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Furstenberg Transformations and Approximate Conjugacy

Huaxin Lin

Abstract. Let α and β be two Furstenberg transformations on 2-torus associated with irrational numbers θ_1, θ_2, integers d_1, d_2 and Lipschitz functions f_1 and f_2. It is shown that α and β are approximately conjugate in a measure theoretical sense if (and only if) $\overline{\theta_1 \pm \theta_2} = 0$ in $\mathbb{R}/\mathbb{Z}$. Closely related to the classification of simple amenable C^*-algebras, it is shown that α and β are approximately K-conjugate if (and only if) $\overline{\theta_1 \pm \theta_2} = 0$ in $\mathbb{R}/\mathbb{Z}$ and $|d_1| = |d_2|$. This is also shown to be equivalent to the condition that the associated crossed product C^*-algebras are isomorphic.

1 Introduction

A celebrated result of Giordano, Putnam and Skau [5] states that two minimal Cantor systems are strongly orbit equivalent if (and only if) the associated crossed product C^*-algebras are isomorphic, and this can also be described by their K-theory. Moreover, two such Cantor systems are topologically orbit equivalent if (and only if) certain parts of the K-theoretical information (namely the tracial range of the K_0-group) of the associated C^*-algebras are the same (up to unital order isomorphism). This note is an attempt to explore the possible analogy of this result in the case when the space is connected.

With the recent rapid development of the classification of amenable simple C^*-algebras of stable rank one ([1–3, 9, 12, 13], to name a few), it has become possible to apply C^*-algebra theory to the study of minimal homeomorphisms of more general spaces. Several versions of approximate conjugacy have been introduced and studied recently (see [14–17, 20]). In [16, 17], minimal homeomorphisms of the product of the Cantor set and the circle were studied. It was shown that if a certain set of K-theoretical information for two minimal homeomorphisms on the product of the Cantor set and the circle is the same, then they are approximately K-conjugate (and the converse also holds).

One of the reasons that the work Giordano, Putnam and Skau was so successful is that the Cantor set is totally disconnected. Perhaps orbit equivalence for Cantor minimal systems may be viewed as something which lives between measure theory and topology. When the space X is connected, the situation is very different. For example, by a result of Sierpinski for connected spaces, two topological orbit equivalent minimal homeomorphisms are in fact flip conjugate (see [19, Proposition5.5]). On

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the other hand, it is known (even in the case of the Cantor minimal systems) that two minimal dynamical systems whose associated crossed product C^*-algebras are isomorphic may not be flip conjugate. Therefore, for connected space, one should not expect that two minimal homeomorphisms are topologically orbit equivalent if their associated crossed product C^*-algebras have order isomorphic K-groups, or if their associated crossed product C^*-algebras are isomorphic.

We are interested in the following two questions:

Question 1: Let X be a (connected) compact metric space and let α and β be two minimal homeomorphisms of X. Let A_α and A_β denote the associated crossed product C^*-algebras. Suppose that the tracial range of $K_0(A_\alpha)$ and that of $K_0(A_\beta)$ are (unitally) order isomorphic. What can one say about the homeomorphisms α and β?

Question 2: Let X be a (connected) compact metric space and let α and β be two minimal homeomorphisms of X. Let A_α and A_β denote the associated crossed product C^*-algebras. Suppose that A_α and A_β are isomorphic in a way preserving the additional K-theoretical information consisting of $(j_\alpha)_*$ and $(j_\beta)_*$ (associated with the natural inclusions of $C(X)$ in A_α and A_β). What can one say about the homeomorphism α and β?

(For the definition of of j_α and j_β, see Definition 2.6. A clarification of this question will be discussed in Remark 2.11.)

Giordano, Putnam and Skau's results answered both questions, namely, α and β are topologically orbit equivalent in the case of Question 1 and α and β are strongly orbit equivalent in the case of Question 2, under the assumption that X is the Cantor set.

The results of [15–17, 20] suggest that the answer to Question 2 should be that α and β are approximately K-conjugate, and for Question 1, that α and β are approximately conjugate in a more measure theoretical sense. However, the spaces studied in the above mentioned articles are not connected. It would be interesting to see answers to Questions 1 and 2 for any connected spaces (other than $\mathbb{T}$).

A classical example of minimal homeomorphisms on the 2-torus $\mathbb{T}^2$ was studied by Furstenberg [4]. Let θ be an irrational number and let $g\colon \mathbb{T} \to \mathbb{T}$ be a continuous map. The Furstenberg transform $\alpha\colon \mathbb{T}^2 \to \mathbb{T}^2$ is defined to be $\alpha(\xi,\zeta) = (\xi e^{i2\pi\theta}, \zeta g(\xi))$ for $\xi \in \mathbb{T}$ and $\zeta \in \mathbb{T}$ with g being homotopically nontrivial. One may write that $\alpha = (\xi e^{i2\pi\theta}, \zeta\xi^d e^{i2\pi f(\xi)})$, where d is an integer and f is a real continuous function in $C(\mathbb{T})$. Let us denote α by $\Phi_{\theta,d,f}$. It is known that α is always minimal if $d \neq 0$ [4]. It is also known that if g satisfies the Lipschitz condition, then α is also uniquely ergodic [4]. This example has in fact been rather intensively studied (for example, [4, 6–8, 21, 24], to name a few).

It was conjectured by R. Ji [6] that $\Phi_{\theta,1,0}$ is conjugate to $\Phi_{\theta,1,f}$. By considering a quasi-discrete spectrum, counter-examples have been constructed by Rouhani [24], showing that $\Phi_{\theta,1,0}$ may not even be flip conjugate to $\Phi_{\theta,1,f}$.

Let $\alpha = \Phi_{\theta_1,d_1,f}$ and $\beta = \Phi_{\theta_2,d_2,f_2}$. In this note, we first show that if $\overline{\theta_1 \pm \theta_2} = 0$ in $\mathbb{R}/\mathbb{Z}$, then α and β are approximately conjugate in a measure theoretical sense (see Definition 2.1). We also show that the converse is true, *i.e.,* if α and β are approx-

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imately conjugate in that sense, then $\overline{\theta_1 \pm \theta_2} = 0$ in $\mathbb{R}/\mathbb{Z}$. Let $A_\alpha = C(\mathbb{T}^2) \rtimes_\alpha \mathbb{Z}$ and $A_\beta = C(\mathbb{T}^2) \rtimes_\beta \mathbb{Z}$ denote the associated crossed products. At least in the case that f_1 and f_2 satisfy the Lipschitz condition, the ranges of $K_0(A_\alpha)$ and $K_0(A_\beta)$ under the tracial map are the same, namely, $\mathbb{Z} + \mathbb{Z}(\theta_1)$ (we still assume that $\overline{\theta_1 \pm \theta_2} = 0$ in $\mathbb{R}/\mathbb{Z}$). This result seems closer to that of the topological orbit equivalence in the Cantor minimal systems.

It has been recently proved [18] that in the uniquely ergodic cases, A_α and A_β are unital simple C^*-algebras with tracial rank zero. Therefore, by the classification of unital simple amenable C^*-algebras with tracial rank zero (see [13]), A_α and A_β are isomorphic as C^*-algebras if and only if $\overline{\theta_1 \pm \theta_2} = 0$ in $\mathbb{R}/\mathbb{Z}$ and $|d_1| = |d_2|$. We will show that α and β are approximately K-conjugate if and only if $\overline{\theta_1 \pm \theta_2} = 0$ in $\mathbb{R}/\mathbb{Z}$ and $|d_1| = |d_2|$. In the process, we will also show that when $f_1 - f_2$ is in a dense subset of the real part of $C(\mathbb{T})$, Φ_{θ,d,f_1} and Φ_{θ,d,f_2} are actually conjugate (see 4.3 below).

The results of this note are very special. However, it is our hope that this special case will lead us to more interesting answers to Questions 1 and 2 and will serve as an invitation to further exploration.

2 The Main results

Definition 2.1 Let X and Y be two compact metric spaces and let $\alpha\colon X \to X$ and $\beta\colon Y \to Y$ be two minimal homeomorphisms. Denote by T_α and T_β the sets of α-invariant and β-invariant normalized Borel measures, respectively. Let us say that α, β are *approximately conjugate in the sense of M1* if there exist two sequences of homeomorphisms $\sigma_n\colon X \to Y$ and $\gamma_n\colon Y \to X$ and affine homeomorphisms $\Lambda_1\colon T_\alpha \to T_\beta$ and $\Lambda_2\colon T_\beta \to T_\alpha$ such that

$$\lim_{n\to\infty} \sup_{\nu \in T_\beta} \nu(\{y \in Y\colon\ \mathrm{dist}(\sigma_n \alpha \circ \sigma_n^{-1}(y), \beta(y)) \geq a\}) = 0, \tag{2.1}$$

$$\lim_{n\to\infty} \sup_{\mu \in T_\alpha} \mu(\{x \in X : \mathrm{dist}(\gamma_n \circ \beta \circ \gamma_n^{-1}(x), \alpha(x)) \geq a\}) = 0 \tag{2.2}$$

for all $a > 0$, and

$$\Lambda_1(\mu)(S) = \mu(\sigma_n^{-1}(S)), \quad \text{and} \quad \Lambda_2^{-1}(\nu)(G) = \nu(\gamma_n^{-1}(G)) \quad n = 1, 2, \dots \tag{2.3}$$

for all Borel sets $S \subset Y$, $G \subset X$ and for all $\mu \in T_\alpha$, $\nu \in T_\beta$.

