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ABSOLUTE VALUES OF TOEPLITZ OPERATORS AND HANKEL OPERATORS

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ABSTRACT. Nehari's theorem for norms of bounded Hankel operators is revisited. Using it, the absolute values of Toeplitz operators are studied. This gives a theorem of Widom and Devinatz for invertible Toeplitz operators.

1. **Introduction.** Let U be the open unit disc in the complex plane and let ∂U be the boundary of U. If f is analytic in U and $\int_{-\pi}^{\pi} \log^+|f(re^{i\theta})|\,d\theta$ is bounded for $0 \leqq r < 1$, then $f(e^{i\theta})$, which we define to be $\lim_{r\to 1} f(re^{i\theta})$, exists almost everywhere on ∂U. If

$$\lim_{r\to 1} \int_{-\pi}^{\pi} \log^+|f(re^{i\theta})|\,d\theta = \int_{-\pi}^{\pi} \log^+|f(e^{i\theta})|\,d\theta,$$

then f is said to be of the class N_+. The set of all boundary functions in N_+ is again denoted by N_+. For $0 < p \leqq \infty$, the Hardy space H^p is defined as $N_+ \cap L^p$ where L^p denotes $L^p(d\theta)$. Put $H_0^p = \{f \in H^p : f(0) = 0\}$.

P denotes the orthogonal projection from L^2 to H^2. Let $\phi \in L^\infty$. We define the Hankel operator H_ϕ on H^2 by $H_\phi f = (I - P)(\phi f)$ and the Toeplitz operator T_ϕ on H^2 $T_\phi f = P(\phi f)$.

In Section 2 we show that if $H_\phi^* H_\phi \leqq T_v$ then there exists a function k in H^∞ such that $|\phi + k|^2 \leqq v$ a.e. If v is constant, this gives Nehari's theorem [7]. In Section 3 we give necessary and sufficient conditions such that there exists a nonzero function h in H^∞ such that $T_\phi^* T_\phi \geqq T_h^* T_h$. If h is constant, this gives a theorem of Widom and Devinatz (cf. [3, p. 187]). In Section 4 we study relations between ϕ and ψ when $H_\phi^* H_\phi = H_\psi^* H_\psi$ and $T_\phi^* T_\phi = T_\psi^* T_\psi$.

In this paper we also consider the above problems when the symbols ϕ are unbounded. For $\phi \in L^2$ we denote by M_ϕ the multiplication operator densely defined on L^2. For $\phi \in L^2$ let H_ϕ denote the Hankel operator densely defined on H^2 by $(H_\phi f, g) = (M_\phi f, g), f \in H^\infty$ and $g \in \bar{H}_0^\infty$ and T_ϕ denote the Toeplitz operator densely defined by $(T_\phi f, g) = (M_\phi f, g), f \in H^\infty$ and $g \in H^\infty$.

For any $\phi \in L^2$, we define $H_\phi^* H_\phi, T_\phi^* T_\phi$ and $T_{|\phi|^2}$ as follows: for any $f, g \in H^\infty$, $(H_\phi^* H_\phi f, g) = (H_\phi f, H_\phi g)$, $(T_\phi^* T_\phi f, g) = (T_\phi f, T_\phi g)$, and $(T_{|\phi|^2} f, g) = (\phi f, \phi g)$. The function $|\phi|^2$ is not necessarily in L^2 but we can define $T_{|\phi|^2}$ as a densely defined operator on H^2. This was pointed out by the referee. When ϕ is in L^∞, both H_ϕ and T_ϕ are bounded linear operators on H^2 which were defined previously. We use the following lemmas several times.

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LEMMA 1. *Suppose* $\phi \in H^\infty$. *If* ϕ *has at least two functions in* H^∞ *which give* $\gamma = \text{dist}(\phi, H^\infty)$, *then there exists a function* k *in* H^∞ *such that* $\log(\gamma - |\phi + k|) \in L^1$.

