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FIXED POINT THEOREMS IN H-SPACES AND EQUILIBRIUM POINTS OF ABSTRACT ECONOMIES

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Abstract

Some fixed point theorems on H-spaces are presented. These theorems are then applied to generalize a theorem of Fan concerning sets with convex sections to H-spaces and to prove the existence of equilibrium points of abstract economies in which the commodity space is an H-space.

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1. Introduction

In a recent paper [10] we have a fixed point theorem which contains our earlier fixed point theorem in [11] as a special case and used this fixed point theorem to prove the existence of equilibrium points of abstract economies. The object of this paper is to extend this theorem to the more general situation of H-spaces. As in [10] we have applied our theorem to obtain a generalized version of a theorem of Fan [3, Theorem 16] concerning sets with convex sections. In the final section we consider the abstract economy in which the commodity space is an H-space and prove by means of our fixed point theorem the existence of equilibrium points of such economies.

To begin with we explain the notion of an H-space introduced by Horvath [5], [6] and [7] and related concepts on H-spaces.

Let X be a topological space and $\mathscr{F}(X)$ the family of all nonempty finite subsets of X. Let $\{F_A\}$ be a family of nonempty contractible subsets of X indexed by $A \in \mathscr{F}(X)$ such that $F_A \subset F'_A$ whenever $A \subset A'$. The pair $(X, \{F_A\})$ is called an H-space. Given an H-space $(X, \{F_A\})$, a nonempty set D is called H-convex if $F_A \subset D$ for each nonempty finite subset A of D. For a nonempty subset K of an H-space we define the H-convex hull of K, denoted by H-co K as

$$\text{H-co}\, K = \bigcap \{D \subset X : D \text{ is H-convex and } D \supset K\}.$$

Then H-co K is H-convex (see [12]) and is the smallest H-convex set containing K. It is also known [12, Lemma 1] that

$$\text{H-co}\, K = \bigcup \{\text{H-co}\, A : A \text{ is a finite subset of } K\}.$$

The following lemma is also proved in [12].

LEMMA 1.1. *The product of any number (finite or infinite) of H-spaces is an H-space and the product of H-convex subsets is H-convex.*

PROOF. Let $\{(X_\alpha, \{F^\alpha_{A_\alpha}\}) : \alpha \in I\}$ be a family of H-spaces where I is a finite or an infinite index set. For each nonempty finite subset A of $X = \prod\{X_\alpha : \alpha \in I\}$, we set $F_A = \prod\{F^\alpha_{A_\alpha} : \alpha \in I\}$ where for each $\alpha \in I$, $A_\alpha = P_\alpha(A)$ and $P_\alpha : X \to X_\alpha$ is the projection of X onto X_α. Now it is easy to see that $(X, \{F_A\})$ is an H-space (for details see [12].

Following Debreu [2] and Shafer and Sonnenschein [9], we will describe an abstract-economy or generalized qualitative game by $\mathscr{E} = \{(X_\alpha, \{F^\alpha_{A_\alpha}\}), A_\alpha, U_\alpha : \alpha \in I\}$, where I is a finite or an infinite set of agents or players; for each $\alpha \in I$, $(X_\alpha, \{F^\alpha_{A_\alpha}\})$ is an H-space, the commodity space; $A_\alpha : X = \prod\{X_\alpha : \alpha \in I\} \to 2^{X_\alpha}$ is the constraint correspondence (set valued mapping) and $U_\alpha : X \to \mathbb{R}$ is the utility or pay off function. We denote the product $\prod\{X_\beta : \beta \subset I \text{ and } \beta \neq \alpha\}$ by $X_{-\alpha}$ and a generic element of $X_{-\alpha}$ by $x_{-\alpha}$. An abstract economy instead of being given by $\{(X_\alpha, \{F^\alpha_{A_\alpha}\}), A_\alpha, P_\alpha : \alpha \in I\}$ may be given by $\mathscr{E} = \{(X_\alpha, \{F^\alpha_{A_\alpha}\}), A_\alpha, P_\alpha : \alpha \in I\}$ where $P_\alpha : X \to 2^{X_\alpha}$ is the preference correspondence. The relationship between the utility function U_α and the preference correspondence P_α can be exhibited by the definition:

