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HANDLES, SEMIHANDLES, AND DESTABILIZATION OF ISOTOPIES

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1. Introduction. Kirby and Siebenmann showed that handles of index other than 3 can be straightened, and then went on to prove an abundance of immensely important results such as triangulation theorems, Hauptvermutungen, and classification and structure theorems. (See [**7**; **6**; **2**; **3**; **8**; **4**; **5**].) Their proofs relied on some nonsimply connected surgery techniques and computations due to Wall, Hsiang, and Shaneson [**10**; **11**; **12**; **1**], which at the time seemed so difficult as to be practically inaccessible to the large number of mathematicians interested in the consequences. The main result presented here, viz., that a stably straightened handle can be straightened, was obtained by the author some time ago in an attempt to prove the stable homeomorphism theorem without the assistance of Wall, Hsiang and Shaneson (cf. [**13**]). Although these attempts have been thwarted by some difficulties in formulating the appropriate obstruction theory, the results and techniques are still quite interesting. Moreover, as an unexpected bonus, it has turned out that they are quite useful, and probably necessary, in classifying PL structures. For since the original handle-straightening results were not valid for index 3, results in the spirit of those presented here would be needed to prove general results which hold without the dimensional restrictions imposed by the work of Wall, Hsiang and Shaneson. Kirby and Siebenmann [**4**; **5**] have now independently announced similar (and somewhat more general) results and indicated their role in their "Classification Programme", though they (as well as their proofs) remain rather obscure in that environment. In fact Kirby and Siebenmann may have possessed clandestine knowledge of such results before the author even knew what a Main Diagram was.

2. Definitions and statements of the main results. Let R^n denote Euclidean n-space with the sum-of-the-absolute-values metric and rB^n the n-cell in R^n of radius r centered at 0. Let $B^n = 1B^n$ and

$$J^{n-1} = \{1\} \times B^{n-1} \cup B^1 \times \partial B^{n-1} \subseteq B^1 \times B^{n-1} = B^n.$$

A (k, n)-*handle* (respectively, *a* (k, n)-*semihandle*) is a TOP homeomorphism $h\colon B^k \times R^n \twoheadrightarrow V^{k+n}$, where V is a PL $(k + n)$-manifold and h is PL on $\partial B^k \times R^n$ (respectively, on $J^{k-1} \times R^n$). Suppose $A \subseteq B^k \times R^n$. We say that a (semi-)

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handle $h\colon B^k \times R^n \twoheadrightarrow V^{k+1}$ can be (*semi-*) *straightened modulo* A if and only if there is an isotopy $h_t\colon B^k \times R^n \twoheadrightarrow V$ $(t \in [0, 1])$ such that $h_0 = h$, h_1 is PL on $B^k \times B^n$, and $h_t = h$ on $\partial B^k \times R^n$ $(J^{k-1} \times R^n)$, on A, and outside a compact set. More generally, h can be *stably* (*semi-*) *straightened modulo* A if and only if $h \times 1_{R^m}$ can be (semi-) straightened modulo $A \times R^m$. If $A = \emptyset$, we suppress it in our terminology.

The following two straightening theorems, both of which avoid the index $\neq 3$ restrictions, are our main results.

THEOREM 1. *For $k + n \geqq 6$ and $k \geqq 1$, every (k, n)-semihandle h can be semistraightened.*

THEOREM 2. *For $k + n \geqq 5$, a (k, n)-handle can be straightened if and only if it can be stably straightened.*

Theorem 1 may seem familiar as it is a disguised version of the "concordance implies isotopy" theorem of [**2**] and [**8**]. Since this result has received somewhat scant treatment in the previous literature, we shall be a little more explicit with it here.

