GENERALISATION OF AN EMBEDDING THEOREM FOR GROUPS

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1. Introduction. Let G be a given group and A, B be two subgroups of G which may or may not coincide. A homomorphism μ which maps A onto B is called a *partial endomorphism* of G. When A coincides with G then we call μ a *total endomorphism* or as it is usually called an *endomorphism* of G. If μ^* is a partial (or total) endomorphism of a supergroup $G^* \supseteq G$, then we say that μ^* *extends*, or *continues*, μ when μ^* is defined for at least all the elements $a \in A$ and moreover $a\mu = a\mu^*$ for all $a \in A$. If the partial endomorphism μ is an isomorphic mapping then we speak of a *partial automorphism* of G.

It is known [3] that any number of partial automorphisms $\mu(\alpha)$ of a group G can be simultaneously extended to inner automorphisms of one and the same supergroup.

When $\mu(\alpha)$ are partial endomorphisms (no longer necessarily isomorphic) then necessary and sufficient conditions for their simultaneous extension to total endomorphisms of a supergroup $G^* \supseteq G$ were derived in [1]. These conditions are in fact a generalisation of a result obtained by B. H. Neumann and Hanna Neumann [4] in the case of extending a single partial endomorphism.

In a recent paper [2] the author has derived necessary and sufficient conditions for a partial endomorphism μ of a group G to be extendable to a total endomorphism μ^* of a supergroup G^* such that μ^* acts as an isomorphism on $G^*(\mu^*)^m$, for some given positive integer m.

In the following work we take a group G and a sequence $\mu(\alpha)$ of partial endomorphisms of G, where α ranges over some well-ordered set Σ, and using transfinite induction we generalise the conditions of [2] to give necessary and sufficient conditions for the simultaneous extension of the $\mu(\alpha)$ to total endomorphisms $\mu^*(\alpha)$ of one and the same group $G^* \supseteq G$ such that $\mu^*(\alpha)$ acts as an isomorphism on $G^*\{\mu^*(\alpha)\}^{n(\alpha)}$, where, for each $\alpha \in \Sigma$, $n(\alpha)$ is a given positive integer.

2. Derivation of the necessary conditions. Let $\mu(\alpha)$, where α ranges over a well-ordered set Σ whose ordinal is σ, be a partial endomorphism of the group G mapping A_α onto B_α, where A_α and B_α are subgroups of G. To find the necessary conditions for $\mu(\alpha)$ to be simultaneously extendable to total endomorphisms $\mu^*(\alpha)$ of one and the same supergroup $G^* \supseteq G$ such that $\mu^*(\alpha)$ acts as an isomorphism on $G^*\{\mu^*(\alpha)\}^{n(\alpha)}$, where $n(\alpha)$, for each $\alpha \in \Sigma$, is a positive integer, we assume that the extension is already established, i.e., we assume that G^*, $\mu^*(\alpha)$ exist with the required properties.

Let Ω^* be the semigroup generated by $\mu^*(\alpha)$; then any $\omega^* \in \Omega^*$ is an endomorphism of G^*. Denote the kernel of $\mu(\alpha)$ by K_α and that of ω^* by $K(\omega^*)$.

The canonic mapping of G^* onto $G^*/K[\mu^*(\alpha)]$ must induce the canonic mapping of A_α onto A_α/K_α. But it also induces the canonic mapping of A_α onto $A_\alpha/[K\{\mu^*(\alpha)\} \cap A_\alpha]$; thus

$$K_\alpha = K[\mu^*(\alpha)] \cap A_\alpha = \text{kernel of } \mu(\alpha).$$

Let Ω be the semigroup freely generated by a set of elements whose ordinal is σ which we denote, conveniently but without ambiguity, by $\mu(\alpha)$. To every word $\omega = \omega[\mu(\alpha)]$ in Ω there corresponds an element $\omega^* = \omega[\mu^*(\alpha)]$ in Ω^*. For every word $\omega \in \Omega$ and its corresponding

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element $\omega^* \in \Omega^*$ we put

$$L(\omega) = K(\omega^*) \cap G.$$

As in [1] we can show that $K(\omega^*)$ $(\omega^* \in \Omega^*)$ are normal subgroups of G^* such that

$$K(\omega^*) \subseteq K(\omega^* \omega_1^*),$$

for any $\omega^*, \omega_1^* \in \Omega^*$, and thus $L(\omega)$ $(\omega \in \Omega)$ are normal subgroups of G for which we can also prove as in [1] that

$$L(\omega) \subseteq L(\omega\omega_1) \quad \text{for any } \omega, \omega_1 \in \Omega,$$
$$L[\mu(\alpha)] \cap A_\alpha \quad \text{is the kernel of } \mu(\alpha),$$
$$[L\{\mu(\alpha)\omega\} \cap A_\alpha]\mu(\alpha) = L(\omega) \cap B_\alpha.$$

Moreover, we prove the following lemma.