A couple of remarks are in order.

(i) Suppose that there exists a homeomorphism $\sigma\colon X \to Y$ such that $\sigma \circ \alpha \circ \sigma^{-1} = \beta$. Define $\Lambda\colon T_\alpha \to T_\beta$ by $\Lambda(\mu)(S) = \mu(\sigma^{-1}(S))$ for all Borel sets $S \subset Y$. It should be noted that

$$\begin{aligned}
\Lambda(\mu)(\beta(S)) &= \mu(\sigma^{-1} \circ \beta(S)) \\
&= \mu(\sigma^{-1} \circ \beta \circ \sigma \circ \sigma^{-1}(S)) \\
&= \mu(\alpha \circ \sigma^{-1}(S)) = m(\sigma^{-1}(S)) = \Lambda(\mu(S))
\end{aligned}$$

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for all Borel sets $S \subset Y$. So $\Lambda(\mu) \in T_\beta$. In particular, conjugate homeomorphisms are approximately conjugate in the sense of M1.

In general, a sequence of homeomorphisms $\{\sigma_n\}$ does not preserve measures even though both (2.1) and (2.2) hold. One could have $\lim_{n\to\infty} \mu(\sigma_n(S)) = 0$. Here we require that $\{\sigma_n\}$ has some consistent information on measure spaces. So in Definition 2.1, one should note that the conditions in (2.3) are an important part of the definition.

(ii) In Definition 2.1, put

$$E_n(a) = \{y \in Y : \mathrm{dist}(\sigma_n\alpha \circ \sigma_n^{-1}(y), \beta(y)) \geq a\}$$

for $a > 0$. Put $S_n(a) = \{x \in X : \mathrm{dist}(\sigma_n \circ \alpha(x), \beta \circ \sigma_n(x)) \geq a\}$. Then

$$\sigma_n^{-1}(E_n(a)) = S_n(a).$$

By (2.1) and (2.4), $\lim_{n\to\infty} \sup_{\mu\in T_\alpha} \mu(S_n(a)) = 0$.

(iii) It is an easy exercise that approximately conjugacy in the sense of M1 is an equivalence relation among minimal homeomorphisms.

Definition 2.2 Let θ be an irrational number and let $g\colon \mathbb{T} \to \mathbb{T}$ be a continuous function with degree $d \neq 0$ (the winding number $d \neq 0$). A Furstenberg transform is a map $\alpha\colon \mathbb{T}^2 \to \mathbb{T}^2$ defined by $\alpha((\xi,\zeta)) = (\xi e^{i2\pi\theta}, \zeta g(\xi))$ for $\xi \in \mathbb{T}$ and $\zeta \in \mathbb{T}$. There is a real function $f \in C(\mathbb{T})$ such that $g(\xi) = \xi^d \exp(i2\pi f(\xi))$ for $\xi \in \mathbb{T}$. The map α is called the Furstenberg transform associated with the irrational number θ, integer d and function f. It will also be denoted by $\Phi_{\theta,d,f}$.

We are interested in the case that $(\mathbb{T}^2, \alpha)$ is uniquely ergodic. It is known that α is always minimal (see [4]). It is also shown in [4] that $(\mathbb{T}^2, \alpha)$ is uniquely ergodic if g has Lipschitz property (or f is Lipschitz). The unique invariant measure is the product of the normalized Lebesgue measure $m_2 = m \times m$. We fix the following metric on $\mathbb{T}^2$:

$$\mathrm{dist}((\xi,\zeta),(\xi',\zeta')) = \sqrt{|\xi\overline{\xi'} - 1|^2 + |\zeta'\overline{\zeta'} - 1|^2},$$

where $\xi, \xi', \zeta, \zeta' \in \mathbb{T}$.

We will keep these notation throughout this note.

Theorem 2.3 *Let $\alpha = \Phi_{\theta_1,d_1,f_1}, \beta = \Phi_{\theta_2,d_2,f_2}\colon \mathbb{T}^2 \to \mathbb{T}^2$ be uniquely ergodic Furstenberg transforms. Then the following are equivalent:*

(i) $\overline{|\theta_1 \pm \theta_2|} = 0$ *in* $\mathbb{R}/\mathbb{Z}$*;*

(ii) *α and β are approximately conjugate in the sense of M1.*

In this case, one can make

$$\lim_{n\to\infty} \sup\{\mathrm{dist}(\sigma_n\alpha \circ \sigma_n^{-1}(y), \beta(y)) : y \in Y)\} = 0,$$

$$\lim_{n\to\infty} \sup\{\mathrm{dist}(\gamma_n \circ \beta \circ \gamma_n^{-1}(x), \alpha(x)) : x \in X\} = 0,$$

if one does not insist that σ_n and γ_n to be continuous everywhere. More precisely, we have the following.

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Theorem 2.4 *Let $\alpha = \Phi_{\theta_1,d_1,f_1}, \beta = \Phi_{\theta_2,d_2,f_2} : \mathbb{T}^2 \to \mathbb{T}^2$ be uniquely ergodic Furstenberg transforms. Then each condition* (i) *or* (ii) *in Theorem 2.3 is also equivalent to the following:*

(iii) *There are sequences of Borel equivalences $\{\sigma_n\}$ and $\{\gamma_n\}$ such that*

$$\lim_{n\to\infty} \sup\{\mathrm{dist}(\sigma_n \circ \alpha \circ \sigma_n^{-1}(y), \beta(y)) : y \in Y)\} = 0,$$
$$\lim_{n\to\infty} \sup\{\mathrm{dist}(\gamma_n \circ \beta \circ \gamma_n^{-1}(x), \alpha(x)) : x \in X\} = 0,$$

and for each n there exists a closed subset $F_n \subset \mathbb{T}$ and $K_n \subset \mathbb{T}$ such that $m(F_n) = 0$ and $m(K_n) = 0$, and σ_n and γ_n are continuous on $\mathbb{T} \times (\mathbb{T} \setminus F_n)$ and on $\mathbb{T} \times (\mathbb{T} \setminus K_n)$ respectively. Moreover,

$$m(\sigma_n(S)) = m(S), \quad m(\gamma_n(G)) = m(G), \quad n = 1, 2, \dots \tag{2.4}$$

for all Borel sets $S \subset \mathbb{T}^2$.

Definition 2.5 Let A be a stably finite unital C^*-algebra and let $T(A)$ be the tracial state space. Denote by Tr the usual (non-normalized) trace on M_k. Define $\rho\colon K_0(A) \to \mathrm{Aff}(T(A))$ by $\rho([p]) = \tau(p)$ for projections in $M_k(A)$, where $\tau = t \otimes \mathrm{Tr}$ and $t \in T(A)$.

Definition 2.6 Let X be a compact metric space and let $\alpha\colon X \to X$ be a minimal homeomorphism. Then the transformation group C^*-algebra, the crossed product, $C(X) \rtimes_\alpha \mathbb{Z}$ will be denoted by A_α. We will use $j_\alpha\colon C(X) \to A_\alpha$ for the natural embedding.

For a unital C^*-algebra A, we let $\mathrm{ad}\, u(a) = u^* a u$ for all $a \in A$. We fix a unitary u_α so that $\mathrm{ad}\, u_\alpha \circ j_\alpha(f) = j_\alpha(f \circ \alpha)$ for all $f \in C(X)$. It should be noted that there are other choices for such u_α. For example, if $z \in C(X)$ is a unitary, then $w = u_\alpha j_\alpha(z)$ is another choice. In fact,

$$\mathrm{ad}\, w(j_\alpha(f)) = j_\alpha(z)^* u_\alpha^*(j_\alpha(f)) u_\alpha j_\alpha(z) = j_\alpha(z^*(f \circ \alpha) z) = j_\alpha(f \circ \alpha) \tag{2.5}$$

for all $f \in C(X)$.

Remark 2.7 Let $X = \mathbb{T}^2$ and let $\alpha = \Phi_{\theta,d,f}$ be a Furstenberg transformation with Lipschitz f. Then A_α is a unital simple C^*-algebra with a unique tracial state.

It is computed (see [23, Example 4.9]) that $K_0(A_\alpha) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ with $\rho(K_0(A_\alpha)) = \mathbb{Z} \oplus \mathbb{Z}(\theta) \subset \mathbb{R}$ and

$$K_0(A_\alpha)_+ = \{m_1 + m_2 + m_3 \in \mathbb{Z}^3 : m_1 + m_3\theta > 0 \text{ or } m_1 = m_2 = m_3 = 0\},$$

where the first two copies of $\mathbb{Z}$ are identified with the image of $K_0(C(\mathbb{T}^2))$ under the embedding $(j_\alpha)_{*0}$, and $K_1(A_\alpha) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}$, and where $\mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}$ is the image of $K_1(C(\mathbb{T}^2))$ under $(j_\alpha)_{*1}$. Let $z_1, z_2 \colon \mathbb{T}^2 \to \mathbb{T}$ be the functions defined by $z_1((\xi, \zeta)) = \xi$ and $z_2((\xi, \zeta)) = \zeta$ $(\xi, \zeta \in \mathbb{T})$. Then $(j_\alpha)_{*1}([z_1])$ is the standard generator of $\mathbb{Z}/d\mathbb{Z}$ and $(j_\alpha)_{*1}([z_2])$ is the standard generator of $\mathbb{Z}$.

