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A STRONGLY CONVERGENT ALGORITHM FOR SOLVING MULTIPLE SET SPLIT EQUALITY EQUILIBRIUM AND FIXED POINT PROBLEMS IN BANACH SPACES

E.C. GODWIN, O.T. MEWOMO AND T.O. ALAKOYA

School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa (220100908@stu.ukzn.ac.za; emmysworld05@yahoo.com; mewomoo@ukzn.ac.za; alakoyat1@ukzn.ac.za; timimaths@gmail.com)

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Abstract In this article, using an Halpern extragradient method, we study a new iterative scheme for finding a common element of the set of solutions of multiple set split equality equilibrium problems consisting of pseudomonotone bifunctions and the set of fixed points for two finite families of Bregman quasi-nonexpansive mappings in the framework of p-uniformly convex Banach spaces, which are also uniformly smooth. For this purpose, we design an algorithm so that it does not depend on prior estimates of the Lipschitz-type constants for the pseudomonotone bifunctions. Furthermore, we present an application of our study for finding a common element of the set of solutions of multiple set split equality variational inequality problems and fixed point sets for two finite families of Bregman quasi-nonexpansive mappings. Finally, we conclude with two numerical experiments to support our proposed algorithm.

Keywords: split feasibility problem; equilibrium problem; fixed-point problem; pseudomonotone bifunction; p-uniformly convex; Bregman distance; quasi-nonexpansive mapping; strong convergence; Banach spaces

2020 Mathematics subject classification: Primary 47H09; 47H10; 49J20; 49J40

1. Introduction

Let E_1 and E_2 be two real Banach spaces with duals E_1^* and E_2^*, respectively, and let C and Q be nonempty closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator. Censor and Elving [7] introduced the concept of split feasibility problem (SFP), which is formulated as

$$\text{find } x^* \in C \quad \text{such that } Ax^* \in Q. \tag{1.1}$$

The SFP has been found useful in solving numerous real-life problems, including medical image reconstruction, phase retrieval, signal processing, radiation therapy treatment planning, among others. See, for example [17, 18, 24] and the references therein.

For solving the SFP (1.1) using Bregman projection, Schöpfer et al. [40] proposed the following method in the framework of p-uniformly convex real Banach spaces: for $x_1 \in E_1$, set

$$x_{n+1} = \Pi_C J_q^{E_1^*} \left[J_{E_1}^p (x_n) - \gamma_n A^* J_{E_2}^p \left(Ax_n - P_Q(Ax_n) \right) \right], \quad n \geq 1, \tag{1.2}$$

where Π_C denotes the Bregman projection from E_1 onto C and J_E^p is the duality mapping. Closely related to the SFP (1.1) is the following split variational inequality problem (SVIP) introduced by Censor et al. [9] in the framework of real Hilbert spaces as follows: find $x^* \in C$, which satisfies the inequality

$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C, \tag{1.3}$$

such that $y^* = Ax^* \in Q$ solves the inequality

$$\langle G(y^*), y - y^* \rangle \geq 0 \quad \forall y \in Q, \tag{1.4}$$

where C and Q are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, $F : H_1 \to H_1$ and $G : H_2 \to H_2$ are two given operators, $A : H_1 \to H_2$ is a bounded linear operator.
Recall that when problems (1.3) and (1.4) are viewed separately, then Equation (1.3) is the classical variational inequality problem (VIP) in H_1 with its solution set VIP(C, F) and Equation (1.4) is another VIP in H_2 with its solution set VIP(Q, G). To solve the SVIP (1.3) and (1.4), Censor et al. [9] put forward the following algorithm. Let $x_1 \in H_1$, the sequence $\{x_n\}$ is generated by

$$x_{n+1} = P_C(I - \lambda U)(x_n + \gamma A^*(P_Q(I - \lambda V) - I)Ax_n), \quad n \geq 1, \tag{1.5}$$

where $\gamma \in (0, 1/L)$ and L is the spectral radius of the operator A^*A. They proved that the above Algorithm (1.5) converges weakly to a solution of the SVIP under the assumption that U, V are α_1, α_2-inverse strongly monotone operators and $\lambda \in (0, 2\alpha)$ (where $\alpha := \min\{\alpha_1, \alpha_2\}$).
Let C be a nonempty, closed and convex subset of a real Banach space E with dual E^*. Let $f : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem (EP) studied by Blum and Oettli [5] is to locate a point $x^* \in C$ such that

$$f(x^*, x) \geq 0, \quad \forall\, x \in C. \tag{1.6}$$

We denote by EP(f) the set of solutions of Equation (1.6). The EP was formerly introduced as the Ky Fan inequality [15]. This class of problem has been extensively studied by numerous scholars because of its several applications. It is well known that many problems arising in economics, optimization and physics can be reduced to problem (1.6).

Moreover, several iterative algorithms have been proposed to solve the EP (1.6) and related optimization problems in both Hilbert and Banach spaces (see [1, 3, 5, 13, 16, 17, 28, 31, 36, 34, 39, 45] and other references contained therein). However, most of the existing results on the EP are of the monotone type.

