristic values of a regular Sturm-Liouville problem on a closed interval $[a, b]$ when b is near a singular point of the differential operator. Similar results have been obtained in [3], [8], and [9] by other methods.

Certain situations in quantum mechanics leading to variational problems of the present type have received considerable attention; see [3] for references leading back to the physical origin of the problem. Related questions have been considered by Schiffer and Spencer [7] in connection with the variation of Green's function and other domain functionals for the Laplacian operator on finite Riemann surfaces.

Our treatment here may be regarded as a model for the more difficult situations that arise in connection with elliptic partial differential equations. We remark that such problems for elliptic operators on surfaces arise when the surface is deformed topologically by cutting a hole or attaching a handle, and adjoining a boundary condition on any new boundary component thereby introduced.

Sturm-Liouville problems will be considered for the formally self-adjoint differential operator M defined by

$$k(s)Mx = -[p(s)x']' + q(s)x \tag{1}$$

on the basic real interval $[a, \omega)$, where a prime denotes differentiation with respect to s. The functions k, p, q are real-

Canad. Math. Bull. vol. 4, no. 3, September 1961

valued piecewise continuous functions on the basic interval with the further properties that k and p are positive-valued, and p is differentiable. The point ω is supposed to be a singularity of M and the possibility that $\omega = \infty$ is not excluded. We shall consider only the case that ω is a limit point singularity in H. Weyl's classification [1], [5]. The limit circle case has been considered elsewhere [10] by the author.

2. Basic and perturbed problems. The following notation will be used

(2) $$(x, y)_s = \int_a^s x(t)\overline{y}(t)k(t)dt, \qquad a \leq s < \omega$$

The symbol (x, y) will be used for the left member of (2) when s has been replaced by ω. The notations $L^2(a, b)$, $L^2(a, \omega)$ will be used for the Lebesgue spaces with respective inner products $(x, y)_b$, (x, y) and norms $||x||_b$, $||x||$. (This consitutes a slight departure from the customary usage.) Let B_a and B_b be the linear boundary operators defined by

$$B_a y = \alpha_o y(a) + \alpha_1 y'(a)$$

$$B_b y = \beta_o(b)y(b) + \beta_1(b)y'(b) \qquad (a < b < \omega),$$

with α_o, α_1 real and not both zero, and $\beta_o(b)$, $\beta_1(b)$ real-valued and not both zero for any b on $[b_o, \omega)$ for some fixed b_o. Let D_b be the set of all complex-valued functions $y \in L^2(a, b)$ which have the following properties: (a) y is differentiable and y' is absolutely continuous on $[a, b]$; (b) $My \in L^2(a, b)$; and (c) y satisfies the homogeneous boundary conditions $B_a y = B_b y = 0$. Consider the regular Sturm-Liouville problem

(3) $$My = ny, \qquad y \in D_b$$

on the interval $[a, b]$. Our problem is to estimate the characteristic values $n = n_b$ and the characteristic functions $y = y_b$ in terms of the characteristic values m and the characteristic functions x of the singular problem

(4) $$Mx = mx, \qquad x \in D,$$

where D is the set of all complex-valued functions $x \in L^2(a, \omega)$

with the properties (a) x is differentiable on $[a,\omega)$ and x' is absolutely continuous on every closed subinterval of this interval; (b) $Mx \in L^2(a,\omega)$; and (c) x satisfies the homogeneous boundary condition $B_a x = 0$. No boundary condition is required at the singular endpoint ω in the limit point case under consideration.

The characteristic value problem (4) is called the basic problem and (3) is called a perturbed problem. The endpoint b is the perturbation parameter. In various problems of practical interest [3], [8], the characteristic values m and characteristic functions x of the basic problem are easily calculated (examples of the latter being Legendre, Laguerre, and Hermite polynomials). Our main purpose is to obtain asymptotic estimates for each characteristic value n_b for b near ω under suitable hypotheses. The results are obtained here by a method communicated by Professor H. F. Bohnenblust, in which a suitable projection mapping on the space $L^2(a,b)$ is considered. Similar results were obtained previously by the author [8], [9] by means of an integral equations approach.

3. Asymptotic estimates for characteristic values. Let j_o be a complex number with $\operatorname{Im} j_o \neq 0$, and let M_o be the differential operator $M - j_o$. Let $h(s)$ be the uniquely determined solution of the boundary value problem

(5) $$M_o h = 0, \qquad B_a h = 1, \qquad B_b h = 0.$$

The uniqueness follows because j_o is not a characteristic value of the real, symmetric operator M on D_b. Thus $M_o g = B_a g = B_b g = 0$ imply that g is the zero function. Such a solution h of (5) is called an M_o measure on $[a,b]$ with respect to the boundary operators B_a, B_b.

