45

Potential approaches to quarkonia

45.1 The Schrödinger equation

As mentioned earlier, when the heavy quark mass m_Q is much larger than the QCD scale $\Lambda_{\rm QCD}$, the running coupling $\alpha_s(m_Q)$ is small implying that at this scale of the order of the Compton wavelengh $\lambda \sim 1/m_Q$, one can safely use perturbative QCD for describing the hadrons. In this case the heavy $\bar{Q}Q$ bound states with the size $\lambda/\alpha_s(m_Q) \ll R_{\rm had} \sim 1$ fermi are hydrogen-like atoms to which ordinary quantum mechanics can be applied. In the *non-relativistic limit* (NR), it is possible to show that the interaction between the two $\bar{Q}$ and Q states can be described by a local potential $V(\vec{r})$, where $\vec{r}$ is the relative coordinate between Q and $\bar{Q}$ (spin is neglected for the moment). The energy levels and wave functions of the bound states can be found by solving the Schrödinger equation in three-dimensions:

$$E_{nl}\Psi_{nlm}(\vec{r}) = \left[-\frac{\hbar^2}{2\mu}\Delta + V(\vec{r})\right]\Psi_{nlm}(\vec{r}) \tag{45.1}$$

where $\mu \equiv m_Q/2$ is the reduced mass of the system; $\Psi_{nlm}(\vec{r})$ is the Schrödinger wave function; $V(\vec{r})$ is the interaction potential and E_{nl} is the energy eigenvalue; n, l and m are respectively the principal quantum number, angular orbital, and eigenvalue of l_z on the z-axis; $\hbar = 1$ in standard units:

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\,, \tag{45.2}$$

is the Laplacian. In the case of central pontential, the wave function can be decomposed into its radial $R_{nl}(r)$ and spherical harmonic $Y_{lm}(\theta,\phi)$ components:

$$\Psi_{nlm}(\vec{r}) = R_{nl}(r)Y_{lm}(\theta,\phi)\,. \tag{45.3}$$

In terms of the reduced wave function:

$$u_{nl}(r) \equiv r\,R_{nl}(r)\,, \tag{45.4}$$

the Schrödinger equation becomes:

$$-\frac{d^2u_{nl}}{dr^2} = -\frac{\hbar^2}{2\mu}\left[E_{nl} - V(r) - \frac{l(l+1)\hbar^2}{2\mu r^2}\right]u_{nl}(r)\,, \tag{45.5}$$

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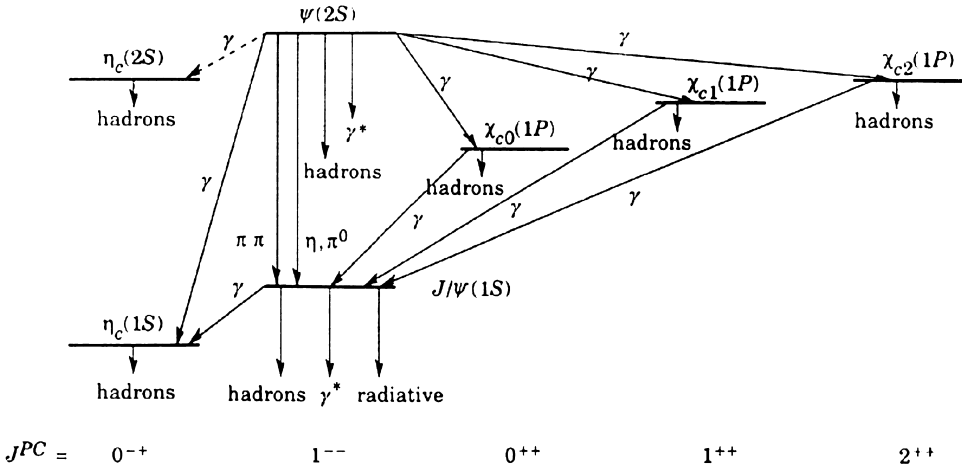

Fig. 45.1. Current situation of the charmonium system and transitions interpreted from some models. Dashed lines are uncertain states. $\gamma*$ refers to processes involving virtual photons, including decays into e^+e^- and $\mu^+\mu^-$.

with the boundary conditions:

$$u_{nl}(0) = 0 \,, \qquad \left.\frac{du_{nl}}{dr}\right|_{u=0} = R_{nl}(0) \,, \tag{45.6}$$

except that even parity solutions are inconsistent with the above boundary conditions. The wave function is normalized such that:

$$\int d^3\vec{r}\,|\Psi_{nlm}(\vec{r})|^2 = \int_0^\infty dr\,|u_{nl}(r)|^2 = 1 \,. \tag{45.7}$$

It shows that the system is now described by the effective potential:

$$V_{\rm eff}(r) \equiv V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2} \,, \tag{45.8}$$

Obviously, the main uncertainty for a quantitative spectroscopy is the choice of the correct $\bar{Q}Q$ potential $V(r)$ as far as its exact form is not yet known from first principles. We show in Fig. 45.1 the spectra of charmonium and in Fig. 45.2 those for the bottomium systems from [16]

45.2 The QCD static Coulomb potential

First, the model has to recover the short distance QCD *Coulomb static potential*. The expression of this potential can be derived from the tree-level scattering amplitude of the process:

$$\mathcal{A}[Q(p_1,\lambda_1,i_1) + \bar{Q}((p_2,\lambda_2,i_2) \to Q(p_1',\lambda_1',i_1') + \bar{Q}((p_2',\lambda_2',i_2')] \,, \tag{45.9}$$

shown at tree level in Fig. 45.3; i, j and λ_i are respectively colour and spinor indices.

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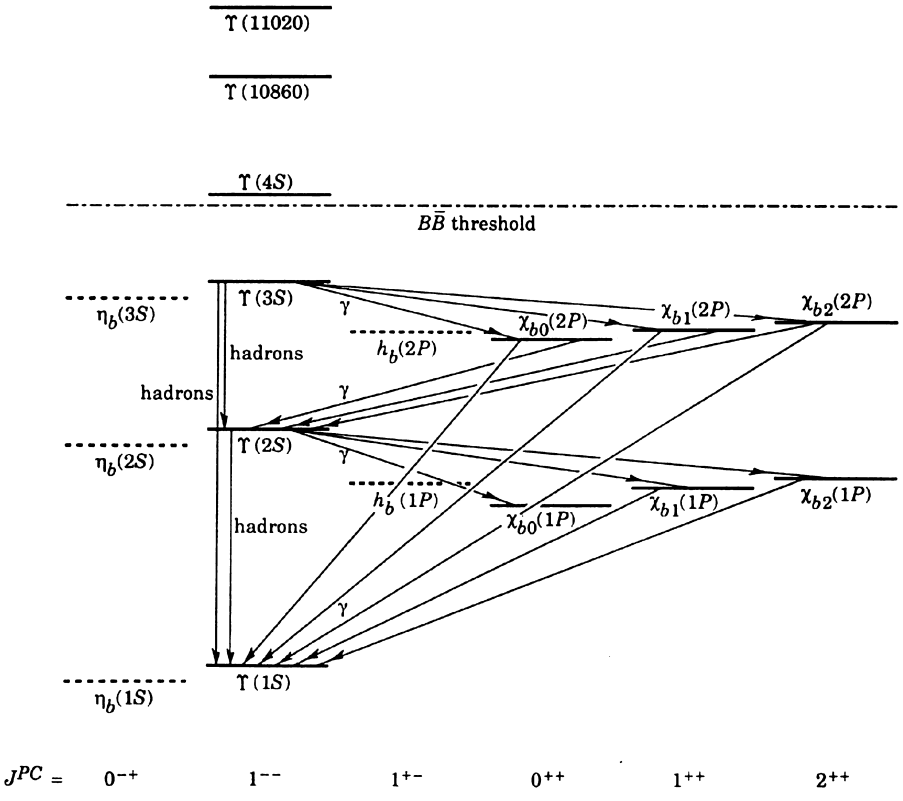

Fig. 45.2. Same as in Fig. 45.1 but for the Bottomium system.