$$P_\alpha(x) = \{y_\alpha \in X_\alpha : U_\alpha([y_\alpha, x_{-\alpha}]) > U_\alpha(x)\}$$

where for each $\alpha \in I$, $x_{-\alpha}$ is the projection of x onto $X_{-\alpha}$ and $[y_\alpha, x_{-\alpha}]$ is the point of X whose αth coordinate is y_α.

In the case of the economy being given by $\mathscr{E} = \{(X_\alpha, \{F^\alpha_{A_\alpha}\}), A_\alpha, U_\alpha : \alpha \in I\}$, a point $\overline{x} \in X$ is called an equilibrium point or a generalized Nash equilibrium point of the economy $\mathscr{E}$ if

$$U_\alpha(\overline{x}) = U_\alpha[\overline{x}_\alpha, \overline{x}_{-\alpha}] = \sup_{z_\alpha \in A_\alpha(\overline{x})} U_\alpha[z_\alpha, \overline{x}_{-\alpha}]$$

for each $\alpha \in I$ where $\overline{x}_\alpha$ and $\overline{x}_{-\alpha}$ are respectively projections of $\overline{x}$ onto X_α and $X_{-\alpha}$. In this case the equilibrium point is the natural extension of the equilibrium point introduced by Nash [8]. Now let

$$\mathscr{E} = \{(X_\alpha, \{F^\alpha_{A_\alpha}\}), A_\alpha, U_\alpha : \alpha \in I\}$$

be an abstract economy and let for each $\alpha \in I$, P_α be obtained as above. Then it can be easily checked that a point $\overline{x} \in X$ is an equilibrium point of $\mathscr{E}$ if and only if for each $\alpha \in I$, $P_\alpha(\overline{x}) \cap A_\alpha(\overline{x}) = \varnothing$ and $\overline{x}_\alpha \in A_\alpha(\overline{x})$. Thus given an abstract economy $\mathscr{E} = \{(X_\alpha, \{F^\alpha_{A_\alpha}\}), P_\alpha, A_\alpha : \alpha \in I\}$ we can define an equilibrium point of $\mathscr{E}$ as follows: a point $\overline{x} \in X$ is said to be an equilibrium point of the abstract economy $\mathscr{E} = \{X_\alpha, P_\alpha, A_\alpha : \alpha \in I\}$ if for each $\alpha \in I$, $P_\alpha(\overline{x}) \cap A_\alpha(\overline{x}) = \varnothing$ and $\overline{x}_\alpha \in A_\alpha(\overline{x})$ where $\overline{x}_\alpha$ is the projection of $\overline{x}$ onto X_α.

For more references on this topic we refer to [1], [2], [4], [9] and [10].

2. Fixed point theorems in H-spaces

The following theorem, the proof of which is contained in the proof of Theorem 1 of Horvath [7], will be the basic tool for our purpose.

THEOREM 2.1. *Let X be a topological space such that for every subset J of $\{0, 1, \dots, n\}$ there is a nonempty contractible subset F_J of X having the property that $F_J \subset F'_j$ whenever $J \subset J'$. Then there is a continuous mapping $g: \Delta_n \to X$ such that $g(\Delta_J) \subset F_J$ for each subset J of $\{0, 1, \dots, n\}$, where Δ_n is the standard n-dimensional simplex with vertices $e_0, e_1, \dots, e_n$ and for any subset J of $\{0, 1, \dots, n\}$ Δ_J $(\subset \Delta_n)$ is the convex hull of the vertices $\{e_j : j \in J\}$.*

THEOREM 2.2. *Let $\mathfrak{X}$ be a compact topological space and $(Y, \{F_A\})$ an H-space. Let $T: X \to 2^Y$ be a set valued mapping such that*

(i) *for each $x \in X$, $T(x)$ is a nonempty H-convex subset of Y,*

(ii) *for each $y \in Y$, $T^{-1}(y) = \{x \in X : y \in T(x)\}$ contains an open subset O_y of X (O_y may be empty for some y),*

(iii) $\bigcup\{O_y : y \in Y\} = X$.