Let X be a compact metric space and let $\alpha, \beta \colon X \to X$ be two minimal homeomorphisms. It is desirable to have two sequences of homeomorphisms $\{\sigma_n\}$ and $\{\gamma_n\}$ on X so that

$$\lim_{n\to\infty} \sup\{\mathrm{dist}(\sigma_n \circ \alpha \circ \sigma_n^{-1}(x), \beta(x)) : x \in X\} = 0, \tag{2.6}$$

$$\lim_{n\to\infty} \sup\{\mathrm{dist}(\gamma_n \circ \beta \circ \gamma_n^{-1}(x), \alpha(x)) : x \in X\} = 0. \tag{2.7}$$

However, when a sequence of maps (such as $\{\sigma_n\}$ and $\{\gamma_n\}$) is involved, one also expects that the maps have something in common. At least, in the Cantor set case, without any consistency on the conjugating maps $\{\sigma_n\}$ and $\{\gamma_n\}$, (2.6) and (2.7) are not so interesting (see [15]). Even though one should not expect these maps will converge in any meaningful way, one hopes that some information about these maps is independent of n. For example, one would like to require that both sequences preserve the measures as in Definition 2.1 as well as in Theorem 2.4(iii). To be more topologically interesting, one may require that both sequences preserve some topological data. For example, in [15], approximate K-conjugacy requires that both sequences preserve K-theory (in the crossed products). We use the following definition in this note.

Definition 2.8 Let X be a compact metric space, and let $\alpha, \beta \colon X \to X$ be two minimal homeomorphisms. Two homeomorphisms α and β are said to be *approximately K-conjugate* if there exist two sequences of homeomorphisms $\{\sigma_n\}$ and $\{\gamma_n\}$ on X such that (2.6) and (2.7) hold, and there exist an isomorphism $\phi \colon A_\alpha \to A_\beta$ and sequences of unitaries $u_n \in A_\beta$ and $v_n \in A_\alpha$ such that

$$\lim_{n\to\infty} \|\mathrm{ad}\ u_n \circ \phi(j_\alpha(f)) - j_\beta(f \circ \sigma_n^{-1})\| = 0$$

for all $f \in C(X)$, and $\lim_{n\to\infty} \|\mathrm{ad}\ u_n \circ \phi(u_\alpha) - u_\beta z_n\| = 0$, for some $z_n \in U(A_\beta)$ such that $\lim_{n\to\infty} \|z_n j_\beta(f) - j_\beta(f) z_n\| = 0$ for all $f \in C(X)$; and

$$\lim_{n\to\infty} \|\mathrm{ad}\ v_n \circ \phi(j_\beta(f)) - j_\alpha(f \circ \gamma_n^{-1})\| = 0$$

for all $f \in C(X)$ and $\lim_{n\to\infty} \|\mathrm{ad}\ v_n \circ \phi(u_\beta) - u_\alpha y_n\| = 0$, where $y_n \in U(A_\alpha)$ such that $\lim_{n\to\infty} \|y_n j_\beta(f) - j_\alpha(f) y_n\| = 0$ for all $f \in C(X)$.

Remark 2.9

(1) In Definition 2.8 one can choose $z_n = u_\beta^*(u_n^* u_\alpha u_n)$. From the discussion that leads to (2.5), one sees that the unitaries z_n and y_n can not be omitted.

(2) The existence of the unitaries $\{u_n\}$ and $\{v_n\}$ implies that $\{\sigma_n\}$ and $\{\gamma_n\}$ preserve the invariant measures. Note also if $p \in M_k(C(X))$ is a projection, then $[j_\beta(p \circ \sigma_n^{-1})] = [j_\beta(p)]$ in $K_0(A_\beta)$ for all large n. In fact, $\{\sigma_n\}$ and $\{\gamma_n\}$ preserve the ordered K-theory (independent of n) and beyond.

Condition (i) in Theorem 2.3 implies that $\rho(A_\alpha)$ and $\rho(A_\beta)$ are (unitally) order isomorphic. In the case that X is the Cantor set, by a theorem of Giordano, Putnam, and Skau, the condition that $\rho(A_\alpha)$ and $\rho(A_\beta)$ are unitally order isomorphic is equivalent to the condition that α and β are topologically orbit equivalent. A similar conclusion is not possible for a connected space, for as mentioned earlier, two topologically orbit equivalent minimal homeomorphisms on connected spaces are flip conjugate. As we stated in the introduction, topological orbit equivalence for minimal Cantor systems seems to be something between measure theory and topology. The conclusion of Theorem 2.3 also has both topological and measure theoretical flavor. However, one cannot require maps $\{\sigma_n\}$ and $\{\gamma_n\}$ in Theorem 2.4(iii) to be homeomorphisms in general. As mentioned in Remark 2.7, without the assumption that maps $\{\sigma_n\}$ and $\{\gamma_n\}$ preserve the measure, the approximate conjugacy in the sense of (2.6) and (2.7) is a rather weak relation for minimal a Cantor system (see [15]). However, it seems that it plays a completely different role for homeomorphisms on the connected spaces.

It is was recently proved [18] that for a uniquely ergodic Furstenberg transformation α, the associated crossed product A_α has tracial rank zero. So the classification theorem (see [12,13]) can be applied. In particular, if $\alpha = \Phi_{\theta_1,d_1,f_1}$ and $\beta = \Phi_{\theta_2,d_2,f_2}$, then $A_\alpha \cong A_\beta$ if and only if $\overline{\theta_1 \pm \theta_2} = 0$ in $\mathbb{R}/\mathbb{Z}$ and $|d_1| = |d_2|$.

Theorem 2.10 *Let $\alpha, \beta\colon \mathbb{T}^2 \to \mathbb{T}^2$ be two uniquely ergodic Furstenberg transforms associated with irrational numbers θ_1, θ_2 and integers $d_1, d_2 \in \mathbb{Z} \setminus \{0\}$, respectively. Then the following are equivalent.*

(i) $A_\alpha \cong A_\beta$.
(ii) $\overline{|\theta_1 \pm \theta_2|} = 0$ *in* $\mathbb{R}/\mathbb{Z}$ *and* $|d_1| = |d_2|$.
(iii) α *and* β *are approximately* K*-conjugate.*
(iv) $(K_0(A_\alpha), K_0(A_\alpha)_0, [1_{A_\alpha}], K_1(A_\alpha)) \cong (K_0(A_\beta), K_0(A_\beta)_+, [1_{A_\beta}], K_1(A_\beta))$.
(v) *There exist two sequences of homeomorphisms $\{\sigma_n\}$ and $\{\gamma_n\}$ on $\mathbb{T}^2$ such that*

$$\lim_{n\to\infty} \sup\{\operatorname{dist}(\sigma_n \circ \alpha \circ \sigma_n^{-1}(x), \beta(x)) : x \in \mathbb{T}^2\} = 0,$$

$$\lim_{n\to\infty} \sup\{\operatorname{dist}(\gamma_n \circ \beta \circ \gamma_n^{-1}(x), \alpha(x)) : x \in \mathbb{T}^2\} = 0,$$

$$m_2(\sigma_n(S)) = m_2(S) \quad \textit{and} \quad m_2(\gamma_n(S)) = m_2(S)$$

for all Borel subsets $S \subset \mathbb{T}^2$.

Remark 2.11 It should be pointed out that, in general, (i) or (iv) does not imply that α and β are approximately K-conjugate. See, for example, [16, Example 9.2]. Let X be a connected compact CW-complex and let $\alpha, \beta\colon X \to X$ be two minimal homeomorphisms such that (X, α) and (X, β) are uniquely ergodic. The right condition in this case, at least when $\rho(K_0(A_\alpha))$ is dense in $K_0(A_\alpha)$, should be the following: there is an order isomorphism

$$\kappa\colon (K_0(A_\alpha), K_0(A_\alpha)_+, [1_{A_\alpha}], K_1(A_\alpha)) \to (K_0(A_\beta)), K_0(A_\beta)_+, [1_{A_\beta}], K_1(A_\beta))$$

and a sequence of isomorphisms $\lambda_n\colon C(X) \to C(X)$ such that $[\kappa \circ j_\alpha \circ \lambda_n)] = [j_\beta]$ in $KL(C(X), A_\beta)$ and $\lim_{n\to\infty} \tau \circ j_\alpha(f \circ \lambda_n) = \tau \circ j_\alpha(f)$, where τ is the unique tracial state of A_α (for a more general case, see [14]).

3 Measure Theoretical Approximate Conjugacies

The following is taken from [19, p. 299] (see also [18, Theorem 2.3]).

Theorem 3.1 *Let X be an infinite compact metric space, and let $h\colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be closed and have nonempty interior. For $y \in Y$ set $r(y) = \min\{n \geq 1 : h^n(y) \in Y\}$. Then $\sup_{y\in Y} r(y) < \infty$. Let $n(0) < n(1) < n(2) < \cdots < n(l)$ be the distinct values in the range of r, and for $0 \leq k \leq l$ set*

$$Y_k = \overline{\{y \in Y : r(y) = n(k)\}} \quad \text{and} \quad Y_k^\circ = \mathrm{int}(\{y \in Y : r(y) = n(k)\}).$$

Then

(i) *the sets $h^j(Y_k^\circ)$ for $0 \leq k \leq l$ and $1 \leq j \leq n(k)$ are disjoint;*
(ii) $\bigcup_{k=1}^{l} \bigcup_{j=1}^{n(k)} h^j(Y_k) = X$;
(iii) $\bigcup_{k=0}^{l} h^{n(k)}(Y_k) = Y$.