PROOF. By a theorem of Adamyan, Arov and Krein (cf. [5, p. 160]) we may assume that $\text{dist}(\phi, H^\infty) = 1$ and $|\phi| = 1$. By the hypothesis, there exists a nonzero $k \in H^\infty$ such that $|\phi + 2k| \leqq 1$ a.e. Since $|1/2\phi + k| \leqq 1/2, 1/4 + \text{Re}\,\bar{\phi}k + |k|^2 \leqq 1/4$ and hence $1 + 2\,\text{Re}\,\bar{\phi}k + |k|^2 + |k|^2 \leqq 1$. This implies that $|\phi + k|^2 + |k|^2 \leqq 1$. Thus $s = 1 - |\phi + k| \geqq 1 - \sqrt{1 - |k|^2} \geqq |k|^2/2$.

LEMMA 2. *For any* ϕ *in* L^2 $H_\phi^* H_\phi + T_\phi^* T_\phi = T_{|\phi|^2}$.

Lemma 2 is well known and obvious.

2. **Norms of Hankel operators.**

The following theorem is an extension of Nehari's [7] to densely defined Hankel operators. If B is a linear operator densely defined on H^2, that is, defined on H^∞, and $(Bf, f) \geqq 0$ for any f in H^∞, then we write $B \geqq 0$.

THEOREM 1. *Suppose* $\phi \in L^2$. *Then*

(1) *For any* k *in* H^2, $H_\phi^* H_\phi \leqq T_{|\phi + k|^2}$.

(2) *If* v *is a nonnegative function in* L^1 *and* $H_\phi^* H_\phi \leqq T_v$, *then there exists a function* k *in* H^2 *such that* $|\phi + k|^2 \leqq v$ *a.e. and hence* $H_\phi^* H_\phi \leqq T_{|\phi + k|^2} \leqq T_v$.

PROOF. (1) If $k \in H^2$ and $f \in H^\infty$ then for any $g \in \bar{H}_0^2$ $(H_\phi f, g) = ((\phi + k)f, g)$. Letting g range over the unit ball in $\bar{H}_0^2$, we get from this $\|H_\phi f\|_2 \leqq \|(\phi + k)f\|_2 = \sqrt{(T_{|\phi + k|^2} f, f)}$. Therefore, since $\|H_\phi f\|_2^2 = (H_\phi^* H_\phi f, f)$, we have $H_\phi^* H_\phi \leqq T_{|\phi + k|^2}$.

(2) For $f \in H^\infty$ and $g \in \bar{H}_0^\infty$

$$\left| \int_{-\pi}^{\pi} \phi f \bar{g} d\theta / 2\pi \right|^2 = |(H_\phi f, g)|^2 \leqq (H_\phi f, H_\phi f)(g, g)$$
$$\leqq (vf, f)(g, g)$$

because $H_\phi^* H_\phi \leqq T_v$. Let $\epsilon > 0$. Since $v + \epsilon$ is then $\geqq \epsilon$ and in L^2, there is an outer function h_ϵ in H^2 with $v + \epsilon = |h_\epsilon|^2$. Then, for $f \in H^\infty$ and $g \in \bar{H}_0^\infty$, we have by the previous relation

$$\left| \int_{-\pi}^{\pi} \phi h_\epsilon^{-1} (h_\epsilon f) \bar{g} d\theta / 2\pi \right|^2 \leqq \|h_\epsilon f\|_2 \|g\|_2.$$

By Nehari's theorem [8] there exists a function l in H^∞ such that $|\phi h_\epsilon^{-1} + l| \leqq 1$ and $|\phi + h_\epsilon l|^2 \leqq v + \epsilon$. By the standard limit process, we can find $k \in H^2$ such that $|\phi + k|^2 \leqq v$. ■

In the proof of (2) of Theorem 1, we can use a lifting theorem of Cotlar and Sadosky [2].

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COROLLARY 1. *If $\phi \in L^\infty$ has at least two functions in H^∞ which give* dist (ϕ, H^∞), *then there exists a function $\psi \in H^\infty$ such that $H_\phi^* H_\phi + H_\psi^* H_\psi \leqq$* dist (ϕ, H^∞).