Then there is a continuous selection $f: X \to Y$ of T such that $f = g \circ \Psi$ where $g: \Delta_n \to Y$ and $\Psi: X \to \Delta_n$ are continuous mappings and n is some positive integer.

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PROOF. Since X is compact, by (ii) and (iii) there is a finite subset $\{y_0, y_1, \dots, y_n\}$ of Y such that $\bigcup\{O_{y_i}: i = 0, 1, \dots, n\} = X$. For each nonempty subset J of $\{0, 1, \dots, n\}$, let $F_J = F_A$, where $A = \{y_i: i \in J\}$. Then clearly $F_J \subset F_{J'}$ whenever $J \subset J'$. Hence by Theorem 2.1 there is a continuous mapping $g: \Delta_n \to Y$ such that $g(\Delta_J) \subset F_J$ for each subset J of $\{0, 1, \dots, n\}$. Now let $\{\Psi_0, \Psi_1, \dots, \Psi_n\}$ be a partition of unity corresponding to the finite covering $\{O_{y_0}, O_{y_1}, \dots, O_{y_n}\}$ of X, that is, Ψ_i is a real valued continuous function defined on X such that Ψ_i vanishes outside O_{y_i}, $0 \le \Psi_i(x) \le 1$ and

$$\sum_{i=0}^{n} \Psi_i(x) = 1$$

for each $x \in X$. We can thus define a mapping $\Psi: X \to \Delta_n$ by

$$\Psi(x) = \sum_{i=0}^{n} \Psi_i(x) e_i,$$

$x \in X$ with $\Psi(x) \in \Delta_{J(x)}$ where $J(x) \subset \{0, 1, \dots, n\}$ is defined by

$$i \in J(x) \Leftrightarrow \Psi_i(x) \neq 0 \Leftrightarrow x \in O_{y_i} \subset T^{-1}(y_i) \Leftrightarrow y_i \in T(x).$$

Hence by H-convexity of $T(x)$, $F_{J(x)} \subset T(x)$ for each $x \in X$. Now the mapping $f = g \circ \Psi$ has the property that for each $x \in X$, $f(x) = g(\Psi(x)) \in g(\Delta_{J(x)}) \subset F_{J(x)} \subset T(x)$. In other words, f is a continuous selection of T.

COROLLARY 2.1. *Let X be a compact topological space and $(Y, \{F_A\})$ an H-space. Let $S, T: X \to 2^Y$ be two set valued mappings such that*

(i) *for each $x \in X$, $S(x) \neq \varnothing$ and for each $y \in Y$, $S^{-1}(y)$ is open,*

(ii) *for each $x \in X$, $S(x) \subset T(x)$,*

(iii) *for each $x \in X$, $T(x)$ is H-convex.*

Then there is a continuous selection $f: X \to Y$ of T.

PROOF. For each $y \in Y$, set $O_y = S^{-1}(y)$. Then since for each $x \in X$, $S(x) \subset T(x)$, it follows that $O_y = S^{-1}(y) \subset T^{-1}(y)$ for each $y \in Y$. Since for each $x \in X$, $S(x) \neq \varnothing$, it follows that $\bigcup\{O_y: y \in Y\} = X$. Hence the conclusion follows from Theorem 2.2.