Lemma 1. *The normal subgroups $L(\omega)$ satisfy the relations*

$$L[\{\mu(\alpha)\}^{n(\alpha)}] = L[\{\mu(\alpha)\}^{n(\alpha)+i}],$$

for any $\alpha \in \Sigma$ and any integer $i > 0$.

Proof. We have

$$K[\{\mu^*(\alpha)\}^{n(\alpha)}] \subseteq K[\{\mu^*(\alpha)\}^{n(\alpha)+1}]. \qquad \text{(i)}$$

If $x \in K[\{\mu^*(\alpha)\}^{n(\alpha)+1}]$, then

$$x\{\mu^*(\alpha)\}^{n(\alpha)+1} = [x\{\mu^*(\alpha)\}^{n(\alpha)}]\mu^*(\alpha) = e,$$

where e is the unit element of G^*. Since $\mu^*(\alpha)$ is an isomorphism on $G^*\{\mu^*(\alpha)\}^{n(\alpha)}$, then

$$x\{\mu^*(\alpha)\}^{n(\alpha)} = e,$$

i.e.,

$$x \in K[\{\mu^*(\alpha)\}^{n(\alpha)}],$$

and thus

$$K[\{\mu^*(\alpha)\}^{n(\alpha)+1}] \subseteq K[\{\mu^*(\alpha)\}^{n(\alpha)}]. \qquad \text{(ii)}$$

(i) and (ii) together give

$$K[\{\mu^*(\alpha)\}^{n(\alpha)}] = K[\{\mu^*(\alpha)\}^{n(\alpha)+1}].$$

Intersecting both sides with G, we get

$$L[\{\mu(\alpha)\}^{n(\alpha)}] = L[\{\mu(\alpha)\}^{n(\alpha)+1}],$$

which proves the lemma when $i = 1$. The proof for $i > 1$ is the same. Thus we have the following theorem.

Theorem 1. *Let $\mu(\alpha)$, where α ranges over a well-ordered set Σ whose ordinal is σ, be a partial endomorphism of a group G mapping the subgroup $A_\alpha \subseteq G$ onto another subgroup $B_\alpha \subseteq G$. Then the necessary conditions for the existence of a supergroup $G^* \supseteq G$ with total endomorphisms $\mu^*(\alpha)$ extending $\mu(\alpha)$ such that $\mu^*(\alpha)$ is an isomorphism on $G^*\{\mu^*(\alpha)\}^{n(\alpha)}$, where $n(\alpha)$, for each $\alpha \in \Sigma$, is a positive integer, is that if we denote by Ω the semigroup freely generated by the $\mu(\alpha)$, then for every $\omega \in \Omega$ there exists a normal subgroup $L(\omega)$ of G such that*

$$L(\omega) \subseteq L(\omega\omega_1) \quad \textit{for all } \omega, \omega_1 \in \Omega, \qquad (2.1)$$

$$L[\{\mu(\alpha)\}^{n(\alpha)}] = L[\{\mu(\alpha)\}^{n(\alpha)+i}], \quad \textit{for any } \alpha \in \Sigma \textit{ and any integer } i > 0, \qquad (2.2)$$

$$L[\mu(\alpha)] \cap A_\alpha \quad \textit{is the kernel of } \mu(\alpha), \qquad (2.3)$$

$$[L\{\mu(\alpha)\omega\} \cap A_\alpha]\mu(\alpha) = L(\omega) \cap B_\alpha, \textit{ for every } \alpha \in \Sigma, \omega \in \Omega. \qquad (2.4)$$

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COROLLARY 1. *If $x\{\mu(\alpha)\}^i$ is defined, then*

$$x\{\mu(\alpha)\}^i = e \text{ if and only if } x \in L[\{\mu(\alpha)\}^i] \cap A_\alpha. \quad \text{......(2.5)}$$