Lemma 3.2 *Let $\alpha\colon \mathbb{T} \to \mathbb{T}$ be a minimal homeomorphism. Let $n > 1$ be an integer. Then there are finitely many pairwise disjoint open arcs $\{J_i = (c_i, d_i) : i = 1, 2, \dots, k\}$ of $\mathbb{T}$ such that*

(i) *$\alpha^j(J_i)$ are pairwise disjoint for $0 \leq j \leq h(i) - 1$ and $i = 1, 2, \dots, k$;*
(ii) *$n \leq h(i)$, $1 \leq i \leq k$;*
(iii) *$\mathbb{T} \setminus \bigcup_{i=1}^{k} \bigcup_{j=0}^{h(i)-1} \alpha^j(J_i))$ is a set of finite many points.*

Proof Fix $x \in \mathbb{T}$ and fix an integer $n > 1$. Since α is minimal, there is a closed arc Y containing x such that $\alpha^j(J)$ are pairwise disjoint for $0 \leq j \leq n$. Set

$$r(y) = \min\{m \geq 1 : \alpha^m(y) \in Y\}.$$

Applying Theorem 3.1, we obtain $n(0) < n(1) < \cdots < n(l)$, $Y_0, Y_1, \dots, Y_l$, and $Y_0^o, Y_1^o, \dots, Y_l^o$ as in Lemma 3.1. Note that $n(0) \geq n$.

Let $X_k = \{y \in Y : r(y) = n(k)\}$, $k = 0, 1, 2, \dots, l$ and let $\Omega = \mathrm{int}\, Y$. Note that $X_0 = Y_0$. Let $V_1 = \alpha^{n(0)}(\Omega) \cap \Omega$. Since Ω is a nonempty arc, so is V_1. Set $S_1 = \alpha^{-n(0)}(V_1)$. Then S_1 is an open sub-arc of Y. Note that $S_1 = Y_0^o$.

Let $V_2 = \alpha^{n(1)}(\Omega) \cap \Omega$. Then V_2 is a nonempty open arc. Set $S_2 = \alpha^{-n(1)}(V_2)$. Then $Y_1^o = S_2 \setminus X_0$. This shows that X_1 is a union of finitely many arcs (possibly neither open nor closed).

Let $V_3 = \alpha^{n(2)}(\Omega) \cap \Omega$. Then V_3 is a nonempty open arc. Set $S_3 = \alpha^{-n(2)}(V_3)$. Then $Y_2^o = S_3 \setminus (X_1 \cup X_2)$. So X_2 is a union of finitely many arcs.

By induction, we conclude that all X_k are unions of finitely many arcs. It follows that Y_k^o is a union of finitely many open arcs and Y_k is the closure of Y_k^o. Note that $\mathbb{T} \setminus \bigcup_{k=0}^{l} \bigcup_{j=0}^{n(j)} Y_k^o$ contains only finitely many points. The lemma then follows. ■

A similar lemma to the one below for the Cantor set appeared in the proof of [20, 4.4]. The following is a version of that for the circle. Note the function ω cannot be made to be continuous on the whole space as in the Cantor set case.

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Lemma 3.3 *Let $\{I_1, I_2, \dots, I_k\}$ be finitely many disjoint open arcs of $\mathbb{T}$ and let $\alpha\colon \mathbb{T} \to \mathbb{T}$ be a homeomorphism such that $\alpha^j(I_i)$ are pairwise disjoint for $1 \le i \le k$ and $0 \le j \le h(i) - 1$. Let $F, G\colon \mathbb{T} \to \mathbb{T}$ be continuous maps. Then, for any $\varepsilon > 0$ and for each i, there are $I_i^{(s)} \subset I_i$ $s = 1, 2, \dots, m(i)$ disjoint open arcs and there is a continuous map $\omega\colon S \to \mathbb{T}$, where $S = \bigcup_{j=0}^{h(i)-1} \bigcup_{i=1}^{k} \bigcup_{s=1}^{m(i)} \alpha^j(I_i^{(s)})$, such that*

(i) $\omega(x) = 0$ *if* $x \in \bigcup_{s,i} I_j^{(s)}$;

(ii) $|[F(\alpha^j(x)) + \omega(\alpha^j(x))] - [\omega(\alpha^{j+1}(x)) + G(\alpha^j(x))]| < \frac{1}{h(i)}$ *for all* $x \in I_i^{(s)}$, $s = 1, 2, \dots, m(i)$, $j = 0, 1, \dots, h(i) - 2$ *(we identify $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$);*

(iii) *in* $\mathbb{R}/\mathbb{Z}$, $|[F(\alpha^{h(i)-1}(x)) + \omega(\alpha^{h(i)-1}(x))] - [G(\alpha^{h(i)-1}(x))]| < \frac{1}{h(i)}$ *for all* $x \in I_i^{(s)}$, $s = 1, 2, \dots, m(i)$, $i = 1, 2, \dots, k$;

(iv) $I_i \setminus \bigcup_{s=1}^{m(i)} I_i^{(s)}$ *contains only finitely many points.*

Moreover, on the closure of each $I_i^{(s)}$, ω can be extended to be a continuous function and (i), (ii) *and* (iii) *remain true for x in the left-closed arcs of $I_i^{(s)}$ if inequalities are replaced by $\le$.*

Proof We identify $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$. Define $\kappa(x) = \sum_{i=0}^{h(i)-1}(F(\alpha^j(x)) - G(\alpha^j(x)))$ for $x \in I_i$, $1 \le i \le k$.

Let $K = \sum_{i=1}^{k} h(i)$. One can break I_i into a disjoint union of finitely many arcs so that the image of κ on each sub-arc is a subset of a proper closed subset of $\mathbb{T}$. Thus, there are for each i, pairwise disjoint open sub-arcs $I_i^{(s)} \subset I_i$, $s = 1, 2, \dots, m(i)$, such that $\kappa|_{\overline{I_j^{(s)}}}$ has image contained in a proper subset of $\mathbb{T}$ such that $I_i \setminus \bigcup_{s=1}^{m}(s) I_i^{(s)}$ contains only finitely many points, $i = 1, 2, \dots, k$. So (iv) now holds.

Define $\tilde{\kappa}(x)$ as follows $\tilde{\kappa}(x) = \kappa(x) + \mathbb{Z}$ and $-1 \le \tilde{\kappa}(x) \le 1$ for $x \in \overline{I_i^{(s)}}$, $1 \le s \le m(i)$, $1 \le i \le k$.

Put $S = \bigcup_{i=1}^{k} \bigcup_{s=1}^{m(i)} I_i^{(s)}$. Define $\eta\colon S \to \mathbb{T}$ as follows:

$$\eta(\alpha^j(x)) = -\frac{j}{h(i)}\tilde{\kappa}(x) + \mathbb{Z}$$

for $x \in \overline{I_i^{(s)}}$, $j = 0, 1, \dots, h(i) - 1$, $1 \le i \le k$. Since the image of κ on each $\overline{I_i^{(s)}}$ is a proper subset of $\mathbb{T}$, we see that η is continuous.

Put $\Omega_0 = \sum_{i=1}^{k} \bigcup_{s=1}^{m(i)} I_i^{(s)}$. Define $\omega\colon S \to \mathbb{T}$ as follows: $\omega(x) = 0$ if $x \in \Omega_0$, and

$$\omega(\alpha^j(x)) = \eta(\alpha^j(x)) + \sum_{l=0}^{j-1}[F(\alpha^l(x)) - G(\alpha^l(x))]$$

for $x \in I_i^{(s)}$, $j = 1, 2, \dots, h(i) - 1$, $1 \le s \le m(i)$, $1 \le i \le k$.

If $x \in I_i^{(s)}$, then

$$\begin{aligned} &|[F(x) + \omega(x)] - [\omega(\alpha(x)) + G(x)]| \\ &\qquad \le \Big| F(x) - \Big[\frac{1}{h(i)}(\tilde{\kappa}(x) + (F(x) - G(x)) + G(x)\Big]\Big| < \frac{1}{h(i)} \end{aligned}$$

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for $1 \le s \le m(i)$ and $1 \le i \le k$.

If $x \in \alpha^j(I_i^{(s)})$ for $j = 1, 2, \dots, h(i) - 2$, then

$$|[F(\alpha^j(x))+\omega(x)]-[\omega(\alpha^{j+1}(x))+G(\alpha^j(x))]| \le |\eta(\alpha^j(x))-\eta(\alpha^{k+1}(x))| < \frac{1}{h(i)}$$

for $1 \le i \le k$.

We verify that in $\mathbb{R}/\mathbb{Z}$,

$$\begin{aligned}|[F(\alpha^{h(i)-1}(x)) + \omega(\alpha^{h(i)-1}(x))] - [G(\alpha^{h(i)-1}(x))]|\\ = |\kappa(x) - \frac{h(i)-1}{h(i)}\tilde{\kappa}(x)| < \frac{1}{h(i)}\end{aligned}$$

for all $x \in I_i$, $1 \le i \le k$.