COROLLARY 2.2. *Let X and Y be as in Theorem 2.2 and $T: X \to 2^Y$ be a set valued mapping satisfying the conditions of Theorem 2.2. Then for any continuous mapping $h: Y \to X$ there exists a point $y_0 \in Y$ such that $y_0 \in T(h(y_0))$.*

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Proof. By Theorem 2.2, there is a continuous mapping $f = g \circ \Psi$ where $g: \Delta_n \to Y$ and $\Psi: X \to \Delta_n$. Hence the continuous mapping $\Psi \circ h \circ g: \Delta_n \to \Delta_n$ has a fixed point q_0 by the Brouwer fixed point theorem. Let $y_0 = g(q_0)$. Then $g \circ \Psi \circ h(y_0) = y_0$. But as $f = g \circ \Psi$ is a selection of T, $y_0 \in T(h(y_0))$.

Corollary 2.3. *Let $(X, \{F_A\})$ be a compact H-space and $T: X \to 2^X$ be a set valued mapping such that*

(i) *for each $x \in X$, $T(x)$ is a nonempty H-convex subset,*

(ii) *for each $y \in Y$, $T^{-1}(y) = \{x \in X: y \in T(x)\}$ contains an open set O_y (O_y may be empty for some y),*

(iii) $\bigcup\{O_y: y \in X\} = X$.

Then there is a point $x_0 \in T(x_0)$.

Proof. If we take $h = I$, the identity mapping on X in Corollary 2.2, we obtain the corollary.

Theorem 2.3. *Let $\{(X_\alpha, \{F_{\alpha_\alpha}\}: \alpha \in I\}$ be a family of compact H-spaces, where I is a finite or infinite index set. Let $X = \prod\{X_\alpha: \alpha \in I\}$. For each $\alpha \in I$, let $T_\alpha: X \to 2^{X_\alpha}$ be a set valued mapping such that*

(i) *for each $x \in X$, $T_\alpha(x)$ is a nonempty H-convex subset of X_α,*

(ii) *for each $x_\alpha \in X_\alpha$, $T_\alpha^{-1}(x_\alpha) = \{y \in X: x_\alpha \in T_\alpha(y)\}$ contains an open subset O_{x_α} of X such that $\bigcup\{O_{x_\alpha}: x_\alpha \in X_\alpha\} = X$ (O_{x_α} may be empty for some x_α). Then there is a point $x \in X$ such that $x \in T(x) = \prod\{T_\alpha(x): \alpha \in I\}$, i.e. $x_\alpha \in T_\alpha(x)$ for each $\alpha \in I$, where x_α is the projection of x into X_α for each $\alpha \in I$.*

Proof. For each $\alpha \in I$, there are by Theorem 2.2 continuous mappings $g_\alpha: \Delta_{n(\alpha)} \to X_\alpha$ and $\Psi_\alpha: X \to \Delta_{n(\alpha)}$ such that the mapping $f_\alpha = g_\alpha \circ \Psi_\alpha: X \to X_\alpha$ is a continuous selection of T_α. Now let $S = \prod\{\Delta_{n(\alpha)}: \alpha \in I\}$. For each $\alpha \in I$, let F_α be the linear hull of the set $\{e_0, e_1, \dots, e_{n(\alpha)}\}$. Then for each $\alpha \in I$, F_α is a locally convex Hausdorff topological vector space as it is finite dimensional. The convex hull $\Delta_{n(\alpha)}$ of the set $\{e_0, e_1, \dots, e_{n(\alpha)}\}$ is a compact convex subset of F_α for each $\alpha \in I$. Thus $S = \prod\{\Delta_{n(\alpha)}: \alpha \in I\}$ is a compact convex subset of the locally convex topological vector space $F = \prod\{F_\alpha: \alpha \in I\}$. Now we define the continuous mappings $g: S \to X$ and $\Psi: X \to S$ by $g(t) = \prod\{g_\alpha(P_\alpha(t)): \alpha \in I\}$, $t \in S$ where $P_\alpha: S \to \Delta_{n(\alpha)}$ is the continuous projection of S onto $\Delta_{n(\alpha)}$ for each $\alpha \in I$ and $\Psi(x) = \prod\{\Psi_\alpha(x): \alpha \in I\}$, $x \in X$. Let I_X be the identity on X. Then by the Tychonoff fixed point theorem the continuous mapping $\Psi \circ I_X \circ g: S \to S$ has a fixed point $q \in S$, i.e., $\Psi \circ I_X g(q) = q$. Let $g(q) = y$. Then