In a covariant Feynman gauge, it is easy to obtain:

$$\mathcal{A} = \frac{1}{4}\sum_a \lambda^a_{i_2 i'_2}\lambda^a_{i_1 i'_1}\left(\frac{g^2}{4\pi^2}\right)\bar{u}(p'_1,\lambda'_1)\gamma^\mu u(p_1,\lambda_1)\frac{g_{\mu\nu}}{k^2}\bar{v}(p_2,\lambda_2)\gamma^\mu v(p'_2,\lambda'_2)\,, \quad (45.10)$$

where λ^a are colour matrices. It can again be rearranged by using the relation:

$$\bar{v}(p_2,\lambda_2)\gamma_\mu v(p'_2,\lambda'_2) = -\bar{u}(p'_2,\lambda'_2)\gamma_\mu u(p_2,\lambda_2)\,. \quad (45.11)$$

The non-relativistic amplitude is related to $\mathcal{A}$ as:

$$\mathcal{T}_{\rm NR} = \frac{1}{4\sqrt{p_{10}p'_{10}p_{20}p'_{20}}}\mathcal{A}\,. \quad (45.12)$$

In the non-relativistic limit:

$$p_0 \equiv \sqrt{\vec{p}^2 + m_Q^2} \simeq m_Q + \frac{\vec{p}^2}{2m_Q} + \frac{\vec{p}^4}{8m_Q^2}\,,$$
$$k^2 = (p_{10}-p_{20})^2 - \vec{k}^2 \simeq -\vec{k}^2 + \frac{\vec{p}^2 - \vec{p}'^2}{4m_Q^2}\,, \quad (45.13)$$

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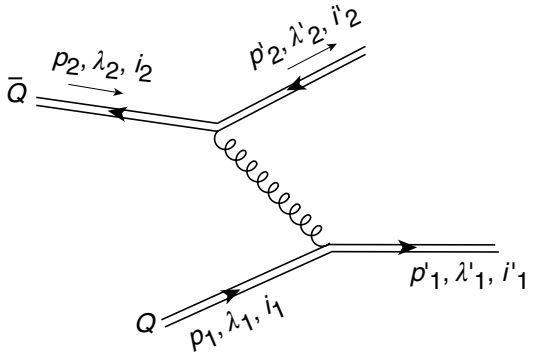

Fig. 45.3. Tree-level diagram for $\bar{Q}Q$ scattering.

and

$$\frac{1}{\sqrt{2p_0}}u(p,\lambda) \simeq \begin{pmatrix} \left(1-\vec{p}^2/4m_Q^2\right)\chi(\lambda) \\ (1/2m_Q)\vec{p}\cdot\vec{\sigma}\,\chi(\lambda) \end{pmatrix} \tag{45.14}$$

where the Pauli matrices $\vec{\sigma}$ act on the two-component spinor $\chi(\lambda_i)$. In the static limit, one can retain only the leading term in Eq. (45.13), and obtains:

$$\begin{aligned} \mathcal{T}_{\rm NR}(Born) &\simeq -\frac{1}{4}\sum_a \lambda^a_{i_2 i'_2}\lambda^a_{i_1 i'_1}\left(\frac{g^2}{4\pi^2}\right)\chi^\dagger(\lambda'_1)\chi^\dagger(\lambda'_2)\frac{1}{\vec{k}^2}\chi(\lambda_1)\chi(\lambda_2) \\ &= -\frac{1}{4}\sum_a \lambda^a_{i_2 i'_2}\lambda^a_{i_1 i'_1}\delta_{\lambda_1\lambda'_1}\delta_{\lambda_2\lambda'_2}\left(\frac{g^2}{4\pi^2}\right)\frac{1}{\vec{k}^2}\,. \end{aligned} \tag{45.15}$$

On the other hand, the non-relativistic amplitude can be related to the potential as:

$$\mathcal{T}_{\rm NR}(Born) = -\frac{1}{4\pi^2}\int d^3\vec{r}\; e^{i\vec{k}\vec{r}}\,\chi^\dagger(\lambda'_1)\chi^\dagger V(\vec{r})(\lambda'_2)\frac{1}{\vec{k}^2}\chi(\lambda_1)\chi\,. \tag{45.16}$$

By identification, taking the inverse Fourier transform, and using:

$$\frac{1}{4}\sum_a \lambda^a_{i_2 i'_2}\lambda^a_{i_1 i'_1} = \begin{cases} -C_F\;: & \text{Singlet} \\ \dfrac{1}{2N_c} = \dfrac{1}{6}\;: & \text{Octet} \end{cases} \tag{45.17}$$

one obtains the expression.[1]

$$\begin{aligned} V(r \ll 1/\Lambda_{\rm QCD}) &= -\left(C_F \equiv \frac{4}{3}\right)\frac{\alpha_s}{r}\;: \qquad \text{Singlet} \\ &= \left(\frac{1}{2N_c}\right)\frac{\alpha_s}{r}\;: \qquad \text{Octet}\,, \end{aligned} \tag{45.18}$$

where the running of the QCD coupling and the form of the potential have been verified on the lattice. Using this form of the potential, the eigenvalue of the previous Schrödinger

[1] We shall only consider the singlet case in the following.

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equation in Eq. (45.5) is the so-called *binding energy*:

$$E_{nl} = 2m_Q - \frac{C_F^2 \alpha_s^2}{4n^2} m_Q \ . \tag{45.19}$$

45.3 Potential models

The model dependence enters into the large distance part ($r \gg 1/\Lambda_{\rm QCD}$) of the potential. Many phenomenological QCD-motivated forms of the potential have been proposed in the literature [12,81–94].

45.3.1 Cornell potential

The simplest phenomenological form is the Cornell linear potential [82]:

$$V(r \gg 1/\Lambda_{\rm QCD}) \simeq \sigma r \tag{45.20}$$

where σ is the QCD string tension.

45.3.2 Richardson potential

In the Richardson potential [83], the QCD coupling in the Coulomb potential is allowed to run, and an interpolating formula for the Fourier transform of the potential has been proposed:

$$\tilde{V}(q) = -\frac{4}{3}\left(\frac{48\pi^2}{33-2n_f}\right)\frac{1}{q^2 \ln\left(1+q^2/\Lambda_{\rm QCD}^2\right)} \ . \tag{45.21}$$

which behaves as:

$$\tilde{V}(q \gg \Lambda_{\rm QCD}) \sim \frac{1}{q^2 \ln\left(q^2/\Lambda_{\rm QCD}^2\right)} \ , \qquad V(q \ll \Lambda_{\rm QCD}) \sim \frac{1}{q^2} \ . \tag{45.22}$$

The charmonium and upsilon spectroscopy fix the parameters to be:

$$\Lambda_{\rm QCD} \approx 400 \text{ MeV} \ , \qquad \sigma \simeq (400 \text{ MeV})^2 \ . \tag{45.23}$$

45.3.3 Martin potential

Some more empirical models are the Martin potential [12,84–86]:

$$V(r) \sim A + Br^n \ , \tag{45.24}$$

where the different terms are fixed from the fit of the rich quarkonia families. The power of the potential is found to be:

$$n \simeq 0.1 \ . \tag{45.25}$$

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Martin's potential is neither Coulombic at short distance nor linear at long distance, but is strongly constrained inside the region $0.1 \sim 1$ fermi. Its slight modification outside this region does not affect the results as the wave function vanishes rapidly.