Let x, w be linearly independent solutions of $Mx = mx$ with $x \in D$, $||x|| = 1$, and $p(xw' - wx') = 1$ [5]. The following assumptions will be made in the sequel; they are satisfied in most situations of practical interest [8].

https://doi.org/10.4153/CMB-1961-027-6 Published online by Cambridge University Press

(i) The singularity ω is not an accumulation point of the zeros of x, w and the quantity $\rho_b = x(b)/w(b)$ is $o(1)$ as $b \to \omega$.

(ii) the norm $||h||_b$ of the M_o measure on $[a, b]$ is a bounded function of b on $[b_o, \omega)$ for some fixed $b_o > a$.

(iii) The boundary operator B_b satisfies the condition that $w(b)B_b x/x(b)B_b w$ is bounded on $[b_o, \omega)$.

The third assumption is a rather mild restriction on the boundary condition at $x = b$. See [8; p. 838] for specific examples when ω is a regular singularity or an irregular singularity of finite rank.

In terms of the linearly independent solutions x, w of $Mx = mx$ described above, define $z(s) = x(s) - (B_b x/B_b w)w(s)$. Then $Mz = mz$, $B_b z = 0$, and $B_a z = O(\rho_b)$ $(b_o \leq b < \omega)$, where assumption (iii) has been used. Let G_b denote the linear integral operator whose kernel is the Green's function on $[a, b]$ associated with the operator M_o and the boundary conditions $B_a y = B_b y = 0$. Define $f = z - (m - j_o)G_b z$, which is a twice differentiable solution of $M_o f = 0$ satisfying $B_a f = O(\rho_b)$ and $B_b f = 0$. Since the M_o measure with respect to B_a, B_b is uniquely determined, it follows that $f(s) = (B_a f)h(s)$ and hence $||f||_b = O(\rho_b)||h||_b$. Since $||h||_b$ is bounded by assumption (ii), there exists a constant C such that $||f||_b \leq C\rho_b ||z||_b$, and hence

(6) $$||z - (m - j_o)G_b z||_b \leq C\rho_b ||z||_b.$$

Let $P(d)$ $(0 < d \leq d_o)$ denote the projection mapping from the Hilbert space $L^2(a, b)$ onto its subspace spanned by all characteristic functions $y^{(n)}$ of (3) with $|n - m| \leq d$. The following fundamental lemma can be proved by applying the Parseval completeness formula to the set $\{y^{(n)}\}$. The details of the proof have been given elsewhere [10] and will be omitted.

LEMMA. For any $Z \in L^2(a,b)$,

$$||Z - P(d)Z||_b \leq (1 + |m - j_o|/d)||Z - (m - j_o)G_b Z||_b.$$

Application of the lemma to $Z = z$ and use of (6) show that there exists a constant A such that

$$||z - P(d)z||_b \leq (A\rho_b/2d)||z||_b$$

on $0 < d \leq d_o$, $b_o \leq b < \omega$. With the choice $d = A\rho_b$, we obtain $||z - P(A\rho_b)z||_b \leq (1/2)||z||_b$. Then $P(A\rho_b)z = 0$ implies that $z = 0$ on $[a, b]$. Hence there exists at least one characteristic value $n = n_b$ satisfying $|n_b - m| \leq A\rho_b$ $(b_o \leq b < \omega)$. It can be deduced from the maximum-minimum principle for characteristic values [2] that there is exactly one. The details of the proof are given in [10] and will not be duplicated here.

THEOREM. Suppose that the singularity ω of (1) is of the limit point type and that the assumptions of this section hold. Then for every characteristic value m of the basic problem (4) there exist constants b_o and A, independent of b, such that a unique characteristic value n_b of the perturbed problem (3) lies in the interval $|n_b - m| \leq A\rho_b$ whenever $b_o \leq b < \omega$.

In particular, there exists a unique n_b such that $n_b \to m$ as $b \to \omega$. In the special case that $p(s) = 1$ and $\omega = 0$ is a regular singularity with real exponents λ, μ $(\lambda > \mu)$ it turns out [8] that $\rho_b = 0(b^{\lambda - \mu})$. The following uniform estimates on $a \leq s \leq b$ for characteristic functions y_b, x associated with n_b, m respectively also can be derived by the method used in [10]:

$$y_b(s) = x(s) - (B_b x/B_b w)w(s) + 0(\rho_b), \qquad b \geq b_o,$$

$$||x|| = ||y_b||_b = 1.$$

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https://doi.org/10.4153/CMB-1961-027-6 Published online by Cambridge University Press