We also note that the last statement follows easily since F and G are continuous functions. ■

Lemma 3.4 *Let $\sigma((\xi,\zeta)) = (\sigma_1(\xi,\zeta), \sigma_2(\xi,\zeta))$ be a Borel equivalence from $\mathbb{T}^2$ to $\mathbb{T}^2$ such that $m_2(\sigma(S)) = m_2(\sigma(S))$ for all Borel set S.*

(i) *Then for any Borel set $S_1, S_2 \subset \mathbb{T}$,*

$$m(\sigma_1((S_1,\zeta)) = m(S_1) \quad \textit{and} \quad m(\sigma_2((\xi,S_2)) = m(S_2)$$

for almost all $\zeta \in \mathbb{T}$ and almost all $\xi \in \mathbb{T}$,

(ii) *If there exists a closed subset $F \subset \mathbb{T}$ with $m(F) = 0$ such that $\sigma_1(-,\zeta)$ is continuous on $\mathbb{T}$ for all $\zeta \in \mathbb{T}$, then for each $\zeta \in \mathbb{T} \setminus F$,*

$$\sigma_1(\xi,\zeta) = \xi g_1(\zeta) \quad \textit{for all } \xi \in \mathbb{T} \tag{3.1}$$

or

$$\sigma_1(\xi,\zeta) = \overline{\xi} g_1(\zeta) \quad \textit{for all } \xi \in T. \tag{3.2}$$

(iii) *If σ is continuous, then there are continuous maps $g_1\colon \mathbb{T} \to \mathbb{T}$ such that*

$$\sigma_1(\xi,\zeta) = \xi g_1(\zeta) \quad \textit{for all } \xi,\zeta \in \mathbb{T}$$

or

$$\sigma_1(\xi,\zeta) = \overline{\xi} g_1(\zeta) \quad \textit{for all } \xi,\zeta \in \mathbb{T}.$$

Proof (i) Let $S_1 \subset \mathbb{T}$ be a Borel subset. Suppose that there exists a measurable subset $E_1 \subset \mathbb{T}$ with positive measure such that $\int_{S_1} \sigma_1(\xi,\zeta)d\xi \ne m(S_1)$ for all $\zeta \in E_1$. Put

$$E_1^+ = \left\{\zeta \in E_1 : \int_{S_1} \sigma_1(\xi,\zeta)\,d\xi > m(S_1)\right\},$$

$$E_1^- = \left\{\zeta \in E_1 : \int_{S_1} \sigma_1(\xi,\zeta)\,d\xi < m(S_1)\right\}.$$

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If $m(E_1^+) > 0$, then

$$m_2(S_1 \times E_1^+) = \int_{S_1 \times E_1^+} \sigma_1(\xi, \zeta)\, d\xi d\zeta > \int_{E_1^+} m(S_1)\, d\zeta = m_2(S_1 \times E_1^+).$$

If $m(E_1^-) > 0$, then

$$m_2(S_1 \times E_1^-) = \int_{S_1 \times E_1^-} \sigma_1(\xi, \zeta)\, d\xi d\zeta < \int_{E_1^-} m(S_1)\, d\zeta = m_2(S_1 \times E_1^-).$$

Neither could be true. The proof for the variable ζ is the same.

(ii) Applying part (i), we have a measurable set $E \subset \mathbb{T}$ with $m(E) = m(\mathbb{T}) = 1$ such that

$$m(\sigma_1((S, \zeta))) = m(S) \tag{3.3}$$

for all Borel subsets $S \subset \mathbb{T}$ and $\zeta \in E$. Thus, if $\zeta \in E \cap (\mathbb{T} \setminus F)$, by (3.3), it is well known that either $\sigma_1(\xi, \zeta) = \xi g_1(\zeta)$ for all $\xi \in \mathbb{T}$ and for some $g_1(\zeta) \in \mathbb{T}$ or

$$\sigma_1(\xi, \zeta) = \bar{\xi} g_1(\zeta)$$

for all $\xi \in \mathbb{T}$ and for some $g_1(\zeta) \in \mathbb{T}$.

(iii) This part follows immediately from (ii). By considering the subset

$$\{(1, \zeta) : \zeta \in \mathbb{T}\}$$

and applying (3.1) and (3.2), we conclude that $\sigma_1(1, \zeta) = g_1(\zeta)$ is a continuous function. Then, by continuity of σ, (ii) follows. ∎

Proof of Theorem 2.4 Let $\sigma((\xi, \zeta)) = (\bar{\xi}, \zeta)$. Then $\sigma \colon \mathbb{T}^2 \to \mathbb{T}^2$ is a homeomorphism and $\sigma^{-1} = \sigma$. One has

$$\begin{aligned}\sigma^{-1} \circ \Phi_{\theta,1,0}\sigma((\xi, \zeta)) &= \sigma^{-1}((\bar{\xi} e^{i2\pi\theta}, \zeta\bar{\xi})) \\ &= (\xi e^{-i2\pi\theta}, \zeta\xi^{-1})\end{aligned}$$

for all $\xi, \zeta \in \mathbb{T}$. It follows that $\Phi_{\theta,1,0}$ and $\Phi_{-\theta,-1,0}$ are conjugate.

We will show that if $\overline{\theta_1 \pm \theta_2} = 0$ in $\mathbb{R}/\mathbb{Z}$, then Theorem 2.4(iii) holds. For convenience, we will say α and β are approximately conjugate in the sense of M2 if (iii) holds. In this part of the proof, we will identify $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$.

Let $\theta \in (0, 1)$ be an irrational number. Since $\Phi_{\theta,1,0}$ and $\Phi_{-\theta,-1,0}$ are conjugate as shown above, it suffices to show that α and β are approximately conjugate in the sense of M2 if $\alpha = \Phi_{\theta,d_1,f_1}$ and $\beta = \Phi_{\theta,d_2,f_2}$.

Let $\varepsilon > 0$. Choose $n > 0$ so that $1/n < \varepsilon$. Let $J_1, J_2, \ldots, J_k$ be the open arcs provided by Lemma 3.2 with the integer n (and $\alpha(t) = t + \theta$ for $t \in \mathbb{R}/\mathbb{Z}$). Put $F(\xi) = \xi^{d_1} \exp(i2\pi f_1(\xi))$ and $G(\xi) = \xi^{d_2} \exp(i2\pi f_2(\xi))$ for $\xi \in \mathbb{T}$. Let ω be the function in Lemma 3.3 with $\alpha(t) = t + \theta$ and $I_i = J_i$, $i = 1, 2, \ldots, k$. By extending ω

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continuously on the left-side closed arcs $J_i^{(s)}$ for each s and i, as the last part of Lemma 3.3, one has

$$|[F(x)+\omega(x)]-[\omega(x+\theta)+G(x)]|\leq\varepsilon \tag{3.4}$$

for all $x\in\mathbb{R}/\mathbb{Z}$. Define $\sigma((x,t))=(x,t+\omega(x))$ for $t,x\in\mathbb{R}/\mathbb{Z}$. Therefore,

$$m_2(\sigma(S))=m_2(S)$$

for all Borel set $S\subset\mathbb{T}^2$. Moreover, σ is continuous except finitely many circles (with the form of $B\times\mathbb{T}$, where B is a finite subset of $\mathbb{T}$).

Now, by (3.4),

$$\operatorname{dist}(\alpha\circ\sigma((x,t)),\sigma\circ\beta((x,t)))\leq|[F(x)+\omega(x)]-[\omega(x+\theta)+G(x)]|<\varepsilon$$

for all $x,t\in\mathbb{R}/\mathbb{Z}$. For the converse, let θ_1 and θ_2 be two irrational numbers such that $\overline{|\theta_1\pm\theta_2|}\neq 0$ in $\mathbb{R}/\mathbb{Z}$.

Since we have shown that Φ_{θ_1,d_1,f_1} and $\Phi_{\theta_1,1,0}$ are approximately conjugate in the sense of M2 above, and Φ_{θ_2,d_2,f_2} and $\Phi_{\theta_2,1,0}$ are approximately conjugate in the sense of M2 above, respectively, it suffices to show that $\Phi_{\theta_1,1,0}$ and $\Phi_{\theta_2,1,0}$ are not approximately conjugate in the sense of M2.

Put $\alpha=\Phi_{\theta_1,1,0}$ and $\beta=\Phi_{\theta_2,1,0}$. Put

$$a=|e^{2\pi(\theta_1-\theta_2)}-1|>0\quad\text{and}\quad b=|e^{2\pi(\theta_1+\theta_2)}-1|>0. \tag{3.5}$$

Let $\varepsilon>0$ such that $\varepsilon<\min\{a/2,b/2\}$. Suppose that there exists a Borel equivalence $\sigma\colon\mathbb{T}^2\to\mathbb{T}^2$ which is continuous on $\mathbb{T}\times(\mathbb{T}\setminus F)$ for some closed subset $F\subset\mathbb{T}$ with $\mu(F)=0$ such that

$$\sup\{\operatorname{dist}(\alpha\circ\sigma(x),\sigma\circ\beta(x)):x\in\mathbb{T}^2\}<\varepsilon \tag{3.6}$$

and $m_2(\sigma(S))=m_2(S)$ for all Borel sets $S\subset\mathbb{T}^2$.