From the concavity properties of the potential [12,84–86]:

$$\frac{dV}{dr} > 0\,, \qquad \frac{d^2V}{dr^2} < 0 \quad \Longrightarrow \quad \frac{d}{dr}\left(\frac{1}{r}\frac{dV}{dr}\right) < 0\,, \tag{45.26}$$

some impressive sets of inequalities can be derived. If n is the number of nodes of the radial wave functions, and l the orbital angular momentum, one has for $n \geq 0$:

$$E(n, l+1) > \frac{1}{2}[E(n+1, l) + E(n, l)]\,, \tag{45.27}$$

which is satisfied by the observed masses:

$$M_{\Upsilon'} - M_{\Upsilon} > M_{\Upsilon''} - M_{\Upsilon'} \ldots \tag{45.28}$$

The flavour independence assumption leads to the concavity relation:

$$2E(\bar{Q}q) > E(\bar{Q}Q) + E(\bar{Q}q)\,, \tag{45.29}$$

which is well satisfied by the observed masses. In particular, one expects to have:

$$M_{B_c} \geq \frac{1}{2}[M_{\psi} + M_{\Upsilon}]\,, \tag{45.30}$$

while the lower bound can also be obtained [86]. Analogous inequalities have also been derived among baryons and mesons.

However, despite the great phenomenological success of various types of potential models, some difficulties arise in attempting to relate them to field theory. Leutwyler and Voloshin criticize the locality of the potentials [90], whilst Bell and Bertlmann [91–93] do not see their flavour independence.

45.4 QCD corrections to the static Coulomb potential: Leutwyler–Voloshin model

In this section, we shall consider the Coulomb static potential given in Eq. (45.18) and we shall investigate the different QCD corrections to it.

45.4.1 Relativistic corrections

In this case, the interaction betwen the Q and $\bar{Q}$ can be described by the Breit-fermi potential describing the positronium e^+e^- bound state (see e.g. Schwinger [319], Bertstetski *et al.* [53]). It gives the relativistic corrections:

$$V^{(0)}(r) = V^{(0)}_{\text{stat}}(r) + V^{(0)}_{\text{rel}} \tag{45.31}$$

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where:

$$V^{(0)}_{\rm rel} \equiv V^{(0)}_{\rm orb} + V^{(0)}_{\rm tens} + V^{(0)}_{LS} + V^{(0)}_{HF} \,, \tag{45.32}$$

which corresponds respectively to the purely orbital (spin independent + kinetic energy), tensor, spin-orbit and hyperfine potentials. They read:

$$\begin{aligned} V^{(0)}_{\rm orb} &= -\frac{1}{4m_Q^3}\Delta^2 + \frac{C_F\alpha_s}{m_Q^2}\frac{1}{r}\Delta \\ V^{(0)}_{\rm tens} &= \frac{C_F\alpha_s}{4m_Q^2}\frac{1}{r^3}S_{12} \\ V^{(0)}_{LS} &= \frac{3C_F\alpha_s}{2m_Q^2}\frac{1}{r^3}\vec{L}\cdot\vec{S} \\ V^{(0)}_{HF} &= \frac{4\pi C_F\alpha_s}{3m_Q^2}\vec{S}^2\delta(\vec{r}) \,. \end{aligned} \tag{45.33}$$

Here $\vec{L}$, $\vec{S}$ and S_{12} are respectively the orbital angular momentum, total spin and tensor operators defined as:

$$\vec{L} = -i\vec{r}\times\Delta \,, \qquad \vec{S} = \frac{1}{2}(\sigma_1+\sigma_2) \,, \qquad S_{12} = 2\sum_{ij}\left(2\frac{r_i r_j}{r^2} - \delta_{ij}\right) S_i S_j \,. \tag{45.34}$$

In Eq. (45.33), one should notice that r^{-1} and Δ do not commute, which is not important as one only considers diagonal matrix element of $r^{-1}\Delta$ between the Ψ states. Another peculiar point is that one has to take the expectation values of terms containing $\vec{L}\vec{S}$ and S_{12} to be zero between states with zero angular momentum as their angular average vanishes. This is despite the fact that the factor $1/r^3$ is singular at the origin.

Noting that, in the Coulombic approximation, from the average value:

$$\langle \vec{k}^2/m_Q^2\rangle_{nl} = \left(\frac{C_F\alpha_s}{2n}\right)^2 \,, \tag{45.35}$$

one can, for example, deduce the shift of the spin-independent energy levels:

$$E^{(0)}_{nl} \to E^{(0)}_{nl} + \left[\delta_{rel}E_{nl} \equiv \langle V^{(0)}_{\rm orb}\rangle_{nl}\right] , \tag{45.36}$$

with:

$$\delta_{rel}E_{nl} = \left(\frac{C_F\alpha_s}{m_Q^2 a^3}\right)\left(\frac{2l+1-4n}{(2l+1)n^4}\right) - \frac{2}{m_Q^3 a^4}\left[\frac{1}{(2l+1)n^4} - \frac{3}{8n^5}\right] , \tag{45.37}$$

where:

$$a \equiv \frac{2}{m_Q C_F\alpha_s} \,, \tag{45.38}$$

is the Bohr radius. Analogously, the hyperfine splittings can be obtained from $\langle V^{(0)}_{HF}\rangle_{n0}$,

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which in the case $n = 1$ gives:

$$M_\Upsilon - M_{\eta_b} = \delta_{HF} E_{10} \equiv \left\langle V_{HF}^{(0)} \right\rangle_{10} = \frac{8 C_F \alpha_s}{3 m_Q^2 a^3} \,. \tag{45.39}$$

45.4.2 Radiative and non-perturbative corrections

Radiative corrections to the previous lowest order relativistic corrections are known. The readers can find a compilation of the results obtained within the $\overline{MS}$ scheme in, for example, [46].