3. Preliminary lemmas. This section consists of a number of relatively simple results which show that our definitions of handle-straightening and semihandle-straightening are equivalent to several alternate but useful types of straightening. This entire study can be accomplished without these lemmas but not without introducing technical encumbrances and sacrificing clarity. It is simply more convenient to bounce back and forth between these various equivalent concepts. Since such bouncing will often be implicit and unnoticeable, this section may be skipped. And at any rate, the proofs will be omitted here and left to a more elaborate exposition elsewhere. Thus these results are listed here mainly to reassure scrutinizing readers.

Lemmas 1 and 2 tell us that if we are willing to sacrifice "compact support", we can obtain (semi-) handles which are PL on all of $B^k \times R^n$.

LEMMA 1. *Let $h\colon B^k \times R^n \twoheadrightarrow V^{k+n}$ be a (k, n)-(semi-) handle which is PL on $B^k \times B^n$ where $k + n \geqq 5$. Then h is isotopic to a PL homeomorphism modulo $\partial B^k \times R^n$ $(J^{k-1} \times R^n)$.*

LEMMA 2. *If $k + n \geqq 5$ and the (k, n)-(semi-) handle h can be (semi-) straightened, then it is isotopic to a PL homeomorphism modulo $\partial B^k \times R^n$ $(J^{k-1} \times R^n)$.*

The next two lemmas allow us to eliminate worries about the condition "PL on a neighborhood of $\partial B^k \times R^n$ $(J^{k-1} \times R^n)$" in high-dimensional (semi-) handle-straightening theory. Some similar results seem to be relied on implicitly in [7] and are made more explicit later in [**2**], although the proof given there appears to be incorrect.

LEMMA 3. *Let $h: B^k \times R^n \twoheadrightarrow V^{k+n}$ be a (k, n)-(semi) handle which is PL on*

$$(B^k - q\dot{B}^k) \times R^n \ ((B^k - (-1, q) \times q\dot{B}^{k-1}) \times R^n)$$

for some $q \in (0, 1)$. If h can be (semi-) straightened, then h can be (semi-) straightened modulo $(B^k - r\dot{B}^k) \times R^n$ $((B^k - (-1, r) \times r\dot{B}^{k-1}) \times R^n)$ for any $r \in (q, 1)$. Also if h is isotopic to a PL homeomorphism modulo $\partial B^k \times R^n$ $(J^{k-1} \times R^n)$, then it is isotopic to a PL homeomorphism modulo $(B^k - r\dot{B}^k) \times R^n$ $((B^k - (-1, r) \times r\dot{B}^{k-1}) \times R^n)$ for any $r \in (q, 1)$.

LEMMA 4. *Let $h: B^k \times R^n \twoheadrightarrow V^{k+n}$ be a (k, n)-(semi-) handle and $q \in (0, 1)$. Then there is a (k, n)-(semi-) handle $g: B^k \times R^n \twoheadrightarrow V^{k+n}$ which is PL on $(B^k - q\dot{B}^k) \times R^n$ $(B^k - (-1, q) \times q\dot{B}^{k-1})$ such that g can be (semi-) straightened if and only if h can, and g is isotopic to a PL homeomorphism modulo $\partial B^k \times R^n$ $(J^{k-1} \times R^n)$ if and only if h is. Moreover, g can be chosen to be isotopic to h modulo $\partial B^k \times R^n$ $(J^{k-1} \times R^n)$.*

4. Proof of Theorem 1. The proof consists of the construction of a large commutative Main Diagram. The left half of the diagram is constructed just as in [**7**] (or [**2**]). Thus α is a PL immersion and e an exponential cover, both chosen so that $\alpha e | B^k \times 2B^n$ makes sense and is in fact the inclusion map. W_0 is just $B^k \times (T^n - D^n)$ with the PL structure induced by $h\alpha$. Now W is just $B^k \times T^n$ with the PL structure obtained by "capping off" that of W_0. (See [**7**] and [**2**]. The argument needed here is more delicate than that of [**7**] and [**2**], since h is not PL on all of $\partial B^k \times T^n$, but the spirit is the same. Also Lemma 4 must be used here.) Then g is just the identity map, but g is