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$g \circ \Psi \circ I_X(y) = y$, i.e. $g \circ \Psi(y) = y \Rightarrow g_\alpha \circ \Psi_\alpha(y) = y_\alpha$ for each $\alpha \in I$. (To see this let $\Psi(y) = t$, then $\Psi_\alpha(y) = P_\alpha(t))$. Hence $y_\alpha = g_\alpha \circ \Psi_\alpha(y) = f_\alpha(y) \subset T_\alpha(y)$ for each $\alpha \in I$.

3. Sets with H-convex sections

Our next theorem generalizes a theorem of Fan [3, Theorem 16] and partly a theorem of the author [13] to the case of an H-space.

THEOREM 3.1. *Let $\{(X_\alpha, \{F_{A_\alpha}\}): \alpha \in I\}$ be a family of compact H-spaces, where I is a finite or an infinite index set. Put $X = \prod\{X_\alpha: \alpha \in I\}$ and $X_{-\alpha} = \prod\{X_\beta: \beta \in I$ and $\beta \neq \alpha\}$. Let $x_{-\alpha}$ denote a point of $X_{-\alpha}$. Let $\{G_\alpha: \alpha \in I\}$ and $\{H_\alpha: \alpha \in I\}$ be two families of subsets of X having the following properties:*

(a) *for each $\alpha \in I$ and each $x_\alpha \in X_\alpha$, the set*

$$H_\alpha(x_\alpha) = \{x_{-\alpha} \in X_{-\alpha}: [x_\alpha, x_{-\alpha}] \in H_\alpha\}$$

is open in $X_{-\alpha}$;

(b) *for each $\alpha \in I$ and each $x_{-\alpha} \in X_{-\alpha}$, the set*

$$H_\alpha(x_{-\alpha}) = \{x_\alpha \in \alpha: [x_\alpha, x_{-\alpha}] \in H_\alpha\}$$

is nonempty and the set

$$G_\alpha(x_{-\alpha}) = \{x_\alpha \in X_\alpha: [x_\alpha, x_{-\alpha}] \in G_\alpha\}$$

contains the convex hull of $H_\alpha(x_{-\alpha})$.

Then $\bigcap\{G_\alpha: \alpha \in I\} \neq \varnothing$.

PROOF. For each $\alpha \in I$ and each $x = \{x_\alpha\} \in X$ we set $T_\alpha(x) = \text{H-co}\, H_\alpha(x_{-\alpha})$ where x_α and $x_{-\alpha}$ are projections of x into X_α and $X_{-\alpha}$. Then for each $\alpha \in I$, $T_\alpha: X \to 2^{X_\alpha}$ is a set valued mapping satisfying (i) of Theorem 2.3 (by virtue of the condition (b)). Now for each $\alpha \in I$ and each $y_\alpha \in X_\alpha$,

$$\begin{aligned} T_\alpha^{-1}(y_\alpha) &= \{x = \{x_\alpha\} \in X: y_\alpha \in T_\alpha(x)\} = \{x = \{x_\alpha\} \in X: y_\alpha \in \text{H-co}\, H_\alpha(x_{-\alpha})\} \\ &\supset \{x = \{x_\alpha\} \in X: y_\alpha \in H_\alpha(x_{-\alpha})\} \\ &= \{x = \{x_\alpha\} \in X: x_{-\alpha} \in H_\alpha(y_\alpha)\} = X_\alpha \times H_\alpha(y_\alpha) = O_{y_\alpha}, \end{aligned}$$