A priori, one may expect that non-perturbative corrections to the static potential have complicated structure. However, as the heavy quarks move in a short distance region $\langle \vec{k}^2 \rangle \sim a \ll 1/\Lambda_{\rm QCD}$, one can be convinced that the first known leading non-perturbative correction in $1/m_Q$ is due to the gluon condensate. Treating the interaction Hamiltonian as a perturbation to the Coulomb potential and using a dipole approximation:

$$H_I = -\frac{g}{2} \left(\lambda_q^a - \lambda_{\bar{q}}^a \right) \vec{x} \vec{E}_a \,, \tag{45.40}$$

where $\vec{E}_a$ is the colour electric field which is related to the gluon condensate as:

$$\langle 0 | g^2 \vec{E}^a \vec{E}_a | 0 \rangle = -\pi \langle G^2 \rangle \,, \tag{45.41}$$

the energy levels are determined by the quadratic Stark-effect of the chromoelectric field:

$$\delta_{NP} E_{nl} = \langle \Psi_{nl} | 0 | H_I \frac{1}{E_n^{(0)} - H_{\rm Coul}^{(8)}} H_I | 0 \rangle | \Psi_{nl} \rangle = \frac{\pi}{18} \langle \alpha_s G^2 \rangle \langle \Psi_{nl} | \vec{x} \frac{1}{E_n^{(0)} - H_{\rm Coul}^{(8)}} \vec{x} | \Psi_{nl} \rangle \,, \tag{45.42}$$

which contributes to the level shift as [90]:

$$\delta_{NP} E_{nl} = m_Q \frac{\epsilon_{nl} n^6 \pi \langle \alpha_s G^2 \rangle}{(m_Q C_F \tilde{\alpha}_s)^4} \,, \tag{45.43}$$

where:

$$\epsilon_{nl} = \frac{2}{9} \frac{1}{n^3 (2l+1)} [(l+1)[F(n,l) - F(-n,l)] + l[F(n,-l-1) - F(-n,-l-1)]] \,, \tag{45.44}$$

with:

$$\begin{aligned} F(n,l) \; &= 2n[n^2 - (l+1)^2] + (n+l+2)(n+l+1) \\ &\times \left[\frac{(n-l)(n+l+3)}{9n+16} + 4 \frac{(2n-l)^2}{9n+8} \right] . \end{aligned} \tag{45.45}$$

Some particular values are:

$$\epsilon_{10} = \frac{624}{425} \,, \qquad \epsilon_{20} = \frac{1051}{663} \,, \qquad \epsilon_{21} = \frac{9929}{9945} \,. \tag{45.46}$$

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The main feature of the result is that the level shift grows like n^6, showing that, even for heavy quarks, the non-perturbative corrections are important for excited states.

45.4.3 Validity range

The validity of the previous result can only be realized if the shift is much smaller than the Schrödinger binding energy in Eq. (45.19), which needs that $n^2/m_Q \ll 1$. Taking $n = 1$, this leads to the condition:

$$m_Q \gg 5 \text{ GeV} , \tag{45.47}$$

indicating that the model is quite inaccurate when applied to the bottomium system.

45.4.4 Some phenomenological applications

Collecting all different corrections, the vector-pseudoscalar mass-difference is [46]:

$$\begin{aligned} M_\Upsilon - M_{\eta_b} = & \frac{8C_F\alpha_s}{3m_Q^2 a^3}[1+\delta_{\alpha_s}+\delta_{NP}]^2 \\ & \times\left(\frac{\alpha_s}{\pi}\right)\left\{1+\left[-\beta_1\left(\ln\frac{a\mu}{2}-1\right)+\frac{21}{4}\left(\ln C_F\tilde{\alpha}_s+1\right)+b_{HF}\right]\right. \\ & \left.+\frac{1161}{8704}\frac{\pi\langle\alpha_s G^2\rangle}{m_Q^4\tilde{\alpha}_s^6}\right\} , \end{aligned} \tag{45.48}$$

where:

$$\begin{aligned} b_{HF} &= \frac{11C_A-9C_F}{18} , \qquad \delta_{\alpha_s} = -\frac{3\beta_1}{2}\left(\frac{\alpha_s}{\pi}\right)\left(\ln\frac{a\mu}{2}-\gamma_E\right) , \\ \delta_{NP} &= \frac{1}{2}\left[\frac{270\,459}{108\,800}+\frac{1\,838\,781}{2\,890\,000}\right]\frac{\pi\langle\alpha_s G^2\rangle}{m_Q^4\tilde{\alpha}_s^6} . \end{aligned} \tag{45.49}$$

The leptonic width of the Υ is:

$$\Gamma(\Upsilon\to e^+e^-) = \Gamma^{(0)}\times[1+\delta_{\alpha_s}+\delta_{NP}]^2\left(1-\frac{4C_F\alpha_s}{\pi}\right) , \tag{45.50}$$

where:

$$\Gamma^{(0)} = 16\pi\left(\frac{Q_b\alpha}{M_\Upsilon}\right)^2|\psi(0)|^2 . \tag{45.51}$$

The wave function is found to be:

$$|\psi(0)|^2 = 2\left[m_Q C_F\tilde{\alpha}_s^3(\mu^2)\right] , \tag{45.52}$$

Using:

$$\Lambda_{\rm QCD}(n_f=4,\ 3\text{ loops}) = (0.23\pm0.06)\text{ GeV} , \qquad \langle\alpha_s G^2\rangle = (0.06\pm0.02)\text{ GeV}^4 , \tag{45.53}$$

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they lead to the numerical predictions:

$$M_\Upsilon - M_{\eta_b} = (47 \pm 13)\ \text{MeV}\ , \qquad \Gamma(\Upsilon \to e^+e^-) \simeq (1.1 \pm 0.3)\ \text{keV}\ . \quad (45.54)$$

The electronic width is quite inaccurate but agrees within the errors with the data (1.32 ± 0.04) keV. The prediction for the mass splitting will be compared with the other QCD-based predictions in subsequent chapters. Some other predictions for the mass splittings are also available [46], which in general are in good agreement with the data. This method has been also used in [90,94,46] for extracting the values of the quark pole masses. We quote below the corresponding values of the $\overline{MS}$ running masses:

$$\bar{m}_b\left(m_b^2\right) = 4440^{+43}_{-28}\ \text{MeV}\ , \qquad \bar{m}_c\left(m_c^2\right) = 1531^{+132}_{-127}\ \text{MeV}\ , \quad (45.55)$$

which are systematically higher than predictions from QCD spectral sum rules methods (see however, [602] and the next chapter on quark masses).

45.5 Bell–Bertlmann equivalent potentials

'Equivalent' potential reproducing the Leutwyler–Voloshin spectrum has been proposed in [93]. Using the form of the Stark effect in Eq. (45.42), in the static limit ($m_Q \to \infty$) where one can neglect the kinetic term p^2/m_Q, the energy denominator becomes a potential difference:

$$E_n^{(0)} - H_{\text{coul}}^8 \overset{m_Q \to \infty}{\to} V_{\text{coul}}^0 - V_{\text{coul}}^8 = \frac{9\beta}{8}\frac{1}{r}\ , \quad (45.56)$$

which leads to the cubic potential:

$$V_{\text{static}} = \frac{4\pi}{81\beta}\langle\alpha_s G^2\rangle r^3\ . \quad (45.57)$$

This potential accounts for large quantum numbers, as the distance in a Coulombic state behaves as:

$$\langle r\rangle_n^{\text{Coul}} \sim n^2\ . \quad (45.58)$$

Corrections of order $1/m_Q$ to this potential approximate low quantum numbers. Therefore, one arrives at the 'equivalent' potential [93]:

$$\delta V(r) = \frac{4\pi}{81\beta}\langle\alpha_s G^2\rangle\left[r^3 - \frac{304}{81}\frac{r^2}{m_Q\beta} + \frac{53}{10}\frac{r}{(m_Q\beta)^2} - \frac{113}{100}\frac{1}{(m_Q\beta)^3}\right], \quad (45.59)$$

which differs from the effective potential models as it is *flavour dependent*.