Proof. If $x \in L[\{\mu(\alpha)\}^i] \cap A_\alpha$, then

$$x\mu(\alpha) \in \{L[\{\mu(\alpha)\}^i] \cap A_\alpha\}\mu(\alpha) = L[\{\mu(\alpha)\}^{i-1}] \cap B_\alpha,$$

from (2.4). Since $x\{\mu(\alpha)\}^2$ is defined, then

$$x\mu(\alpha) \in L[\{\mu(\alpha)\}^{i-1}] \cap B_\alpha \cap A_\alpha \subseteq L[\{\mu(\alpha)\}^{i-1}] \cap A_\alpha,$$

and
$$x\{\mu(\alpha)\}^2 \in \{L[\{\mu(\alpha)\}^{i-1}] \cap A_\alpha\}\mu(\alpha) = L[\{\mu(\alpha)\}^{i-2}] \cap B_\alpha.$$

Repeating this procedure a finite number of times, we get

$$x\{\mu(\alpha)\}^{i-1} \in L[\mu(\alpha)] \cap A_\alpha,$$

from which, because of (2.3), it follows that

$$x\{\mu(\alpha)\}^i = e.$$

Conversely, if $x\{\mu(\alpha)\}^i = e$, then

$$x\{\mu(\alpha)\}^{i-1} \in L[\mu(\alpha)] \cap A_\alpha.$$

But also $x\{\mu(\alpha)\}^{i-1} \in B_\alpha$; thus

$$\begin{aligned} x\{\mu(\alpha)\}^{i-1} &\in L[\mu(\alpha)] \cap A_\alpha \cap B_\beta \\ &\subseteq L[\mu(\alpha)] \cap B_\alpha \\ &= \{L[\{\mu(\alpha)\}^2] \cap A_\alpha\}\mu(\alpha). \end{aligned}$$

Thus
$$x\{\mu(\alpha)\}^{i-2} \in L[\{\mu(\alpha)\}^2] \cap A_\alpha.$$

The proof can then be completed by induction.

COROLLARY 2. *If $x\{\mu(\alpha)\}^{n(\alpha)+1}$ is defined, then*

$$x\{\mu(\alpha)\}^{n(\alpha)+1} = e \text{ implies that } x\{\mu(\alpha)\}^{n(\alpha)} = e, \quad \text{......(2.6)}$$

Proof. $x\{\mu(\alpha)\}^{n(\alpha)+1} = e$ implies, by Corollary 1, that

$$x \in L[\{\mu(\alpha)\}^{n(\alpha)+1}] \cap A_\alpha = L[\{\mu(\alpha)\}^{n(\alpha)}] \cap A_\alpha,$$

from (2.2). Thus, by Corollary 1 again,

$$x\{\mu(\alpha)\}^{n(\alpha)} = e.$$

3. Important lemma. Before proving that conditions (2.1)–(2.4) are also sufficient, we prove the following important lemma which will be required later on.

LEMMA 2. *Suppose that we have a sequence of groups*

$$G_1, G_2, \ldots, G_\lambda, \ldots,$$

defined for every λ in a well-ordered set Λ, such that $G_\lambda \subseteq G_\pi$ whenever $\lambda, \pi \in \Lambda$ and $\lambda < \pi$.

Let $\mu(\lambda, \alpha)$, where $\lambda \in \Lambda$ and α ranges over an index set Σ, be a partial endomorphism of G_λ which maps the subgroup $A_{\lambda,\alpha} \subseteq G_\lambda$ onto a second subgroup $B_{\lambda,\alpha} \subseteq G_\lambda$, such that

$$A_{\lambda,\alpha} \subseteq A_{\pi,\alpha}, \quad B_{\lambda,\alpha} \subseteq B_{\pi,\alpha}$$

and $\mu(\pi, \alpha)$ extends $\mu(\lambda, \alpha)$ wherever $\lambda < \pi$.