$$\begin{tikzcd}
& B^k\times pB^n \arrow[rr, "G"] & & B^k\times pB^n & B^k\times B^n \arrow[l, "G"'] \arrow[d, "G"] \\
& B^k\times R^n \arrow[u, "\Lambda"] \arrow[d, "e"] \arrow[rr, "F"] & & B^k\times R^n \arrow[u, "\Lambda"'] \arrow[d, "e"] & B^k\times pB^n \arrow[ul, "\psi"] \arrow[dddll, bend left, "\varphi"] \\
B^k\times 2B^n \arrow[uur, "\subseteq"] \arrow[ur, "\subseteq"'] \arrow[ddr, "\subseteq"'] & B^k\times T^n \arrow[r, "g"] & W & B^k\times T^n \arrow[l, "\tilde{g}"'] & \\
& B^k\times(T^n-D^n) \arrow[u, "\subseteq"] \arrow[d, "\alpha"] \arrow[r, "l"] & W_0 \arrow[u, "\subseteq"'] \arrow[d, "h\alpha"] & & \\
& B^k\times R^n \arrow[r, "h"] & V & &
\end{tikzcd}$$

Main diagram

PL only on $J^{k-1} \times T^n$ (or a neighborhood thereof) insofar as we know. The PL homeomorphism $\tilde{g}$ is obtained by recalling that $k \geqq 1$ and viewing W as an S-cobordism between $g(\{-1\} \times B^{k-1} \times T^n)$ and $g(\{+1\} \times B^{k-1} \times T^n)$ which is trivial over the boundary. The relative S-cobordism theorem [**9**] applies to give $\tilde{g}$. Note that $\tilde{g}$ can be taken to be g on $J^{k-1} \times T^n$. F is a PL homeomorphism which lifts $\tilde{g}^{-1}g$. Unique lifting shows that F is bounded and can be taken to be the identity on $J^{k-1} \times R^n$. Next $p > 3$ is chosen so that $F(B^k \times 2B^n) \subseteq B^k \times (p-1)\dot{B}^n$. The map Λ is a version of Kirby's "fold-down" embedding which fixes $B^k \times (p-1)B^n$ and has $B^k \times pB^n - \{(-1, 0)\} \times p\partial B^n$ as its image (cf. [**2**]. Here $(-1, 0) \in B^1 \times B^{k-1} = B^k$ and $p\partial B^n = \partial(pB^n)$.). The construction of Λ and the boundedness of F allow us to define a homeomorphism G: $B^k \times pB^n \twoheadrightarrow B^k \times pB^n$ such that $G \circ \Lambda = \Lambda \circ F$ and $G = \text{id}$ on $J_0 = J^{k-1} \times pB^n \cup B^k \times p\partial B^n$. Using a careful double engulfing argument, we can construct a PL embedding ψ: $B^k \times pB^n \to G(B^k \times 2\dot{B}^n)$ such that $\psi = \text{id}$ on $G(B^k \times B^n)$ and

$$\psi^{-1}(\{-1\} \times B^{k-1} \times pB^n) = \{-1\} \times B^{k-1} \times pB^n.$$

Let $\varphi = h\alpha\tilde{g}e\,\Lambda^{-1}\psi$. Then φ is a well-defined PL embedding and $\varphi = hG^{-1}$ on $G(B^k \times B^n)$. Let γ: $B^k \times pB^n \times I \twoheadrightarrow B^k \times B^n \times I$ be the isotopy from G to 1 modulo J_0 which is defined by coning over $B^k \times pB^n \times 0 \cup J_0 \times I$ from the point $(-1, 0, 1) \in B^1 \times (B^{k-1} \times pB^n) \times I$. It can be easily checked that the desired isotopy h_t may be defined by $h_t = \varphi\gamma_t G^{-1}\varphi h$ on $h^{-1}(\text{Im}\varphi)$ and $h_t = h$ elsewhere.