By Lemma 3.4, there exists a Borel set $E\subset\mathbb{T}$ such that $m(\mathbb{T}\setminus E)=0$ and for each $\zeta\in E$, $\sigma_1((\xi,\zeta))=\xi g_1(\zeta)$ or $\sigma_1((\xi,\zeta))=\bar{\xi}g_1(\zeta)$, for all $\xi\in\mathbb{T}$. Thus, for $\zeta\in E$, by (3.6), we have either

$$|\xi g_1(\zeta)e^{i2\pi\theta_1}-\xi g_1(\zeta\xi)e^{i2\pi\theta_2}|<\varepsilon,\quad|\bar{\xi}g_1(\zeta)e^{i2\pi\theta_1}-\bar{\xi}g_1(\zeta\xi)e^{-i2\pi\theta_2}|<\varepsilon,$$
$$|\xi g_1(\zeta)e^{i2\pi\theta_1}-\bar{\xi}g_1(\zeta\xi)e^{-i2\pi\theta_2}|<\varepsilon,\quad\text{or}\quad|\bar{\xi}g_1(\zeta)e^{i2\pi\theta_1}-\xi g_1(\zeta\xi)e^{i2\pi\theta_2}|<\varepsilon$$

for all $\xi\in\mathbb{T}$.

Choose $\xi=1$ for all $\zeta\in E$. One computes that either

$$a=|g_1(\zeta)e^{2i\pi\theta_1}-e^{2i\pi\theta_2}g_1(\zeta)|<\varepsilon\quad\text{or}\quad b=|g_1(\zeta)e^{2i\pi\theta_1}-e^{-2i\pi\theta_2}g_1(\zeta)|<\varepsilon.$$

By (3.5), neither is possible. ∎

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Proof of Theorem 2.3 (i) $\Rightarrow$ (ii): We will modify the relevant part of the proof of Theorem 2.4. Let $\varepsilon > 0$. By the proof of Theorem 2.4, there exists a finite subset $B \subset \mathbb{T}$ and a function $\omega\colon \mathbb{T} \to \mathbb{T}$ which is continuous on $\mathbb{T} \setminus B$ such that

$$\operatorname{dist}(\sigma \circ \alpha((x,t)), \beta \circ \sigma((x,t))) < \varepsilon$$

for all $x, t \in \mathbb{R}/\mathbb{Z}$, where $\sigma((x,t)) = (x, t + \omega(x))$ for all $x, t \in \mathbb{R}/\mathbb{Z}$. There is an open subset $G \subset \mathbb{T}$ such that $B \subset G$ and $\mathrm{m}(G) < \varepsilon$. There is a continuous function ω_0 from $\mathbb{T}$ to $\mathbb{R}/\mathbb{Z}$ such that $\omega_0(x) = \omega(x)$ for $x \in \mathbb{T} \setminus G$. Now define $\sigma_0((x,t)) = (x, t + \omega_0(x))$ for $x, t \in \mathbb{R}/\mathbb{Z}$. Then

$$\{(x,t) : \operatorname{dist}(\sigma_0 \circ \alpha((x,t)), \beta \circ \sigma_0((x,t))) \geq \varepsilon\} \subset \mathbb{T} \times G.$$

It follows that $m_2(\{(x,t) : \operatorname{dist}(\sigma_0 \circ \alpha((x,t)), \beta \circ \sigma_0((x,t))) \geq \varepsilon\}) < \varepsilon$. This proves (i) $\Rightarrow$ (ii).

To see (ii) $\Rightarrow$ (i), let θ_1 and θ_2 be two irrational numbers such that $\overline{|\theta_1 \pm \theta_2|} \neq 0$ in $\mathbb{R}/\mathbb{Z}$.

Since we have shown that Φ_{θ_1,d_1,f_1} and $\Phi_{\theta_1,1,0}$ are approximately conjugate in the sense of M1, and Φ_{θ_2,d_2,f_2} and $\Phi_{\theta_2,1,0}$ are approximately conjugate in the sense of M1, respectively, it suffices to show that $\Phi_{\theta_1,1,0}$ and $\Phi_{\theta_2,1,0}$ are not approximately conjugate in the sense of M1.

Put $\alpha = \Phi_{\theta_1,1,0}$ and $\beta = \Phi_{\theta_2,1,0}$. Suppose that $\sigma\colon \mathbb{T}^2 \to \mathbb{T}^2$ is a homeomorphism such that $m_2(\sigma(S)) = m_2(S)$. Write $\sigma((\xi,\zeta)) = (\sigma_1(\xi,\zeta), \sigma_2(\xi,\zeta))$. It follows from Theorem 3.4(iii) that either $\sigma_1(\xi,\zeta) = \xi g_1(\zeta)$ or $\sigma_1(\xi,\zeta) = \bar{\xi} g_1(\zeta)$ for all $\xi, \zeta \in \mathbb{T}$, where $g_1\colon \mathbb{T} \to \mathbb{T}$ is a continuous map. Put

$$a = |e^{2\pi(\theta_1-\theta_2)} - 1| > 0 \quad \text{and} \quad b = |e^{2i\pi(\theta_1+\theta_2)} - 1| > 0.$$

Let $0 < \varepsilon < \min\{1/4, a/4, b/4\}$. Let $z \in C(\mathbb{T})$ be defined by $z((\xi,\zeta)) = \xi$ for $\xi \in \mathbb{T}$. If α and β are approximately conjugate in the sense of M1, then there exists σ described above such that $\int_{\mathbb{T}}\int_{\mathbb{T}} |z(\alpha \circ \sigma(\xi,\zeta) - z(\sigma \circ \beta(\xi,\zeta))|\, d\xi d\zeta < \varepsilon^4/4$. One then computes that $\int_{\mathbb{T}} |z(\alpha \circ \sigma(\xi,\zeta) - z(\sigma \circ \beta(\xi,\zeta))|\, d\xi < \varepsilon^2$ for all $\zeta \in E$, where E is a measurable set such that $m(E) > 1 - \varepsilon$.

We now assume that $\sigma_1(\xi,\zeta) = \xi g_1(\zeta)$ for all $\xi, \zeta \in \mathbb{T}$.

For $\zeta \in E$,

$$\varepsilon^2 > \int_{\mathbb{T}} |\xi g_1(\zeta) e^{i2\pi\theta_1} - \xi e^{i2\pi\theta_2} g_1(\zeta\xi)|\, d\xi = \int_{\mathbb{T}} |g_1(\zeta) e^{i2\pi(\theta_1-\theta_2)} - g_1(\zeta\xi)|\, d\xi.$$

By considering the constant function $F_1(\xi) = g_1(\zeta)e^{i2\pi(\theta_1-\theta_2)}$ and the function $F_2(\xi) = g_1(\zeta\xi)$ and by translating by $\bar{\zeta}$, we obtain

$$\int_{\mathbb{T}} |g_1(\zeta) e^{i2\pi(\theta_1-\theta_2)} - g_1(\xi)|\, d\xi < \varepsilon^2 \tag{3.7}$$

for all $\zeta \in E$. Fix $\zeta_0 \in E$ and let

$$E_{\zeta_0} = \{\xi \in \mathbb{T} : |g_1(\zeta_0) e^{i2\pi(\theta_1-\theta_2)} - g_1(\xi)| < \varepsilon\}.$$

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Then by (3.7), we compute that $m(E_{\zeta_0}) > 1 - \varepsilon$. Therefore $m(E_{\zeta_0} \cap E) > 0$. If $\zeta_1 \in E_{\zeta_0} \cap E$, by (3.7), we have that

$$\begin{aligned}
\varepsilon^2 &> \int_{\mathbb{T}} |g_1(\zeta_1)e^{i2\pi(\theta_1-\theta_2)} - g_1(\xi)|\,d\xi \\
&\geq |g_1(\zeta_0)e^{i2\pi(\theta_1-\theta_2)} - g_1(\zeta_0)e^{i4\pi(\theta_1-\theta_2)}| - |g_1(\zeta_0)e^{i4\pi(\theta_1-\theta_2)} - g_1(\zeta_1)e^{i2\pi(\theta_1-\theta_2)}| \\
&\qquad - \int_{\mathbb{T}} |g_1(\zeta_0)e^{i2\pi(\theta_1-\theta_2)} - g_1(\xi)|\,d\xi \\
&> |e^{i2\pi(\theta_1-\theta_2)} - 1| - \varepsilon - \varepsilon^2 > a/2.
\end{aligned}$$

By the choice of ε, this is impossible.