Let G_λ also contain the normal subgroups $L(\lambda, \omega)$ for every ω in the semigroup Ω freely generated by $\mu(\alpha)$, such that $L(\lambda, \omega) \subseteq L(\pi, \omega)$ whenever $\lambda < \pi$. Let, for all $\lambda \in \Lambda$,

$$L(\lambda, \omega) \subseteq L(\lambda, \omega\omega_1) \quad \textit{for all } \omega, \omega_1 \in \Omega, \quad \text{......(3.1)}$$

M

$$L[\lambda, \{\mu(\alpha)\}^{n(\alpha)}] = L[\lambda, \{\mu(\alpha)\}^{n(\alpha)+i}] \quad \textit{for any integer } i > 0, \qquad (3.2)$$

$$L[\lambda, \mu(\alpha)] \cap A_{\lambda,\alpha} \quad \textit{be the kernel of } \mu(\lambda, \alpha), \qquad (3.3)$$

$$[L\{\lambda, \mu(\alpha)\omega\} \cap A_{\lambda,\alpha}]\mu(\lambda, \alpha) = L(\lambda, \omega) \cap B_{\lambda,\alpha}, \qquad (3.4)$$

and put

$$G' = \bigcup_{\lambda \in \Lambda} G_\lambda, \quad A'_\alpha = \bigcup_{\lambda \in \Lambda} A_{\lambda,\alpha}, \quad B'_\alpha = \bigcup_{\lambda \in \Lambda} B_{\lambda,\alpha}, \quad L'(\omega) = \bigcup_{\lambda \in \Lambda} L(\lambda, \omega).$$

Define $\mu'(\alpha)$ *to map* A'_α *onto* B'_α *as follows. If* $x \in A'_\alpha$, *that is to say* $x \in A_{\lambda,\alpha}$ *for some suitable* $\lambda \in \Lambda$, *we put*

$$x\mu'(\alpha) = x\mu(\lambda, \alpha).$$

Then G', A'_α, B'_α, $L'(\omega)$ *and* $\mu'(\omega)$ *satisfy the following relations:*

$$L'(\omega) \subseteq L'(\omega\omega_1) \quad \textit{for all } \omega, \omega_1 \in \Omega, \qquad (3.5)$$

$$L'[\{\mu(\alpha)\}^{n(\alpha)}] = L'[\{\mu(\alpha)\}^{n(\alpha)+i}] \quad \textit{for any integer } i > 0, \qquad (3.6)$$

$$L'[\mu(\alpha)] \cap A'_\alpha \quad \textit{is the kernel of } \mu'(\alpha), \qquad (3.7)$$

$$[L'\{\mu(\alpha)\omega\} \cap A'_\alpha]\mu'(\alpha) = L'(\omega) \cap B'_\alpha. \qquad (3.8)$$

Proof. If $l \in L'(\omega)$, then

$$l \in L(\lambda, \omega) \subseteq L(\lambda, \omega\omega_1) \subseteq L'(\omega\omega_1) \quad \text{for some } \lambda \in \Lambda,$$

which proves (3.5).

If $x \in L'[\{\mu(\alpha)\}^{n(\alpha)+1}]$, then

$$x \in L[\lambda, \{\mu(\alpha)\}^{n(\alpha)+1}] = L[\lambda, \{\mu(\alpha)\}^{n(\alpha)}] \subseteq L'[\{\mu(\alpha)\}^{n(\alpha)}],$$

for some suitable λ. Thus

$$L'[\{\mu(\alpha)\}^{n(\alpha)+1}] \subseteq L'[\{\mu(\alpha)\}^{n(\alpha)}];$$

but also

$$L'[\{\mu(\alpha)\}^{n(\alpha)}] \subseteq L'[\{\mu(\alpha)\}^{n(\alpha)+1}]$$

from (3.5). These two relations prove (3.6) when $i = 1$. The proof for $i > 1$ is the same.

To prove (3.7) we notice that

$$L'[\mu(\alpha)] \cap A'_\alpha = \left[\bigcup_{\lambda \in \Lambda} L\{\lambda, \mu(\alpha)\}\right] \cap \left[\bigcup_{\pi \in \Lambda} A_{\pi,\alpha}\right] = \bigcup_{\lambda, \pi \in \Lambda} [L\{\lambda, \mu(\alpha)\} \cap A_{\pi,\alpha}].$$

Let $x \in L'[\mu(\alpha)] \cap A'_\alpha$; then

$$x \in L[\lambda, \mu(\alpha)] \cap A_{\pi,\alpha} \quad \text{for some} \quad \lambda, \pi \in \Lambda,$$
$$\subseteq L[\tau, \mu(\alpha)] \cap A_{\tau,\alpha}, \quad \text{where} \quad \tau = \max(\lambda, \pi).$$