Another 'equivalent' potential has been proposed in [91–93] for interpreting the QCD spectral sum rule non-relativistic moments (see Part X, QCD spectral sum rules):

$$\mathcal{R}(\tau_N) = -\frac{d}{d\tau_N}\ln\left[\mathcal{M}(\tau_N) \equiv \int dE\ e^{-E\tau_N}\text{Im}\Pi(E)\right] \overset{\tau\to\infty}{\longrightarrow} E_0\ , \quad (45.60)$$

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in a potential theory, where E_0 is the energy of the ground state. $\mathrm{Im}\Pi(E)$ is the spectral function which can be parametrized within the potential theory as:

$$\mathrm{Im}\Pi(E) = \frac{3}{8m_Q^2}\sum_n 4\pi|\psi_n(0)|^2\delta(E-E_n)\,, \tag{45.61}$$

which shows that the moments $\mathcal{M}$ is nothing but the time-dependent Green function:

$$\mathcal{M}(\tau_N) = \frac{3}{8m_Q^2}4\pi\,\langle\vec{x}|e^{-H\tau_N}|\vec{x}\rangle|_{\vec{x}=0}\,, \tag{45.62}$$

where $H \equiv p^2/m_Q + V$ is the total Hamiltonian of the system. Perturbing the kinetic term by the potential with respect to the time τ_N:

$$e^{-H\tau_N} = e^{-\frac{p^2}{m_Q}\tau_N} - \int_0^{\tau_N} d\tau_N' e^{-\frac{p^2}{m_Q}(\tau_N-\tau_N')} V e^{-\frac{p^2}{m_Q}\tau_N'}\,, \tag{45.63}$$

one obtains for a power-like potential: $V = \sum_n v_n r^n$:

$$\mathcal{M}(\tau_N) = \frac{3}{8m_Q^2}4\pi\left(\frac{m_Q}{4\pi\,\tau_N}\right)^{3/2}\left[1-\sum_n v_n\Gamma\left(\frac{n}{2}+1\right)m_Q\left(\frac{\tau_N}{m_Q}\right)^{n/2+1}\right]. \tag{45.64}$$

An identification of this term with the QCD moments in Eq. (49.45) leads to the 'equivalent' potential:

$$V(r) = -\frac{4}{3}\frac{\alpha_s}{r} + \frac{\pi}{144}\langle\alpha_s G^2\rangle m_Q r^4\,, \tag{45.65}$$

which differs from the effective potential models and from the previous Leutwyler–Voloshin 'equivalent' potential. The main feature of the BB 'equivalent' potentials is that they are *flavour dependent* in contrast to the effective potential models.

45.6 Stochastic vacuum model

We have seen previously that for excited states the Voloshin–Leutwyler approach [90] cannnot be applied as n^2/m_Q is no longer smaller than 1. It has been noted in [603], that this is due to the fact that the correlators $\langle G_{\mu\nu}(x)G_{\alpha\beta}(x)\rangle$ have been taken to be independent of x, although they should decrease exponentially for large spacelike x. Splitting the field strength $G^a_{\mu\nu}$ into a chromomagnetic piece $\mathbf{B}_a$ and a chromoelectric one $\mathbf{E}^i_a = G^{0i}$, one can show that in the non-relativistic limit, the spin-independent piece of the splitting will only involve $\mathbf{E}^i_a$. Therefore, the non-perturbative correlator reads:

$$\langle\mathbf{E}(x)\mathbf{E}(0)\rangle = \frac{1}{12}\left[\delta_{ij}\Delta(x) + x_i x_j\frac{\partial}{x_\mu}\frac{\partial}{x_\mu}D_1(x^2)\right] \tag{45.66}$$

with:

$$\Delta(x) \equiv D(x^2) + D_1(x^2) + x^2\frac{\partial}{x_\mu}\frac{\partial}{x_\mu}D_1(x^2)\,. \tag{45.67}$$

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If one neglects the x dependence of the correlator, the only surviving part is:

$$\Delta(0) = 2\pi \langle \alpha_s G^2 \rangle \ . \tag{45.68}$$

Therefore, Eq. (45.66) indicates that one can derive the Voloshin–Leutwyler formula for small n, but one can also obtain another potential for large n if one takes into account the x dependence of the correlator.

Defining the correlation time T_Q for quarks as:

$$\langle x_i(\tau_1) x_j(\tau_2) \rangle_{nl} = \frac{\delta_{ij}}{3} \langle \vec{x}_{nl}^2 \rangle \exp\left[-\frac{|\tau_1 - \tau_2|}{T_Q} \right] , \tag{45.69}$$

and the one T_G for gluons:

$$\left\langle \mathbf{E}_a^i(\tau_1) \mathbf{E}_b^j(\tau_2) \right\rangle_E = \frac{\delta^{ij}}{3} \frac{\delta_{ab}}{8} \langle \mathbf{E}^2 \rangle \exp\left[-\frac{|\tau_1 - \tau_2|}{T_G} \right] , \tag{45.70}$$

one can also find that the sum rule approach within the Bell–Bertlmann 'equivalent' potential is applicable for $T_G \gg T_Q$ [604]. These features are the basis of the *stochastic model* discussed in details in [605,51].

45.6.1 The model

One assumes that the quarks and the gluons background fields fluctuate stochastically according to a Markov process. Let us consider the stochastic variable $\xi(t)$ depending on one or several variables t. It will be distributed according to some distribution which fixes the vacuum expectation values:

$$\langle \xi(t) \rangle \ , \qquad \langle \xi(t_1) \xi(t_2) \rangle \ , \ldots \tag{45.71}$$

The *cumulants* or *linked clusters* are defined by:

$$\left\langle \mathcal{P} \exp \int dt \ [\xi(t)] \right\rangle = \exp\left[\int dt \ \langle\langle \xi(t) \rangle\rangle + \frac{1}{2!} \int \int dt_1 dt_2 \ \langle\langle \xi(t_1) \xi(t_2) \rangle\rangle + \cdots \right] , \tag{45.72}$$

where the path ordering prescription:

$$\Phi(t, t', C) \equiv \left\langle \mathcal{P} \exp \int_C dt \ [\xi(t)] \right\rangle = \lim_{t_{i+1} - t_i \to 0} \prod_{i=1}^{N+1} \exp\left[\xi\left(\frac{t_{i+1} + t_i}{2} \right) (t_{i+1} - t_i) \right] , \tag{45.73}$$

with t_i are ordered points on the path C with $t_N = t'$ and $t_1 = t$, should be introduced if the stochastic variable $\xi(t)$ are non-commuting. In the case of commuting stochastic variables

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which we shall consider here, an expansion of Eq. (45.72) gives:

$$\begin{aligned}
&\langle\langle\xi(t)\rangle\rangle = \langle\xi(t)\rangle \,, \\
&\langle\langle\xi(t_1)\xi(t_2)\rangle\rangle = \langle\xi(t_1)\xi(t_2)\rangle - \langle\langle\xi(t_1)\rangle\rangle\langle\langle\xi(t_2)\rangle\rangle \,, \\
&\ldots
\end{aligned} \tag{45.74}$$

A *centered Gaussian process* is a process where only the $n = 2$ cumulants occur, that is, all expectation values can be determined by the correlators with $n = 2$:

$$\begin{aligned}
&\langle\xi(t)\rangle = \langle\xi(t_1)\xi(t_2)\xi(t_3)\rangle = \cdots = 0 \,, \\
&\langle\xi(t_1)\xi(t_2)\rangle = \langle\langle\xi(t_1)\xi(t_2)\rangle\rangle \,, \\
&\ldots
\end{aligned} \tag{45.75}$$

It can be shown [605,51] that assumptions where the contributions of low frequency fields can be described by a functional integral (stochastic process) with converging clusters leads to an area law of the Wilson loop and then to linear confinement for static sources.