say, which is by (a) an open subset of X. Finally let $x = \{x_\alpha\} \in X$ be arbitrary. Then by the first part of the condition (b) there is $y_\alpha \in H_\alpha(x_{-\alpha})$, i.e. $x \in O_{y_\alpha}$. Thus for each $\alpha \in I$, $\bigcup\{O_{y_\alpha}: y_\alpha \in X_\alpha\} = X$ and hence the condition (ii) of Theorem 2.3 is also fulfilled. Hence by Theorem 2.3 there exists a point $x = \{x_\alpha\} \in X$ such that for each $\alpha \in I$, $x_\alpha \in T_\alpha(x) = \text{H-co}\, H_\alpha(x_{-\alpha}) \subset G_\alpha(x_{-\alpha})$, i.e. $x = [x_\alpha, x_{-\alpha}] \in G_\alpha$ for each $\alpha \in I$, i.e. $x \in \bigcap\{G_\alpha : \alpha \in I\}$.

4. Existence of equilibrium point of an abstract economy

In the present section we apply our Theorem 2.3 to prove the existence of an equilibrium point of an abstract economy $\mathscr{E} = \{X_\alpha, P_\alpha, A_\alpha : \alpha \in I\}$ as described in Section 1. We recollect that $X = \prod\{X_\alpha : \alpha \in I\}$ and X_α is an H-space for each $\alpha \in I$.

THEOREM 4.1. *Let $\mathscr{E} = \{X_\alpha, P_\alpha, A_\alpha : \alpha \in I\}$ be an abstract economy such that for each $\alpha \in I$, the following conditions hold*:

(i) *X_α is compact*;

(ii) *for each $x_\alpha \in X$, $A_\alpha(x)$ is nonempty and H-convex valued*;

(iii) *for each $x_\alpha \in X_\alpha$, $\{P_\alpha^{-1}(x_\alpha) \cup F_\alpha\} \cap A_\alpha^{-1}(x_\alpha)$ contains an open subset O_{x_α} of X such that $\bigcup\{O_{x_\alpha} : \alpha \in I\} = X$, where $F_\alpha = \{x \in X : P_\alpha(x) \cap A_\alpha(x) = \varnothing\}$*;

(iv) *for each $x = \{x_\alpha\} \in X$, $x_\alpha \notin$ H-co $P_\alpha(x)$ for each $\alpha \in I$.*

Then $\mathscr{E}$ has an equilibrium point.

(Note that each X_α being an H-space is H-convex.)

PROOF. For each $\alpha \in I$, let $G_\alpha = \{x \in X : P_\alpha(x) \cap A_\alpha(x)\} \neq \varnothing$; i.e. $G_\alpha = F_\alpha^c$ and for each $x \in X$, let $I(x) = \{\alpha \in I : P_\alpha(x) \cap A_\alpha(x) \neq \varnothing\}$. For each $\alpha \in I$, we now define the set valued mapping $T_\alpha : X \to 2^{X_\alpha}$ by

$$T_\alpha(x) = \begin{cases} (\text{H-co}\, P_\alpha(x)) \cap A_\alpha(x), & \text{for } \alpha \in I(x), \\ A_\alpha(x), & \text{for } \alpha \notin I(x). \end{cases}$$