PROOF. By hypothesis, there exists $k \in H^\infty$ such that $\psi = (\gamma^2 - |\phi + k|^2)^{1/2} > 0$ a.e., where $\gamma =$ dist (ϕ, H^∞). Hence by (1) of Theorem 1 $H_\phi^* H_\phi + H_\psi^* H_\psi \leqq T_{|\phi+k|^2} + T_{\gamma^2 - |\phi+k|^2} \leqq \gamma^2$. ∎

3. **Left invertible Toeplitz operators.** There exists a nonzero function h in H^2 such that $T_{|\phi|^2} \geqq T_h^* T_h$ if and only if $|\phi| \geqq |h|$. In general $T_{|\phi|^2} \geqq T_\phi^* T_\phi$ and so the following theorem is interesting.

THEOREM 2. *Suppose $\phi \in L^2$. Then the following are equivalent.*

(1) *There exists a nonzero outer function h in H^2 such that $T_\phi^* T_\phi \geqq T_h^* T_h$.*

(2) log $|\phi|$ *is integrable, and there exists a function l* in H^2 *such that $|\phi| \geqq |\phi + l|$* and $|\phi| \not\equiv |\phi + l|$.

(3) ϕ *has the form: $\phi = \phi_0 g$, where ϕ_0 is unimodular and g is an outer function in H^2. Moreover there exists a nonzero function k in H^∞ such that $\|\phi_0 + k\|_\infty \leqq 1$.*

PROOF. (1) ⇒ (2). By the hypothesis and Lemma 2,

$$H_\phi^* H_\phi \leqq T_{|\phi|^2 - |h|^2}.$$

By (2) of Theorem 1 there exists a nonzero function $l \in H^2$ such that $|\phi + l|^2 \leqq |\phi|^2 - |h|^2$. Hence log $|\phi| \in L^1$ and (2) is valid.

(2) ⇒ (3). Since log $|\phi|$ is integrable, ϕ has the form in (3): $\phi = \phi_0 g$. Since $|\phi| \geqq |\phi + l|$ and $|\phi| \neq |\phi + l|$ for some $l \in H^2$, l is nonzero and $1 \geqq |\phi_0 + g^{-1} l|$. Put $k = g^{-1} l$ then (3) follows.

(3) ⇒ (1). By the hypothesis and Lemma 1 there exists a function $e \in H^2$ such that log $(1 - |\phi_0 + e|^2) \in L^1$. Hence log $(|\phi|^2 - |\phi + ge|^2) \in L^1$. Since ge is in H^2, by (1) of Theorem 1 $H_\phi^* H_\phi \leqq T_{|\phi + ge|^2}$. By Lemma 2 $T_\phi^* T_\phi \geqq T_{|\phi|^2 - |\phi + ge|^2}$. Since log $(|\phi|^2 - |\phi + ge|^2) \in L^1$, there exists a nonzero function $h \in H^2$ such that $|\phi|^2 - |\phi + ge|^2 = |h|^2$ and hence $T_\phi^* T_\phi \geqq T_h^* T_h$. ∎

COROLLARY 2. *Suppose ϕ is a unimodular. Then there exists a nonzero function h in H^2 such that $T_\phi^* T_\phi \geqq T_h^* T_h$ if and only if ϕ has the form : $\phi = f/\bar{f}$ for some nonzero function f in H^2.*

PROOF. By a lemma of Koosis (cf. [5, pp. 161–163]), there exists a nonzero function k in H^∞ such that $\|\phi + k\|_\infty \leqq 1$ if and only if $\phi = f/\bar{f}$ for some nonzero function f in H^2. Hence Theorem 2 implies the corollary. ∎

The following is a corollary of the proof of Theorem 2 and generalizes a theorem of Devinatz and Widom ([3, p. 187] and [2]) to unbounded symbols.