5. Proof of Theorem 2. Theorem 2 follows immediately from Theorem 2′ below and induction (and Lemma 2).

THEOREM 2′. *Let h: $B^k \times R^n \twoheadrightarrow V^{k+n}$ be a (k, n)-handle such that $h \times 1_R$ is isotopic modulo $\partial B^k \times R^{n+1}$ to a PL homeomorphism. Then h can be straightened.*

Proof of Theorem 2′. There is an isotopy H_t: $B^k \times R^{n+1} \twoheadrightarrow V \times R$ (which we choose to parameterize by $B^1 = [-1, 1]$ so that $-1 \leqq t \leqq 1$) such that $H_{-1} = h \times 1$, H_1 is PL, and $H_t = h \times 1$ on $\partial B^k \times R^{n+1}$ for $-1 \leqq t \leqq 1$. For $(t, x) \in B^1 \times B^k \times R^{n+1}$, $H(t, x) = (t, H_t(x))$ defines a $(k+1, n+1)$-semihandle H: $B^{k+1} \times R^{n+1} \twoheadrightarrow B^1 \times V \times R$. So Theorem 1 yields an isotopy

$$G_t\colon B^{k+1} \times R^{n+1} \twoheadrightarrow B^1 \twoheadrightarrow B^1 \times V \times R \quad (t \in I)$$

such that $G_0 = H$, G_1 is PL on $B^{k+1} \times qB^{n+1}$ for some $q > 1$, and $G_t = H$ on $J^k \times R^{n+1} \cup B^{k+1} \times (R^{n+1} - rB^{n+1})$ for some $r > q$. (See Figure 1. Note that $H|\{-1\} \times B^k \times R^{n+1}$ is not PL, though Figure 1 obscures this fact.)

Now let $M = \{-1\} \times B^k \times q\dot{B}^n \times [1, r]$, which is a copy of $B^{k+1} \times R^n$, and let $g = G_1|M$: $M \twoheadrightarrow G_1(M)$. Then g is PL on $J_1 = \{-1\} \times B^k \times q\dot{B}^n \times \{1\} \cup \{-1\} \times \partial B^k \times q\dot{B}^n \times [1, r]$. And although g is not necessarily PL on $\{-1\} \times B^k \times q\dot{B}^n \times \{r\}$, we at least know that $N = g(\{-1\} \times B^k \times q\dot{B}^n \times \{r\})$ is an open subset of $\{-1\} \times V \times \{r\}$. So N inherits a PL structure from $\{-1\} \times V \times \{r\}$ and is a PL locally flat submanifold of $B^1 \times V \times R$ (as $\{-1\} \times V \times \{r\}$ is). Consequently $\partial G_1(M) = G_1(\partial M)$ is a PL locally flat