Then for each $x \in X$, $T_\alpha(x)$ is nonempty and H-convex valued and for each $y_\alpha \in X_\alpha$, it can be easily seen that

$$\begin{aligned} T_\alpha^{-1}(y_\alpha) &= [\{(\text{H-co}\, P_\alpha)^{-1}(y_\alpha) \cap A_\alpha^{-1}(y_\alpha)\} \cap G_\alpha] \cup [A_\alpha^{-1}(y_\alpha) \cap F_\alpha] \\ &\supset [\{P_\alpha^{-1}(y_\alpha) \cap A_\alpha^{-1}(y_\alpha)\} \cap G_\alpha] \cup [A_\alpha^{-1}(y_\alpha) \cap F_\alpha] \\ &= [P_\alpha^{-1}(y_\alpha) \cap A_\alpha^{-1}(y_\alpha)] \cup [A_\alpha^{-1}(y_\alpha) \cap F_\alpha] \\ &= [P_\alpha^{-1}(y_\alpha) \cup F_\alpha] \cap A_\alpha^{-1}(y_\alpha) = O_{y_\alpha}, \end{aligned}$$

say.

Thus in view of the condition (iii), for each $y_\alpha \in X_\alpha$, $T_\alpha^{-1}(y_\alpha)$ contains an open subset O_{y_α} such that $\bigcup\{O_{y_\alpha} : y_\alpha \in X_\alpha\} = X$. Hence by Theorem

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2.3 there exists a point $x = \{x_\alpha\} \in X$ such that $x_\alpha \in T_\alpha(x)$ for each $\alpha \in I$. By condition (iv) and the definition of T_α, it easily follows that x is an equilibrium point of $\mathscr{E}$.

REMARK 4.1. Condition (iii) of the theorem can be replaced by the stronger condition:

(iii)$'$ for each $x_\alpha \in X_\alpha$, $\{P_\alpha^{-1}(x_\alpha) \cup F_\alpha\} \cap A_\alpha^{-1}(x_\alpha)$ is relatively open in X (for example, see [10, Remark 3.1]).

COROLLARY 4.1. *Let $\mathscr{E} = \{X_\alpha, P_\alpha, A_\alpha : \alpha \in I\}$ be an abstract economy such that for each $\alpha \in I$ the following conditions hold:*

(i) *X_α is compact;*

(ii) *for $x \in X$, $A_\alpha(x)$ is nonempty and H-convex valued;*

(iii) *the set $G_\alpha = \{x \in X : P_\alpha(x) \cap A_\alpha(x) \neq \varnothing\}$ is a closed subset of X;*

(iv) *for each $y_\alpha \in X_\alpha$, $P_\alpha^{-1}(y_\alpha)$ is a relatively open subset in G_α and $A_\alpha^{-1}(y_\alpha)$ is an open subset of X;*

(v) *for each $x = \{x_\alpha\} \in X$, $x_\alpha \notin \text{H-co}\, P_\alpha(x)$ for each $\alpha \in I$.*

Then there is an equilibrium point of the economy $\mathscr{E}$.

PROOF. By condition (iv), $P_\alpha(y_\alpha) = G_\alpha \cap U_\alpha$ for some open subset U_α of X. Thus $P_\alpha^{-1}(y_\alpha) \cup F_\alpha^{-} = (G_\alpha \cap U_\alpha) \cup F_\alpha = X \cap (U_\alpha \cup F_\alpha)$. Hence

$$\{P_\alpha^{-1}(y_\alpha) \cup F_\alpha\} \cap A_\alpha^{-1}(y_\alpha) = (U_\alpha \cup F_\alpha) \cap A_\alpha^{-1}(y_\alpha)$$

is an open subset of X. Now the corollary follows from Theorem 4.1 and Remark 4.1.

We now conclude by the following remark.

REMARK 4.2. As in [10] we can define a qualitative game by $\{X_\alpha, P_\alpha : \alpha \in I\}$ where index set I is finite or infinite and for each $\alpha \in I$, X_α is an H-space and $P_\alpha : X = \prod\{X_\alpha : \alpha \in I\} \to 2^{X_\alpha}$ is a set valued mapping. If convexity is replaced by H-convexity and convex hull is replaced by H-convex hull, then results similar to [10, Theorem 3.2 and Corollary 3.2] concerning the maximal element of a qualitative game can similarly be stated and proved.

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