Thus, by (3.3),

$$x\mu'(\alpha) = x\mu(\tau, \alpha) = e.$$

Conversely, if x lies in the kernel of $\mu'(\alpha)$, then

$$x\mu'(\alpha) = x\mu(\lambda, \alpha) = e, \quad \text{for some} \quad \lambda \in \Lambda,$$

and thus

$$x \in L[\lambda, \mu(\alpha)] \cap A_{\lambda,\alpha} \subseteq L'[\mu(\alpha)] \cap A'_\alpha.$$

This completes the proof of (3.7).

Finally, to prove (3.8), we note that

$$L'[\mu(\alpha)\omega] \cap A'_\alpha = \bigcup_{\lambda, \pi \in \Lambda} [L\{\lambda, \mu(\alpha)\omega\} \cap A_{\pi,\alpha}],$$
$$L'(\omega) \cap B'_\alpha = \bigcup_{\lambda, \pi \in \Lambda} [L(\lambda, \omega) \cap B_{\pi,\alpha}].$$

If $x \in L'[\mu(\alpha)\omega] \cap A'_\alpha$, then

$$x \in L[\tau, \mu(\alpha)\omega] \cap A_{\tau,\alpha} \quad \text{for some} \quad \tau \in \Lambda,$$
$$x\mu'(\alpha) = x\mu(\tau, \alpha) \in L(\tau, \omega) \cap B_{\tau,\alpha},$$

by (3.4). Thus

$$[L'\{\mu(\alpha)\omega\} \cap A'_\alpha]\mu'(\alpha) \subseteq L'(\omega) \cap B'_\alpha. \quad \text{.............................(iii)}$$

If, on the other hand, $y \in L'(\omega) \cap B'_\alpha$, then

$$y \in L(\lambda, \omega) \cap B_{\pi,\alpha} \subseteq L(\tau, \omega) \cap B_{\tau,\alpha},$$

where $\tau = \max(\lambda, \pi)$, and there exists an element $x \in L[\tau, \mu(\alpha)\omega] \cap A_{\tau,\alpha}$ such that

$$y = x\mu(\tau, \alpha) = x\mu'(\alpha).$$

Thus

$$L'(\omega) \cap B'_\alpha \subseteq [L'\{\mu(\alpha)\omega\} \cap A'_\alpha]\mu'(\alpha). \quad \text{.............................(iv)}$$

(iii) and (iv) together prove (3.8).

This completes the proof of Lemma 2.

4. Sufficient conditions. Let α be an arbitrary element in Σ. Put

$$H = G/L[\mu(\alpha)].$$

Then H contains a subgroup

$$B'_\alpha = A_\alpha \cup L[\mu(\alpha)]/L[\mu(\alpha)] \cong A_\alpha/A_\alpha \cap L[\mu(\alpha) \cong B_\alpha.$$

The mapping: $aL[\mu(\alpha)] \in B'_\alpha$ corresponds to $a\mu(\alpha) \in B_\alpha$, where $a \in A_\alpha$ defines an isomorphism between B'_α and B_α. Let G_α be the free product of G and H with B_α and B'_α amalgamated according to this isomorphism, i.e., let

$$G_\alpha = \{G * H\,;\, B_\alpha = A_\alpha \cup L[\mu(\alpha)]/L\mu(\alpha)\}.$$

Denote by $\nu(\alpha)$ the canonic mapping of G onto H. $\nu(\alpha)$ extends $\mu(\alpha)$. For every $\omega \in \Omega$, we define

$$M(\omega) = [L\{\mu(\alpha)\omega\}\nu(\alpha) \cup L(\omega)]^{G_\alpha},$$

where X^Y denotes the normal closure of X in Y.