For that purpose, we consider the Wegner–Wilson loop in a pure gauge theory:

$$W[C] = \int \mathcal{D}A_\mu \; e^{-\frac{1}{4}G^2_{\mu\nu}(x)} \exp\left[ig \int_C A^\mu(x) dx_\mu \right] . \tag{45.76}$$

Denoting:

$$\langle \ldots \rangle_A = \int \cdots \prod_{k<\mu} dA_\mu \; e^{-\frac{1}{4}G^2_{\mu\nu}(x)} \,, \tag{45.77}$$

these low frequency contributions to the Wilson loop are defined as:

$$\begin{aligned}
W[C]_A &= \left\langle \exp\left[ig \int_C A^\mu(x) dx_\mu \right] \right\rangle_A \\
&= \left\langle \exp\left[ig \int_{\mathcal{F}} G^{\mu\nu}(x) d\sigma_{\mu\nu}(x) \right] \right\rangle_A .
\end{aligned} \tag{45.78}$$

$\mathcal{F}$ is an area whose border is the loop C; $d\sigma_{\mu\nu}(x)$ $(\mu < \nu)$ is the surface element of $\mathcal{F}$ at point x, and $G_{\mu\nu}$ is the field strength. One has used the Stokes theorem which transforms the line into surface integral. These low frequency contributions are given by the cluster expansion:

$$W[C]_A = \exp\left[-\frac{g^2}{2!} \int d\sigma_{\mu\nu}(x) d\sigma_{\alpha\beta}(x') \, \langle\langle G^{\mu\nu}(x) G^{\alpha\beta}(x')\rangle\rangle + \cdots \right] , \tag{45.79}$$

$\langle G^{\mu\nu}\rangle = 0$ due to Lorentz invariance. Lorentz and translational invariances also yield the most general decomposition:

$$\begin{aligned}
\langle G_{\mu\nu}(x_1) G_{\alpha\beta}(x_1')\rangle_A = \frac{1}{12}\langle G^2\rangle \Bigg\{ &(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}) D(x^2)\kappa + \Bigg[\Bigg(\frac{1}{2}\frac{\partial}{\partial x_\mu}(x_\alpha\delta_{\nu\beta} - x_\beta\delta_{\nu\alpha}) \\
&+ \frac{1}{2}\frac{\partial}{\partial x_\nu}(x_\beta\delta_{\mu\alpha} - x_\alpha\delta_{\mu\beta}) \Bigg) D_1(x^2)(1-\kappa) \Bigg] \Bigg\} ,
\end{aligned} \tag{45.80}$$

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where $\langle G^2 \rangle \equiv \langle G_{\mu\nu}(0) G^{\mu\nu}(0) \rangle$ and κ a parameter. One can insert this expression into the cluster expansion in Eq. (45.79). Assuming that the correlator falls off for $|x - x'| > \lambda$ and using $|\mathcal{F}| \gg \lambda^2$, one obtains:

$$W[C]_A \simeq \exp[-\kappa (g^2 G^2) \lambda^2 |\mathcal{F}| K [1 + \mathcal{O}(1/\mathcal{F})]] \,, \tag{45.81}$$

where the constant K depends on the shape of the scalar function $D(x^2)$. To leading order (two-cluster) of the cluster expansion, one can notice that the result is proportional to κ as D_1 in Eq. (45.80) does not contribute to the term proportional to the area loop. The assumption of a convergent cluster expansion thus leads to the area law of the Wilson loop if the tensor structure with D in Eq. (45.80) does not vanish. Thus leading apparently to a natural linear confinement for Abelian theory as well. However, one can show that the use of the Maxwell equations for QED:

$$\partial^\alpha \epsilon_{\alpha\mu\nu\beta} G^{\mu\nu} = 0 \,, \tag{45.82}$$

implies that:

$$\kappa = 0 \,, \tag{45.83}$$

and hence that we have no area law and then no confinement. For QCD, we have instead:

$$\partial_\alpha \epsilon_{\alpha\mu\nu\beta} G^a_{\mu\nu} = -ig \epsilon_{\alpha\mu\nu\beta} f_{abc} A^b_\mu A^c_\nu \neq 0 \,, \tag{45.84}$$

indicating that there is no reason why κ should be equal to zero. A lattice measurement shows that, for QCD, the correlator $D(x^2)$ is dominant as one finds [606]:

$$\kappa \approx 0.74 \,. \tag{45.85}$$

45.6.2 Application to the static potential

We shall discuss here the application of the model to static potential.[2] Let us consider the gauge-invariant non-local operator:

$$\mathcal{O}(x, x') \equiv \bar{Q}(x) \phi(x, x') Q(x') \,, \tag{45.86}$$

where $Q(x)$ is the heavy quark field, and $\phi(x, x')$ is the string along the straightline:

$$\phi(x, x') = \mathcal{P} \exp \left[ig \int_C A^\mu [x + \lambda(x' - x)] \left(x' - x\right)_\mu d\lambda \right] \,. \tag{45.87}$$

Applying the previous operator to the vacuum state of hadrons leads to a gauge- and Lorentz-invariant state composed of a quark at a position x and an antiquark at a position x'. The evolution of this operator is given by the Green's function:

$$G(x, x'; y, y') = \int \mathcal{D}A \mathcal{D}Q \mathcal{D}\bar{Q} \, e^{-\mathcal{S}} \mathcal{O}(x, x') \mathcal{O}^\dagger(y, y') \,. \tag{45.88}$$

[2] Some other applications of the model can be found in more specialized reviews (see e.g. [51]).

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The QCD action is:

$$\mathcal{S} = \int dx\ \bar{Q}(x)[i\gamma_\mu\,(\partial^\mu + igA^\mu) - m_Q]Q(x) + \mathcal{S}_{YM}\ , \tag{45.89}$$

with:

$$\mathcal{S}_{YM} \equiv (1/4)G^a_{\mu\nu}G^{\mu\nu}_a\ . \tag{45.90}$$

Doing the integration over fermion fields, one obtains:

$$G(x, x'; y, y') = \int \mathcal{D}A\ e^{-\mathcal{S}_{YM}}\mathrm{Det[A]Tr}[S(x', y'; A)\phi(y', y)S(y, x; A)]\ , \tag{45.91}$$

where:

$$S(x, y, A) = \frac{\delta(x-y)}{[i\gamma_\mu\,(\partial^\mu + igA^\mu) - m_Q]}\ , \tag{45.92}$$

is the quark propagator in external field; Det[A] is the functional determinant from the quark field integration, and describes internal fermion loops as power series of g. Using $i\gamma_0 = \gamma_4$, one obtains to leading order in $1/m_Q$ and g:

$$\begin{aligned} S(x, y, A) \simeq \phi(x, y)\delta(\vec{x} - \vec{y}) \times \Bigg[& e^{-m_Q(x_4 - y_4)}\theta(x_4 - y_4)\left(\frac{1+\gamma_0}{2}\right) \\ & + e^{-m_Q(y_4 - x_4)}\theta(y_4 - x_4)\left(\frac{1-\gamma_0}{2}\right)\Bigg] + \mathcal{O}\left(\frac{1}{m_Q}\right) \end{aligned} \tag{45.93}$$