COROLLARY 3. *Suppose $\phi \in L^2$. Then the following are equivalent.*

(1) *There exists a function h in H^2 such that $T_\phi^* T_\phi \geqq T_h^* T_h$ and h^{-1}* is in H^∞.

(2) *There exists a function l in H^2 and a positive constant ϵ such that $|\phi| \geqq \epsilon + |\phi + l|$ a.e.*

(3) *ϕ has the form: $\phi = \phi_0 g$ where ϕ_0 is a unimodular function and g is an outer function in H^2. Moreover g^{-1} is in H^∞ and* dist $(\phi_0, H^\infty) < 1$.

PROOF. (1) $\Rightarrow$ (2). There exists a positive constant ϵ_1 such that $T_h^* T_h \geqq T_{\epsilon_1}^* T_{\epsilon_1}$. As in the proof of Theorem 2, there exists a function $l \in H^2$ such that $|\phi + l|^2 \leqq |\phi|^2 - \epsilon_1^2$. This implies (2).

(2) $\Rightarrow$ (3). Since $|g| \geqq |\epsilon|, g^{-1}$ belongs to H^∞ and $1 \geqq |g^{-1}| + |\phi_0 + g^{-1} l|$. This implies dist $(\phi_0, H^\infty) < 1$.

(3) $\Rightarrow$ (1). There exists a function $e \in H^2$ and a positive constant ϵ such that $\epsilon + |\phi + eg|^2 \leqq |\phi|^2$. Hence $T_\phi^* T_\phi \geqq T_\epsilon^* T_\epsilon$ and this implies (1).

By a theorem of Douglas [4, Theorem 1], when $\phi \in L^\infty$, range $[T_\phi^*] \supset$ range $[T_h^*]$ if and only if $T_\phi^* T_\phi \geqq T_{\lambda h}^* T_{\lambda h}$ for some $\lambda > 0$. Hence Theorem 2 gives necessary and sufficient conditions for that range $[T_\phi^*]$ contains range $[T_h^*]$ for some nonzero function h in H^∞.

If $T_\phi^* T_\phi \geqq T_h^* T_h$ for some outer function h in H^2, then $T_\phi^* T_\phi \geqq T_u$ where $u = |h|^2$. From this view point, we wish to generalize Theorem 2.

THEOREM 3. *Suppose $\phi \in L^2$. There exists a nonnegative, nonzero function u in L^1 such that $T_\phi^* T_\phi \geqq T_u$ if and only if there exists a nonzero function h in H^2 such that $T_\phi^* T_\phi \geqq T_h^* T_h$.*

PROOF. By the remark above, it is sufficient to show the part of 'only if'. If $T_\phi^* T_\phi \geqq T_u$, by Lemma 2 $T_{|\phi|^2 - u} \geqq H_\phi^* H_\phi$. By (2) of Theorem 1 there exists a function g in H^2 such that $|\phi|^2 - u \geqq |\phi + g|^2$. Since u is nonzero, the g is nonzero and $2|\phi||g| \geqq 2Re\bar{\phi}(-g) \geqq |g|^2$. Hence log $|\phi|$ is integrable and so $\phi = \phi_0 k$ where ϕ_0 is a unimodular and k is an outer function in H^2. This implies $1 \geqq \|\phi_0 + k^{-1} g\|_\infty$. Now Theorem 2 proves the 'only if' part. ∎

4. **Absolute values of H_ϕ and T_ϕ.** In this section we are interested in the converse inequality: $T_\phi^* T_\phi \leqq T_h^* T_h$ where $\phi \in L^2$ and $h \in H^2$. Then we will consider when two Toeplitz operators have the same absolute values.