Lemma 3. *If the subgroups $L(\omega)$ are replaced by $M(\omega)$, then the relation* (2.2) *will be preserved.*

Proof. Applying (2.2) we get

$$\begin{aligned} M[\{\mu(\alpha)\}^{n(\alpha)}] &= \{L[\{\mu(\alpha)\}^{n(\alpha)+1}]\nu(\alpha) \cup L[\{\mu(\alpha)\}^{n(\alpha)}]\}^{G_\alpha} \\ &= \{L[\{\mu(\alpha)\}^{n(\alpha)+2}]\nu(\alpha) \cup L[\{\mu(\alpha)\}^{n(\alpha)+1}]\}^{G_\alpha} \\ &= M[\{\mu(\alpha)\}^{n(\alpha)+1}]. \end{aligned}$$

This, together with what was already proved in [1], shows that $\mu(\alpha)$ is extended to a partial endomorphism $\nu(\alpha)$ of a supergroup G_α of G which maps G onto $G/L[\mu(\alpha)]$, such that when we put

$$G_\alpha\,;\, G, G/L[\mu(\alpha)], \nu(\alpha)\,;\, A_\beta, B_\beta, \mu(\beta) \text{ for all } \beta(\neq \alpha) \in \Sigma \text{ and } M(\omega)$$

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in the place of

$$G\,;\, A_\alpha,\, B_\alpha,\, \mu(\alpha)\,;\, A_\beta,\, B_\beta,\, \mu(\beta) \text{ for all } \beta(\neq \alpha) \in \Sigma \text{ and } L(\omega),$$

relations (2.1)–(2.4) will be satisfied. As a corollary, relation (2.6) will also be satisfied.

We shall describe this process of embedding G in G_α by saying that G_α is obtained from G by an *α-extension*.

Now, for any $\lambda \in \Sigma$, we define a group G_λ as follows. If we denote the first element of Σ by 0, then we construct G_1 by 0-extension from G.

Inductively, if G_λ for $\lambda \in \Sigma$ is defined and thus contains normal subgroups $L(\lambda, \omega)$ and contains for every $\alpha \in \Sigma$ a subgroup $A_{\lambda,\alpha}$ mapped homomorphically by $\mu(\lambda, \alpha)$ onto another subgroup $B_{\lambda,\alpha}$ of G_λ, then we construct $G_{\lambda+1}$ by a λ-extension from G_λ, that is, we form

$$G_{\lambda+1} = \{G_\lambda * G_\lambda/L[\lambda, \mu(\lambda)]\,;\; B_{\lambda,\lambda} = A_{\lambda,\lambda} \cup L[\lambda, \mu(\lambda)]/L[\lambda, \mu(\lambda)]\}.$$

$G_{\lambda+1}$ contains, for every $\alpha \in \Sigma$, a subgroup $A_{\lambda+1,\alpha}$ mapped homomorphically by $\mu(\lambda+1, \alpha)$ onto the subgroup $B_{\lambda+1,\alpha}$ of $G_{\lambda+1}$, where

$$A_{\lambda+1,\lambda} = G_\lambda,$$
$$B_{\lambda+1,\lambda} = G_\lambda/L[\lambda, \mu(\lambda)],$$
$$\mu(\lambda+1, \lambda) \text{ is the canonic mapping of } G_\lambda \text{ on } G_\lambda/L[\lambda, \mu(\lambda)]$$

and

$$\left.\begin{aligned} A_{\lambda+1,\alpha} &= A_{\lambda,\alpha}, \\ B_{\lambda+1,\alpha} &= B_{\lambda,\alpha}, \\ \mu(\lambda+1, \alpha) &\text{ is } \mu(\lambda, \alpha)\,. \end{aligned}\right\} \text{ when } \alpha \neq \lambda.$$

Define

$$L(\lambda+1, \omega) = [L\{\lambda, \mu(\lambda)\omega\}\mu(\lambda+1, \lambda) \cup L(\lambda, \omega)]^{G_{\lambda+1}}$$

for every $\omega \in \Omega$. Then, according to [1] and Lemma 1, $G_{\lambda+1}$, $A_{\lambda+1,\alpha}$, $B_{\lambda+1,\alpha}$, $\mu(\lambda+1, \alpha)$ and $L(\lambda+1, \omega)$ satisfy the relations (2.1)–(2.4) and, as a corollary, also relation (2.6).