Therefore, to this approximation, the Green's function becomes:

$$\begin{aligned} G[(\vec{x}, 0), (\vec{x}', 0); (\vec{y}, T), (\vec{y}', T)] &\simeq \delta(\vec{x} - \vec{y})\delta(\vec{x}' - \vec{y}')e^{-2m_Q T} \\ &\quad \times \int \mathcal{D}A\ e^{-\mathcal{S}_{YM}}\mathbf{Tr}[\phi(y, x)\phi(x, x')\phi(x', y')\phi(y', y)] \\ &\equiv \delta(\vec{x} - \vec{y})\delta(\vec{x}' - \vec{y}')e^{-2m_Q T}\,\mathbf{Tr}\ W[L]\ , \end{aligned} \tag{45.94}$$

where the Wegner–Wilson loop has been defined in Eq. (45.76). The second loop integral entering in Eq. (45.76) is defined along the rectangle with corners $[(\vec{x}, 0), (\vec{x}', 0), (\vec{y}', T), (\vec{y}, T)]$. Using the fact that the Green's function scales like $e^{-E_n T}$, where E_n is the energy of the system, one can deduce:

$$E_n = -\lim_{T\to\infty}\frac{1}{T}\ln \mathbf{Tr}\ W[L] + 2m_Q\ . \tag{45.95}$$

The term $2m_Q$ is the rest energy of the two quarks, while the first term can be identified with the potential $V(\vec{x} - \vec{x}')$ of the system. Evaluating the rectangular Wegner–Wilson loop using a strong coupling or lattice calculations, one finds in terms of the string tension σ [491]:

$$V(\vec{x} - \vec{x}') = \sigma(\vec{x} - \vec{x}')\ . \tag{45.96}$$

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Using the cluster decomposition in Eq. (45.79), one obtains, from the area law of the Wegner–Wilson loop, the spin-independent part of the potential:

$$V_0(r) = \frac{1}{24N_c}\langle g^2 G^2\rangle \left[r\int_0^r d\rho \int_{-\infty}^{+\infty} d\tau\ D(\tau^2+\rho^2) + \int_0^r \rho d\rho \int_{-\infty}^{+\infty} d\tau\ \left[-D(\tau^2+\rho^2) + \frac{1}{2}D_1(\tau^2+\rho^2)\right]\right], \quad (45.97)$$

where:

$$\lim_{r\to\infty} V_0(r) \sim \sigma r\ , \quad (45.98)$$

corresponding to the standard linear potential. At short distance:

$$\lim_{r\to 0} V_0(r) \sim r^2\ , \quad (45.99)$$

which recovers the form expected from renormalon calculations (see previous chapter on renormalons).

However, one should notice that this result is only valid for:

$$T^{-1} \gg E_n\ , \quad (45.100)$$

which corresponds to the regime where r is small but the state is located on average at a large distance from the centre of mass. Collecting the previous result, the full non-relativistic potential, from the stochastic model, is:

$$V(r) = -C_F\frac{\alpha_s}{r} + V_0(r)\ . \quad (45.101)$$

To this expression one can add spin-dependent corrections to order $1/m_Q^2$ (see e.g. [51]), which can also be expressed in terms of the correlators $D(x)$ and $D_1(x)$. One should notice that the spin-dependent part of the confining potential is known phenomenologically to be specifically different from the Coulomb potential, while good results are obtained if one assumes that the confining potential leads to the same spin-dependent force as a scalar exchange [607]. Radiative corrections can also be included into the Coulombic potential. Predictions for the higher excited mass splittings using the model are quite sucessful.

45.7 Non-relativistic effective theories for quarkonia

In a previous section, we have anticipated the different regimes (short and long distances) appearing in the $\bar{Q}Q$ system. In the present approach, it is convenient to introduce two UV scales $\Lambda_{1,2}$ which characterize such regimes, and which are ordered by the heavy quark velocity $v \ll 1$:

- The quark mass m_Q is called the hard scale.
- The momentum $m_Q v$ is the soft scale (S).
- The binding energy $m_Q v^2$ is the ultrasoft scale (US).

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Therefore, one can have the regime hierarchy:

$$m_Q v^2, \Lambda_{\rm QCD} \ll \Lambda_1 \ll m_Q v \ll \Lambda_2 \ll m_Q \ . \tag{45.102}$$

In this way, Λ_1 is the cut-off of the quark energy and of the gluon energy and momentum, whilst Λ_2 is the cut-off of the relative momentum $\vec{p}$ of the quark-antiquark system. For a Coulombic system, one has:

$$v \sim \alpha_s \ . \tag{45.103}$$

As the two scales are largely separated, one can (in principle) integrate out the UV scales step by step: after integrating out the heavy quark mass m_Q, one obtains the usual non-relativistic QCD (NRQCD) effective theory [608]. The Lagrangian of NRQCD is written in terms of an expansion in $1/m_Q$. Potential NRQCD (pNRQCD) is obtained by integrating out from NRQCD the soft scale S [609]. In this way, the Lagrangian of pNRQCD is expressed as an expansion in terms of $1/m_Q$ and of the relative coordinate $\vec{r}$ (*multipole expansion*) of the $\bar{Q}$ and Q.

The integration of the degrees of freedom is done using matching conditions (see e.g. [610,611,612,613] for details), namely by comparing the shell amplitudes order by order in QCD and NRQCD. The matching from QCD to NRQCD can always be done perturbatively since, by definition of the heavy quark, $m_Q \gg \Lambda_{\rm QCD}$. The matching from NRQCD to pNRQCD can only be carried out perturbatively when $m_Q v \gg \Lambda_{\rm QCD}$. This condition is assumed to be satisfied in the following discussion. Therefore, the matching coefficients in both NRQCD and pNRQCD can be computed order by order in α_s. The non-analytical behaviour in $1/m_Q$ appears through logs in the matching coefficients of the NRQCD Lagrangian:

$$C_H \sim A\alpha_s \left(\ln \frac{m_Q}{\mu} + B \right), \tag{45.104}$$

where μ denotes the matching scale between QCD and NRQCD. In practice, one can choose:

$$\frac{\Lambda_2^2}{m_Q} \ll \Lambda_1 \ , \tag{45.105}$$

If one denotes by Λ_{mp} any scale below Λ_1, the relevant small dimensionless scales involved in the analysis are:

$$\frac{p}{m_Q} \ , \quad \frac{1}{r m_Q} \ , \quad \text{and} \quad \Lambda_{mp} r \ll 1 \ . \tag{45.106}$$

Decomposing the $\bar{Q}Q$ state into a singlet $S(\vec{R}, \vec{r}, t)$ and an octet $O(\vec{R}, \vec{r}, t)$ states ($\vec{R} \equiv (\vec{x}_1 + \vec{x}_2)/2$, $r \equiv (\vec{x}_1 + \vec{x}_2)$), the minimal form in terms of the derivatives of the

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pNRQCD Lagrangian reads:

$$\begin{aligned}\mathcal{L}_{\rm pNRQCD} = & -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu\,a} \\ & + {\rm Tr}\left\{ {\rm S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m_Q} + \frac{\mathbf{p}^4}{4m_Q^3} - V_s^{(0)}(r) - \frac{V_s^{(1)}}{m_Q} - \frac{V_s^{(2)}}{m_Q^2} + \cdots \right) {\rm S} \right. \\ & \left. + {\rm O}^\dagger \left(i D_0 - \frac{\mathbf{p}^2}{m} - V_o^{(0)}(r) + \cdots \right) {\rm O} \right\} \\ & + g V_A(r) {\rm Tr}\{ {\rm O}^\dagger \mathbf{r}\cdot\mathbf{E}\,{\rm S} + {\rm S}^\dagger \mathbf{r}\cdot\mathbf{E}\,{\rm O}\} + g\frac{V_B(r)}{2}{\rm Tr}\{ {\rm O}^\dagger \mathbf{r}\cdot\mathbf{E}\,{\rm O} + {\rm O}^\dagger {\rm O}\mathbf{r}\cdot\mathbf{E}\} \,, \end{aligned} \tag{45.107}$$

where the dots indicate higher-order potentials in the $1/m_Q$ expansion; $\mathbf{p} \equiv \vec{p}$; E is the chromoelectric field. One has neglected the centre-of-mass variables R and only kept $O(r)$ terms in the multipole expansion.