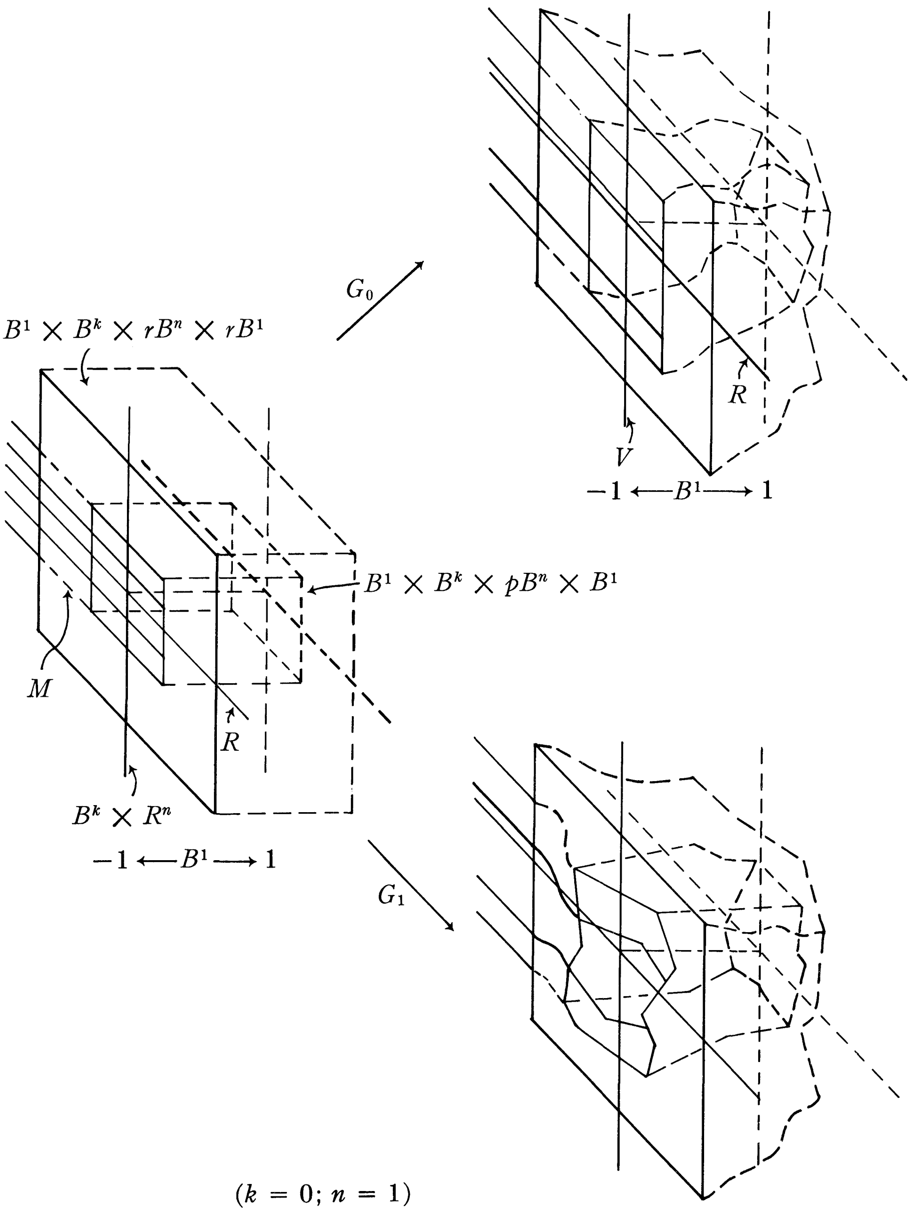

FIGURE 1

submanifold of $B^1 \times V \times R$ and $G_1(M)$ inherits a PL manifold structure thereby. Thus g: $M \twoheadrightarrow G_1(M)$ can be viewed as a $(k + 1, n)$-semihandle. Applying Theorem 1 once again, we obtain an isotopy g_t: $M \twoheadrightarrow G_1(M)$ such that $g_0 = g$, g_1 is PL on $\{-1\} \times B^k \times B^n \times [1, r]$ and $g_t = g$ on $J_1 \cup \{-1\} \times B^k \times (q\dot{B}^n - pB^n) \times [1, r]$ for some $p > 0$. This last condition allows us to extend g_t to an isotopy

$$\bar{g}_t: \{-1\} \times B^k \times R^n \times \{r\} \twoheadrightarrow \{1\} \times V \times \{r\}$$

via H outside M. (Figure 2 is recommended at this point.) A simple check verifies that $\bar{g}_t$ essentially straightens h (It actually straightens $\{-1\} \times h \times \{r\}$.).

6. Consequences. We shall present here a couple of the direct consequences of the preceding theorems. Less direct applications will be developed in another paper. (Also see [**4**] and [**5**].)