If π is a limit ordinal and G_λ, $A_{\lambda,\alpha}$, $B_{\lambda,\alpha}$, $\mu(\lambda, \alpha)$, and $L(\lambda, \omega)$ are defined for all $\lambda < \pi$, put

$$G_\pi = \bigcup_{\lambda<\pi} G_\lambda, \quad A_{\pi,\alpha} = \bigcup_{\lambda<\pi} A_{\lambda,\alpha}, \quad B_{\pi,\alpha} = \bigcup_{\lambda<\pi} B_{\lambda,\alpha}, \quad L(\pi, \omega) = \bigcup_{\lambda<\pi} L(\lambda, \omega)$$

and define $\mu(\pi, \alpha)$ to map $A_{\pi,\alpha}$ onto $B_{\pi,\alpha}$ in the following way. If $a \in A_{\pi,\alpha}$, i.e., if $a \in A_{\lambda\,\alpha}$ for some $\lambda < \pi$, we put

$$a\mu(\pi, \alpha) = a\mu(\lambda, \alpha).$$

Then, by Lemma 2, G_π, $A_{\pi,\alpha}$, $B_{\pi,\alpha}$, $\mu(\pi, \alpha)$ and $L(\pi, \omega)$ satisfy conditions (2.1)–(2.4) and hence (2.6) also.

If σ is the ordinal type of Σ, then we continue this process until we form G_σ.

Put $${}^0G = G, \quad A_\alpha = {}^0A_\alpha, \quad \mu(\alpha) = \mu_0(\alpha),$$

$${}^1G = {}^0G_\sigma, \quad {}^0A_{\sigma,\alpha} = {}^1A_\alpha, \quad \mu_0(\sigma, \alpha) = \mu_1(\alpha),$$

and form inductively

$${}^nG = {}^{n-1}G_\sigma, \quad {}^{n-1}A_{\sigma,\alpha} = {}^nA_\alpha, \quad \mu_{n-1}(\sigma, \alpha) = \mu_n(\alpha),$$

for any positive integer n. Let

$$G^* = \bigcup_n {}^nG.$$

For any $\alpha \in \Sigma$ we define a mapping $\mu^*(\alpha)$ as follows. If $g \in G^*$, that is if $g \in {}^nG = {}^{n-1}G_\sigma$ for some suitable n, then we put

$$g\mu^*(\alpha) = g\mu_n(\alpha).$$

Thus the $\mu^*(\alpha)$ will become total endomorphisms of G^* which extend the $\mu(\alpha)$.

Moreover, if $g \in G^*$ and $g\{\mu^*(\alpha)\}^{n(\alpha)+1} = e$, then

$$g\{\mu_n(\alpha)\}^{n(\alpha)+1} = e, \text{ for some suitable } n,$$

which implies, by a relation corresponding to (2.6), that

$$g\{\mu_n(\alpha)\}^{n(\alpha)} = e,$$

and thus

$$g\{\mu^*(\alpha)\}^{n(\alpha)} = e.$$

This proves that $\mu^*(\alpha)$ is an isomorphism on $G^*\{\mu^*(\alpha)\}^{n(\alpha)}$. This completes the proof of the following theorem.

THEOREM 2. *The necessary conditions* (2.1)–(2.4) *of Theorem* 1 *are also sufficient conditions for the existence of the required extension.*

5. Special case. In this section we give the following theorem which is an immediate consequence of Theorem 2.

THEOREM 3. *With the previous notation, in order that* $\mu(\alpha)$ *should be extendable to one and the same group, such that* $\mu^*(\alpha)$ *is an isomorphism on* $G^*\{\mu^*(\alpha)\}^{n(\alpha)}$, *it is sufficient that there exists for each* $\alpha \in \Sigma$, *a sequence*

$$L(\alpha, 1) \subseteq L(\alpha, 2) \subseteq \dots \subseteq L[\alpha, n(\alpha)] = L[\alpha, n(\alpha)+1] = \dots$$

of normal subgroups in G, *such that*

$$L(\alpha, 1) \cap A_\alpha \text{ is the kernel of } \mu(\alpha), \quad \text{(5.1)}$$

$$[L(\alpha, i+1) \cap A_\alpha]\mu(\alpha) = L(\alpha, i) \cap B_\alpha, \quad \text{(5.2)}$$

$$L(\alpha, i) \cap B_\beta = e \quad \text{(5.3)}$$

for $i = 1, 2, \dots, \alpha, \beta \in \Sigma$ *and* $\alpha \neq \beta$.

For then we can satisfy conditions (2.1)–(2.4) by putting

$$L[\{\mu(\alpha)\}^i\omega] = L[\{\mu(\alpha)\}^i] = L(\alpha, i),$$

or $\omega = \mu(\beta)\omega'$, $\beta \neq \alpha$.

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