Now we assume that $\sigma_1(\xi, \zeta) = \overline{\xi} g_1(\zeta)$ for all $\xi, \zeta \in \mathbb{T}$. As above, one has

$$\varepsilon^2 > \int_{\mathbb{T}} |\overline{\xi} g_1(\zeta)e^{i2\pi\theta_1} - \overline{\xi} e^{-i2\pi\theta_2} g_1(\zeta\xi)|\,d\xi = \int_{\mathbb{T}} |g_1(\zeta)e^{i2\pi(\theta_1+\theta_2)} - g_1(\zeta\xi)|\,d\xi.$$

The same argument used above leads us to $\varepsilon^2 > |e^{i2\pi(\theta_1+\theta_2)} - 1| - \varepsilon - \varepsilon^2 > \frac{b}{2}$. This would violate the choice of ε. ∎

4 Approximate K-Conjugacy

Lemma 4.1 *Let $\theta \in [0, 1]$ be an irrational number and let*

$$V = \{a \sin kt + b \cos mt : a, b \in \mathbb{R}, k, m \in \mathbb{Z}, t \in [0, 2\pi]\}.$$

Then, for every $f \in V$, there exists $g \in C(\mathbb{T})$ such that

$$f(t) = g(t) - g(t + \theta) \quad \text{for all } t \in [0, 2\pi].$$

Proof Put $V_0 = \{f(t) = g(t) - g(t + \theta) : g \in C(\mathbb{T})\}$. It is clear that V_0 is a (real) vector space.

One has two elementary inequalities: for any integer $n > 1$,

$$\Big|\sum_{k=1}^{n} \sin k\theta\Big| = \frac{|\cos\frac{\theta}{2} - \cos(n + \frac{1}{2})\theta|}{2\sin\frac{\theta}{2}} \leq \frac{1}{\sin\frac{\theta}{2}},$$

and similarly

$$\Big|\sum_{k=1}^{n} \cos k\theta\Big| \leq \frac{1}{\sin\frac{\theta}{2}}.$$

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Now, for any integer $m \in \mathbb{Z}$,

$$\begin{aligned}\Big|\sum_{k=0}^{n} \sin(mt + k\theta)\Big| &= \Big|\cos mt\Big(\sum_{k=0}^{n} \sin k\theta\Big) + \sin mt\Big(\sum_{k=0}^{n} \cos k\theta\Big)\Big| \\ &\le \Big|\sum_{k=0}^{n} \sin k\theta\Big| + \Big|\sum_{k=0}^{n} \cos k\theta\Big| \\ &\le 1 + \frac{2}{\sin\frac{\theta}{2}}\end{aligned}$$

for all $t \in \mathbb{R}$. Now, since $t \mapsto t + \theta$ ($t \in \mathbb{R}/\mathbb{Z}$) is a minimal homeomorphism on $\mathbb{T}$, by a lemma of Furstenberg [4, Lemma 5.2], there is $g \in C(\mathbb{T})$ (real) such that $\sin mt = g(t) - g(t + \theta)$ (for $t \in [0, 2\pi]$). It follows that $\sin mt \in V_0$. Similarly, $\cos mt \in V_0$. Since V_0 is a real vector space, $V \subset V_0$. ■

The above lemma can be proved directly by some trigonometric identities and the function g in the proof may be chosen to be in V.

Lemma 4.2 *Let $\theta \in [0, 1]$ be an irrational number and let $f \in C(\mathbb{T})$ be a real function. Then for any $\varepsilon > 0$, there exists a continuous map $g\colon \mathbb{T} \to \mathbb{T}$ such that*

$$|\exp(i2\pi f(\xi))g(\xi)\overline{g(\xi \cdot e^{i2\pi\theta})} - 1| < \varepsilon$$

for all $\xi \in \mathbb{T}$. Moreover, if $d \neq 0$ is an integer, g may be chosen to have the form

$$g(\xi) = \xi^{kd} \exp(i2\pi g_0(\xi)),$$

where $k \in \mathbb{Z}$ and $g_0 \in C(\mathbb{T})$ is a real function.

Proof Note that $\mathbb{Z}+\mathbb{Z}(\theta)$ is dense in $\mathbb{R}$. Thus $[\mathbb{Z}+\mathbb{Z}(\theta)]/\mathbb{Z}$ is dense in $\mathbb{R}/\mathbb{Z}$. Therefore, for any $a \in \mathbb{R}$, there exists an integer $k \in \mathbb{Z}$ such that $|e^{i2\pi a}e^{-i2\pi k\theta} - 1| < \varepsilon$. Hence

$$|e^{i2\pi a}e^{i2\pi kt}e^{-i2\pi k(t+\theta)} - 1| = |e^{i2\pi a}e^{-i2\pi k\theta} - 1| < \varepsilon$$

for all $t \in [0, 1]$.

Let V be as in the proof of Lemma 4.1 and let $f_0 \in V$. Apply Lemma 4.1 and choose a real $g_0 \in C(\mathbb{T})$ such that $f_0(t) = g_0(t + \theta) - g_0(t)$ for all $t \in \mathbb{R}/\mathbb{Z}$. Let $f = a + f_0$ and $g(t) = \exp(i2\pi(kt + g_0(t)))$ for $t \in \mathbb{R}/\mathbb{Z}$. Then

$$\begin{aligned}&|\exp(i2\pi f(t))g(t)\overline{g(t + \theta)} - 1| \\ &\quad= |\exp\{i2\pi f(t)\}\exp\{i2\pi(kt + g_0(t))\}\exp\{-i2\pi[k(t + \theta) + g_0(t + \theta)]\} - 1| \\ &\quad= |\exp\{i2\pi(a + f_0)\}\exp\{i2\pi(-k\theta + g_0(t) - g_0(t + \theta))\} - 1| \\ &\quad= |\exp\{i2\pi(a - k\theta)\} - 1| < \varepsilon\end{aligned}$$

for all $t \in \mathbb{R}/\mathbb{Z}$.

By the Stone–Weierstrass theorem, the set of real trigonometric polynomials is dense in the real part of $C(\mathbb{T})$. Thus the first part of the lemma follows.

To see the last part of the lemma, we only need to note that $d\theta$ is also an irrational number and $\mathbb{Z}d\theta$ is dense in $\mathbb{R}/\mathbb{Z}$. ∎

Proof of Theorem 2.10 That (ii) $\Leftrightarrow$ (iv) follows from the computation in [23, Example 4.9] (see Remark 2.7) and (i) $\Leftrightarrow$ (iv) follows from the classification theorem in [13] as mentioned at the end of Remark 2.9. It is also clear that (iii) $\Rightarrow$ (v).

It remains to show (ii) $\Rightarrow$ (iii) and (v) $\Rightarrow$ (ii).

We will first show (ii) $\Rightarrow$ (iii).

As in the proof of Theorem 2.3, $\Phi_{\theta,d,0}$ and $\Phi_{-\theta,-d,0}$ are conjugate. Put $\sigma((\xi,\zeta)) = (\xi,\overline{\zeta})$. Then $\sigma\colon \mathbb{T}^2 \to \mathbb{T}^2$ is a homeomorphism and $\sigma^{-1} = \sigma$. One checks that

$$\begin{aligned}\sigma^{-1}\circ\Phi_{\theta,d,0}\circ\sigma((\xi,\zeta)) &= \sigma^{-1}((\xi e^{2i\pi\theta},\overline{\zeta}\xi^{d}))\\ &= (\xi e^{2i\pi\theta},\zeta\xi^{-d}) = \Phi_{\theta,-d,0}((\xi,\zeta))\end{aligned}$$

for all $\xi,\zeta\in\mathbb{T}$. Therefore $\Phi_{\theta,d,0}$ and $\Phi_{\theta,-d,0}$ are conjugate. So $\Phi_{-\theta,-d,0}$ and $\Phi_{-\theta,d,0}$ are conjugate. Combining this with the fact which was mentioned above, that $\Phi_{\theta,d,0}$ and $\Phi_{-\theta,-d,0}$ are conjugate, we conclude that $\Phi_{\theta,d,0}$ and $\Phi_{-\theta,d,0}$ are conjugate. Thus, to complete the proof, it suffices to show that $\alpha = \Phi_{\theta,d,f_1}$ and $\beta = \Phi_{\theta,d,f_2}$ are approximately K-conjugate for any real continuous Lipschitz functions f_1 and f_2.

It follows from [18, Theorem 4.6] that both A_α and A_β have tracial rank zero. By the K-theory computation in Remark 2.7, there is an order isomorphism

$$\kappa : (K_0(A_\alpha),K_0(A_\alpha)_+,[1_{A_\alpha}],K_1(A_\alpha)) \to (K_0(A_\beta),K_0(A_\beta)_+,[1_{A_\beta}],K_1(A_\beta))$$

such that $\kappa([u_\alpha]) = [u_\beta]$. By the classification theorem [13], there exists a unital isomorphism $\phi\colon A_\alpha \to A_\beta$ such that $[\phi] = \kappa$.

Let $f(\xi) = f_2(\xi) - f_1(\xi)$ for $\xi\in\mathbb{T}$. Fix $\delta > 0$. By applying Lemma 4.2, we obtain

$$g(\xi) = \xi^{kd}\exp(2i\pi g_0(\xi))$$

for $\xi\in\mathbb{T}$, where $g_0\in C(\mathbb{T})$ is a real function such that

$$|\exp(2i\pi f(\xi))g(\xi)\overline{g(\xi e^{2i\pi\theta})} - 1| < \delta \tag{4.1}$$

for all $\xi\in\mathbb{T}$. Define $\sigma((\xi,\zeta)) = (\xi,\zeta g(\xi))$ for all $(\xi,\zeta)\in\mathbb{T}^2$. Then

$$\begin{aligned}\sigma\circ\alpha((\xi,\zeta)) &= (\xi e^{2i\pi\theta},\zeta\xi^{d}e^{2i\pi f_1(\xi)}g(\xi e^{2i\pi\theta})),\\ \beta\circ\sigma((\xi,\zeta)) &= (\xi e^{2i\pi\theta},\zeta g(\xi)\xi^{d}e^{2i\pi f_2(\xi)})\end{aligned}$$

for all $\xi,\zeta\in\mathbb{T}$. Using (4.1), we estimate that $\operatorname{dist}(\sigma\circ\alpha((\xi,\zeta)),\beta\circ\sigma((\xi,\zeta))) < \delta$ for all $(\xi,\zeta)\in\mathbb{T}^2$.