THEOREM 4. *Let ϕ be a function in L^2. There exists a nonzero function h in H^2 such that $T_\phi^* T_\phi \leqq T_h^* T_h$ if and only if $|\phi| \leqq |h|$.*

PROOF. If $|\phi| \leqq |h|$ then $T_{|\phi|^2} \leqq T_h^* T_h$ and hence $T_\phi^* T_\phi \leqq T_h^* T_h$ by Lemma 2. Conversely suppose $T_\phi^* T_\phi \leqq T_h^* T_h$. For any $\epsilon > 0$, there exists an outer function $h_\epsilon \in H^2$ such that $|h_\epsilon|^2 = |h|^2 + \epsilon$. Then for any $f \in H^\infty$ $\|P(\phi f)\|_2 \leqq \|h_\epsilon f\|_2$. If $g = h_\epsilon^{-1} f$ then $g \in H^\infty$ and hence $\|P(\phi h_\epsilon^{-1} f)\|_2 \leqq \|f\|_2$. Thus sup $\{|\int \phi h_\epsilon^{-1} f \bar{g} d\theta / 2\pi|; f \in H^\infty, g \in H^\infty, \|f\|_2 \leqq 1$ and $\|g\|_2 \leqq 1\} \leqq 1$. Put $A = \{f\bar{g};\ f \in H^\infty, g \in H^\infty,$

$\|f\|_2 \leqq 1$ and $\|g\|_2 \leqq 1\}$ and $B = \{s \in L^\infty ; \|s\|_1 \leqq 1$ and $\log|s| \in L^1\}$. If we show that A is dense in B and then A is dense in the unit ball of L^1, $\|\phi h_\epsilon^{-1}\|_\infty \leqq 1$ and $|\phi| \leqq |h_\epsilon|$. As $\epsilon \to 0$ $|\phi| \leqq |h|$. If $s \in B$ then there exists an outer function $g \in H^\infty$ such that $s = s_0 g\bar{g}$, $|s_0| = 1$ and $\|g\|_2 \leqq 1$. s_0 can be uniformly approximated by the set of quotients of inner functions [5, p. 217]. This implies that A is dense in B. ∎

THEOREM 5. *Suppose ϕ and ψ are in L^2.*

(1) *If $T_\phi^* T_\phi = T_\psi^* T_\psi$ then $|\phi| = |\psi|$.*

(2) *Suppose* $\log|\phi|$ *is integrable. If $T_\phi^* T_\phi = T_\psi^* T_\psi$, then $\phi = \phi_0 h$ and $\psi = \psi_0 h$, where h is an outer function in H^2, and both ϕ_0 and ψ_0 are unimodular. Moreover $T_{\phi_0}^* T_{\phi_0} = T_{\psi_0}^* T_{\psi_0}$.*

(3) *Suppose ϕ and ψ are unimodular. If $T_\phi^* T_\phi = T_\psi^* T_\psi$ then for any g in H^∞ there exists a function f in H^∞ such that $|\phi + g| \geqq |\psi + f|$.*

PROOF. (1) For any $\epsilon > 0$ there exists an outer function $h_\epsilon \in H^2$ such that $|\psi| + \epsilon = |h_\epsilon|$. By Theorem 4 $T_\psi^* T_\psi \leqq T_{h_\epsilon}^* T_{h_\epsilon}$ and so $T_\phi^* T_\phi \leqq T_{h_\epsilon}^* T_{h_\epsilon}$. Again by Theorem 4 $|\phi| \leqq |h_\epsilon| = |\psi| + \epsilon$ and $|\phi| \leqq |\psi|$ because ϵ is arbitrary. Thus $|\phi| = |\psi|$.

(2) If $\log|\phi| \in L^1$, by (1) ϕ and ψ have the forms: $\phi = \phi_0 h$ and $\psi = \psi_0 h$. Hence $T_h^*(T_{\phi_0}^* T_{\phi_0} - T_{\psi_0}^* T_{\psi_0}) T_h = 0$. Since T_h has the dense range, $T_{\phi_0}^* T_{\phi_0} = T_{\psi_0}^* T_{\psi_0}$. (3) Since ϕ and ψ are unimodular, by Lemma 2 $T_\phi^* T_\phi = T_\psi^* T_\psi$ implies $H_\phi^* H_\phi = H_\psi^* H_\psi$. Theorem 1 implies (3). ∎

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