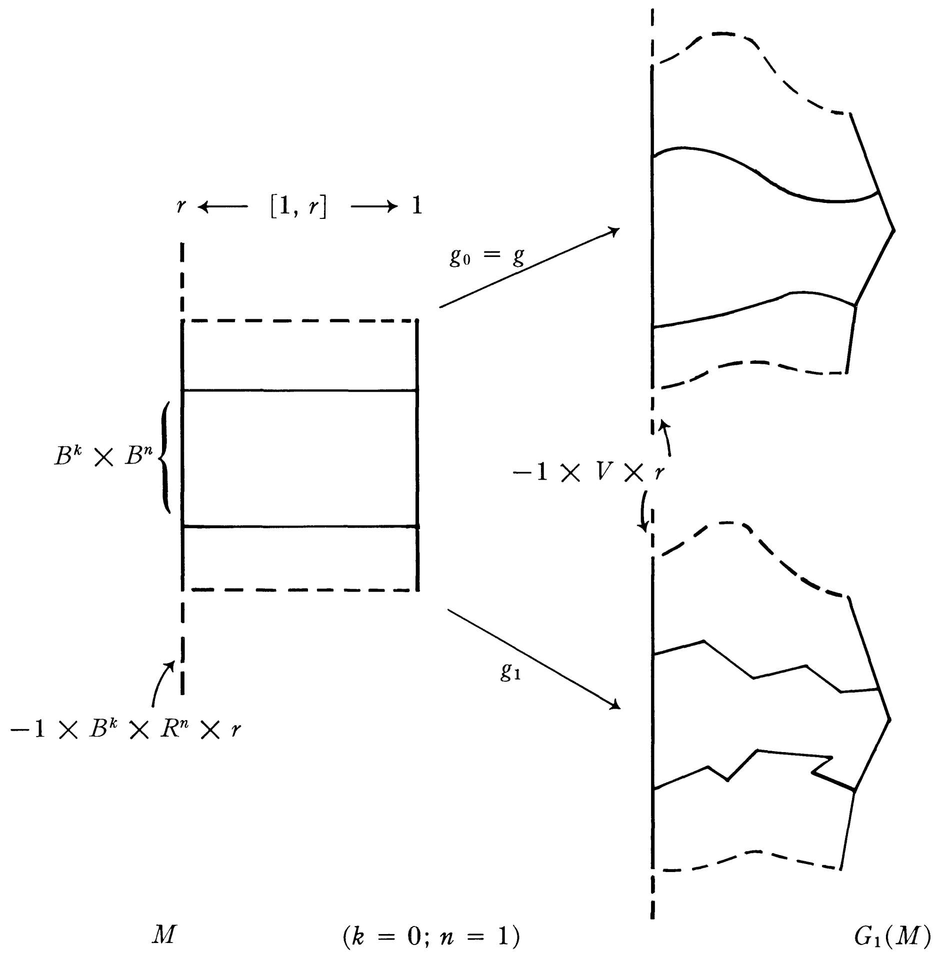

FIGURE 2

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THEOREM 3. *If each (k, n)-handle can be straightened, then so can each (k, m)-handle, provided $m \leqq n$ and $k + m \geqq 5$.*

THEOREM 4. *Every stably stable homeomorphism $h: R^n \twoheadrightarrow R^n$ is stable, for all $n \geqq 5$.*

Proof. If $h \times 1_{R^q}$ is stable, it is isotopic to a PL homeomorphism. So h can be straightened by Theorems 2 and 2′, and hence is isotopic to 1_{R^n}. Thus h is stable by Theorem 2 of [**3**].

We conclude this paper with an amusing proof of a converse of Lemma 2. The converse can also be obtained via a direct (?) but equally amusing application of the Main Diagram argument.

PROPOSITION. *If $k + n \geqq 5$ and the (k, n)-handle h is isotopic to a PL homeomorphism modulo $\partial B^k \times R^n$, then h can be straightened.*

Proof. Since $h \times 1_R$ is also isotopic to a PL homeomorphism modulo $\partial B^k \times R^{n+1}$, we can apply Theorem 2′ to straighten h.

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