Note that σ is homotopic to Φ_{θ,kd,g_0}. As the computation in [23, Example 4.9] (see Remark 2.7), we have $\sigma_{*0} = \mathrm{id}_{*0}$ on $K_0(C(\mathbb{T}^2))$ and σ_{*1} on $K_1(C(\mathbb{T}^2)) \cong \mathbb{Z} \oplus \mathbb{Z}$ is represented by the matrix

$$\begin{pmatrix} 1 & kd \\ 0 & 1 \end{pmatrix}.$$

It induces the identity map from $\mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}$ onto $\mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}$. Therefore

$$h_{*i} = (\phi \circ j_\alpha)_{*i} = (j_\beta)_{*i}, \quad i = 0, 1,$$

where $h\colon C(\mathbb{T}^2) \to A_\beta$ is defined by $h(f) = \phi \circ j_\alpha(f \circ \sigma)$ for $f \in C(\mathbb{T}^2)$. Since $K_i(C(\mathbb{T}^2))$ is free, we in fact have that

$$[h] = [\phi \circ j_\alpha] = [j_\beta] \text{ in } KL(C(\mathbb{T}^2), A_\beta). \tag{4.2}$$

We also note that

$$\tau \circ h(f) = \tau \circ \phi \circ j_\alpha(f) = \tau \circ j_\beta(f) \tag{4.3}$$

for all $f \in C(\mathbb{T}^2)$, where τ is the unique tracial state on A_β.

Let $\mathcal{F}_1 = \mathcal{F} \cup \{f \circ \sigma^{-1} : f \in \mathcal{F}\} \cup \{f(\sigma \circ \alpha\sigma^{-1}) : f \in \mathcal{F}\}$. By (4.2) and (4.3), and by [14, Theorem 3.4], there exists a unitary $W \in A_\beta$ such that

$$W j_\beta(f) W^* \approx_{\varepsilon/3} \phi \circ j_\alpha(f \circ \sigma) \text{ on } \mathcal{F}_1.$$

In particular, if $f \in \mathcal{F}$,

$$W^*\phi \circ j_\alpha(f) W \approx_{\varepsilon/3} j_\beta(f \circ \sigma^{-1}),$$

$$W^*\phi \circ j_\alpha(f(\sigma \circ \alpha)) W \approx_{ep/3} j_\beta(f(\sigma \circ \alpha \circ \sigma^{-1})).$$

Therefore

$$\begin{aligned} W^*\phi(u_\alpha^*) W j_\beta(f) W^* \phi(u_\alpha) W &\approx_{\varepsilon/3} W^*\phi(u_\alpha^*) \phi \circ j_\alpha(f \circ \sigma) \phi(u_\alpha) W \\ &= W^* \phi \circ j_\alpha(f \circ \sigma \circ \alpha) W \\ &\approx_{\varepsilon/3} j_\beta(f \circ \sigma \circ \alpha \circ \sigma^{-1}) \end{aligned}$$

for all $f \in \mathcal{F}$. It follows that, with sufficiently small δ,

$$\mathrm{ad}\,(W^*\phi(u_\alpha)W) \circ (j_\beta(f)) \approx_\varepsilon j_\beta(f \circ \beta)$$

for all $f \in \mathcal{F}$. Put $z = (W^*\phi(u_\alpha)W)u_\beta^*$. Then, $z \in U(A_\beta)$ and $j_\beta(f)z \approx_\varepsilon z j_\beta(f)$ for all $f \in \mathcal{F}$. This shows that α and β are approximately K-conjugate.

Now we consider (v) $\Rightarrow$ (ii). Suppose that $\alpha = \Phi_{\theta_1,d_1,f_1}$ and $\beta = \Phi_{\theta_2,d_2,f_2}$ are two Furstenberg transformations. Suppose that there exist sequences of homeomorphisms $\{\sigma_n\}$ and $\{\gamma_n\}$ on $\mathbb{T}^2$ such that for all Borel sets $S \subset \mathbb{T}^2$,

$$\lim_{n\to\infty} \sup\{\mathrm{dist}(\sigma_n \circ \alpha \circ \sigma_n^{-1}((\xi,\zeta)), \beta((\xi,\zeta))) : (\xi,\zeta) \in \mathbb{T}^2\} = 0,$$

$$\lim_{n\to\infty} \sup\{\mathrm{dist}(\gamma_n \circ \beta \circ \gamma_n^{-1}((\xi,\zeta)), \alpha((\xi,\zeta))) : (\xi,\zeta) \in \mathbb{T}^2\} = 0,$$

$$m_2(\sigma_n(S)) = m_2(S), \quad m_2(\gamma_n(S)) = m_2(S).$$

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It follows from Theorem 2.3 that $\overline{\theta_1 \pm \theta_2} = 0$ in $\mathbb{R}/\mathbb{Z}$.

Write $\sigma_n((\xi,\zeta)) = (G_1^{(n)}((\xi,\zeta)), G_2^{(n)}((\xi,\zeta))$ for all $(\xi,\zeta) \in \mathbb{T}^2$. It follows from Lemma 3.4 that there are continuous maps $g_1, g_2 \colon \mathbb{T} \to \mathbb{T}$ such that $G_1^{(n)}((\xi,\zeta)) = \tilde{\xi} g_1^{(n)}(\zeta)$ and $G_2^{(n)}((\xi,\zeta)) = \tilde{\zeta} g_2^{(n)}(\xi)$ for all $\xi, \zeta \in \mathbb{T}$, where $\tilde{\xi} = \xi$ for all $\xi \in \mathbb{T}$ or $\tilde{\xi} = \overline{\xi}$ for all $\xi \in T$, $\tilde{\zeta} = \zeta$ for all $\zeta \in \mathbb{T}$ or $\tilde{\zeta} = \overline{\zeta}$ for all $\zeta \in \mathbb{T}$. We have

$$\sigma_n \circ \alpha((\xi,\zeta)) = (\tilde{\xi} e^{\pm 2i\pi\theta_1} g_1^{(n)}(\zeta\xi^{d_1} e^{2i\pi f_1(\xi)}), \tilde{\zeta}\xi^{\pm d_1} e^{\pm i\pi f_1(\xi)} g_2^{(n)}(\xi e^{2i\pi\theta_1})),$$

$$\beta \circ \sigma_n((\xi,\zeta)) = (\tilde{\xi} g_1^{(n)}(\zeta) e^{2i\pi\theta_2}, \tilde{\zeta} g_2^{(n)}(\xi)\xi^{\pm d_2} g_1^{(n)}(\zeta)^{d_2} e^{2i\pi f_2(\tilde{\xi} g_1^{(n)}(\zeta))})$$

for all $\xi, \zeta \in \mathbb{T}$.

We compute that for all sufficiently large n,

$$|\xi^{\pm d_1 \pm d_2} e^{2i\pi(f_2(\tilde{\xi} g_1^{(n)}(\zeta)) \pm f_1(\xi))} g_2^{(n)}(\xi) \overline{g_2^{(n)}(\xi e^{2i\pi\theta})} g_1^{(n)}(\zeta)^{d_2} - 1| < 1/2 \tag{4.4}$$

for all $\xi, \zeta \in \mathbb{T}$. Fix $\zeta \in \mathbb{T}$ and let ξ vary in $\mathbb{T}$. It follows that for fixed $\zeta \in \mathbb{T}$,

$$\xi^{\pm d_1 \pm d_2} e^{2i\pi(f_2(\tilde{\xi} g_1^{(n)}(\zeta)) \pm f_1(\xi))} g_2^{(n)}(\xi) \overline{g_2^{(n)}(\xi e^{2i\pi\theta})} g_1^{(n)}(\zeta)^{d_2}$$

is homotopically trivial as a unitary in $C(\mathbb{T})$. Since $g_2^{(n)}(\xi)\overline{g_2^{(n)}(\xi e^{2i\pi\theta})}$ is homotopically trivial, (4.4) implies that $\xi^{\pm d_1 \pm d_2}$ is homotopically trivial. However, that can only happen when $|d_1| = |d_2|$. ∎

Corollary 4.3 *Let*

$V_1 = \{m_1\theta + m_2 + a\cos m_2 t + b \sin m_4 t : m_1, m_2, m_3, m_4 \in \mathbb{Z}, a, b \in \mathbb{R}, t \in [0, 2\pi]\}.$

Let $\alpha = \Phi_{\theta,d,f_1}$ and $\beta = \Phi_{\theta,d,f_2}$ such that $f_1 - f_2 \in V_1$. Then α and β are conjugate.

Note that V_1 is dense in the real part of $C(\mathbb{T})$.

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Department of Mathematics
East China Normal University
Shanghai
China
and (current)
Department of Mathematics
University of Oregon
Eugene, OR 97403-1222
U.S.A.
e-mail: hlin@uoregon.edu