The structure of the potentials up to $O(1/m^2)$ are:

- To order $1/m_Q^0$, one has the Coulomb potential.

$$V_s^{(0)}(r) \equiv -C_F \frac{\tilde{\alpha}_s(r)}{r} \,. \tag{45.108}$$

- To order $1/m_Q$, and using dimensions plus time reversal $V_s^{(1)}(r)$, one can only have the following structure:

$$V_s^{(1)} \equiv -\frac{C_F C_A D_s^{(1)}}{2r^2} \,, \quad C_A = N_c \,. \tag{45.109}$$

- To order $1/m_Q^2$, and using the present accuracy for the matching, one obtains the following potential:

$$\begin{aligned} V_s^{(2)} = & -\frac{C_F D_{1,s}^{(2)}}{2}\left\{\frac{1}{r},\mathbf{p}^2\right\} + \frac{C_F D_{2,s}^{(2)}}{2}\frac{1}{r^3}\mathbf{L}^2 + \pi C_F D_{d,s}^{(2)}\delta^{(3)}(\mathbf{r}) + \frac{4\pi C_F D_{S^2,s}^{(2)}}{3}\mathbf{S}^2\delta^{(3)}(\mathbf{r}) \\ & + \frac{3C_F D_{LS,s}^{(2)}}{2}\frac{1}{r^3}\mathbf{L}\cdot\mathbf{S} + \frac{C_F D_{S_{12},s}^{(2)}}{4}\frac{1}{r^3}S_{12}(\hat{\mathbf{r}}) \,. \end{aligned} \tag{45.110}$$

Note that $\mathbf{p}$ appears analytically in the potentials, with a power ($\mathbf{p}^n$), which is constrained by the power in $1/m_Q$. The different matching coefficients : $\tilde{\alpha}$, $D^{(1)}$, $D^{(2)}$ ... in pNRQCD can be obtained by performing the matching between NRQCD and pNRQCD. A detailed description of the procedure can be found in [609,612,613]. They read:

$$\begin{aligned} \tilde{\alpha}_s(r,\mu) = \alpha_{\rm s}(r)\Bigg\{ & 1 + (a_1 + 2\gamma_E\beta_0)\frac{\alpha_{\rm s}(r)}{4\pi} + \left[\gamma_E\left(a_1\beta_0 + \frac{\beta_1}{2}\right)\right. \\ & \left. + \left(\frac{\pi^2}{12} + \gamma_E^2\right)\beta_0^2 + \frac{a_2}{4}\right]\frac{\alpha_{\rm s}^2(r)}{4\pi^2} + \frac{C_A^3}{12}\frac{\alpha_{\rm s}^3(r)}{\pi}\ln r\mu \Bigg\} \,. \end{aligned} \tag{45.111}$$

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In terms of the β function defined in the first part of the this book (Table 11.1), they read for $SU(n)_f$ flavours:

$$\beta_0 \equiv -2\beta_1 = 11 - \frac{2}{3}n_f\,, \qquad \beta_1 \equiv -8\beta_2 = 2\left(51 - \frac{19}{3}n_f\right)\,. \tag{45.112}$$

The one- and two-loop coefficients a_1, a_2 have been obtained in [614]. They read:

$$a_1 = \frac{31}{9}\,C_A - \frac{20}{9}\,T_F n_f\,, \tag{45.113}$$

$$\begin{aligned} a_2 = &\left(\frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3}\zeta_3\right) C_A^2 \\ &- \left(\frac{55}{3} - 16\zeta_3\right) C_F T_F n_f + \frac{400}{81}\,T_F^2 n_f^2 \\ &- \left(\frac{1798}{81} + \frac{56}{3}\,\zeta_3\right) C_A T_F n_f\,, \end{aligned} \tag{45.114}$$

respectively. For $SU(3)_c$, one has $T_F = 1/2$:

$$\begin{aligned} \tilde{\alpha}_s = \alpha_s^{\overline{MS}}\Bigg\{1 &+ \left(\frac{\alpha_s^{\overline{MS}}}{\pi}\right)(2.6 - 0.3n_f) \\ &+ \left(\frac{\alpha_s^{\overline{MS}}}{\pi}\right)^2 (53.4 - 7.2n_f + 0.2n_f^2) + \cdots\Bigg\}\,. \end{aligned} \tag{45.115}$$

which shows that the convergence of the QCD series is not good. The other coefficients are [615,611–613,616]:

$$\begin{aligned} D_s^{(1)} &= \alpha_s^2(r)\left\{1 + \frac{2}{3}(4C_F + 2C_A)\frac{\alpha_s}{\pi}\ln r\mu\right\}\,, \\ D_{1,s}^{(2)} &= \alpha_s(r)\left\{1 + \frac{4}{3}C_A\frac{\alpha_s}{\pi}\ln r\mu\right\}\,; \\ D_{2,s}^{(2)} &= \alpha_s(r)\,, \\ D_{d,s}^{(2)} &\simeq \alpha_s(r)\left\{1 + \frac{\alpha_s}{\pi}\left[\frac{2C_F}{3} + \frac{17C_A}{3}\right]\ln m_Q r + \frac{16}{3}\frac{\alpha_s}{\pi}\left(\frac{C_A}{2} - C_F\right)\ln r\mu\right\}\,, \\ D_{S^2,s}^{(2)} &\simeq \alpha_s(r)\left(1 - \frac{7C_A}{4}\frac{\alpha_s}{\pi}\ln m_Q r\right)\,, \\ D_{LS,s}^{(2)} &\simeq \alpha_s(r)\left(1 - \frac{2C_A}{3}\frac{\alpha_s}{\pi}\ln m_Q r\right)\,, \\ D_{S_{12},s}^{(2)} &\simeq \alpha_s(r)\left(1 - C_A\frac{\alpha_s}{\pi}\ln m_Q r\right)\,. \end{aligned} \tag{45.116}$$

The previous results can e.g. be used to compute the $\mathcal{O}(m_Q\alpha_s^5)$ corrections to the heavy quarkonium spectrum. The N^3LL correction to the energy shift of the $\Upsilon(1S)$ is found to

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be [609]:

$$\delta E_{n=1;\, l=0;\, j=1} \simeq \frac{1730}{81\pi} m_b \alpha_s^4(\mu)\alpha_s(\mu') \ln\left[1/\alpha_s(\mu')\right] \simeq (80 \sim 100)\ \text{MeV}\ , \quad (45.117)$$

where we note that μ is the matching scale from QCD to NRQCD: $m_Q v < \mu < m_Q$, while $m_Q v^2 < \mu' < m_Q v$ is the one from NRQCD to pNRQCD. One should notice that the correction is relatively large, and the perturbative series has a bad convergence. This convergence might be improved by working with a renormalon-free quark mass definition other than the pole mass, which will then facilitate the main motivation of the approach for exploring the dynamics of the quark-antiquark bound states using the perturbative approach.

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