In 2011, Moudafi [30] extended the SVIP Equations (1.3) and (1.4) to split equilibrium problem (shortly, SEP), which is defined as

$$\begin{aligned} &\text{find } x^* \in C \quad \text{such that } f_1(x^*, y) \geq 0, \quad \forall\, y \in C \\ &\text{and } y^* = Ax^* \in Q \quad \text{such that } f_2(y^*, z) \geq 0, \quad \forall\, z \in Q, \end{aligned} \tag{1.7}$$

where C and Q are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, $f_1 : C \times C \to \mathbb{R}$ and $f_2 : Q \times Q \to \mathbb{R}$ are bifunctions with a bounded linear operator $A : H_1 \to H_2$.

In addition, for solving the SEP (1.7), the author in [20] put forward a proximal technique without product space formulation as follows:

$$\begin{aligned} &x^0 \in C; \quad \{\rho_n\} \subset (0, \infty); \quad \mu > 0 \\ &f_1(y_n, y) + \frac{1}{\rho_n}\langle y - y_n, y_n - x_n\rangle \geq 0 \quad \forall y \in C, \\ &f_2(u_n, v) + \frac{1}{\rho_n}\langle v - u_n, u_n - Ay_n\rangle \geq 0 \quad \forall v \in Q, \\ &x_{n+1} = P_C(y_n + \mu A^*(u_n - Ay_n)), \quad \forall n \geq 0, \end{aligned}$$

where A^* is the adjoint of A. The author in [20] obtained a weak convergence result when the bifunctions f_1 and f_2 are monotone on C and Q, respectively. Since then, several iterative schemes have been proposed when the bifunctions are either monotone or pseudomonotone, see for example [4, 14, 21–23, 25] and check also the references therein.

In Section 6.1 of Censor et al. [9], the authors proposed an improvement of the SVIP (1.3) and (1.4), which they called multiple set SVIP (MSSVIP), which is formulated as follows:

$$\begin{aligned} &\text{find } x^* \in C := \bigcap_{i=1}^{N} C_i \quad \text{such that } \langle F_i(x^*), y - x^*\rangle \geq 0, \quad \forall y \in C_i, \\ &\qquad i = 1, 2, \ldots, N \\ &\text{and such that } y^* = Ax^* \in Q := \bigcap_{j=1}^{M} Q_j \text{ solves } \langle G_j(y^*), z - y^*\rangle \geq 0, \quad \forall z \in Q_j, \\ &\qquad j = 1, 2, \ldots, M, \end{aligned}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $F_i : H_1 \to H_1$, $i = 1, 2, \ldots, N$ and $G_j : H_2 \to H_2$, $j = 1, 2, \ldots, M$ are given operators and C_i, $i = 1, 2, \ldots, N$ and Q_j, $j = 1, 2, \ldots, M$ are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. They proposed an algorithm for solving the problem and proved

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that the sequence generated by the proposed iterative scheme converges weakly to the solution set of MSSVIP when $F_i,\ i=1,2,\dots,N$ and $G_j,\ j=1,2,\dots,M$ are inverse strongly monotone operators.

Moudafi [31] proposed a new SFP, which he called split equality problem (SE$_q$P). Let H_1, H_2 and H_3 be real Hilbert spaces. $A: H_1 \to H_3$ and $B: H_2 \to H_3$ be two bounded linear operators, and $C \subset H_1$ and $Q \subset H_2$ be two nonempty, closed and convex sets. The SE$_q$P is formulated as follows:

$$\text{find } x^* \in C, \qquad y^* \in Q \quad \text{such that } Ax^* = By^*. \tag{1.8}$$

Let $T: E \to E$ be a mapping. We denote the fixed point set of T by $\mathrm{F}(T)$; that is $\mathrm{F}(T) := \{x \in E : Tx = x\}$. The *Fixed Point Problem* has application in various fields, such as optimization theory, economics, game theory, as well as in establishing the existence of solutions of several physical problems arising in differential and integral equations [19, 32, 34, 42].

If $C := F(S)$ and $Q := F(T)$ in Equation (1.8), where $S: H_1 \to H_1$ and $T: H_2 \to H_2$ are two nonlinear mappings, then the SE$_q$P becomes the split equality fixed point problem. Motivated and inspired by the above mentioned results, we introduce and study in the framework of p-uniformly convex Banach space an extension of the MSSVIP to multiple set split equality equilibrium and common fixed points problem of Bregman quasi-nonexpansive mappings. Using Bregman distance, we make use of the Halpern extragradient technique for solving the pseudomonotone EP, which guarantees strong convergence. We design our algorithm in such a way that it does not depend on the prior estimates of the Lipschitz-like constants.

We organize the rest of this article as follows: Section 2 presents preliminaries and some existing results, § 3 is the design of our iterative method, whereas § 4 focuses on the convergence analysis of the proposed algorithm. In § 5, we apply our result to solve a certain class of variational inequality problems. We present some numerical experiments in § 6 and conclude with some final remarks in § 7.

2. Preliminaries

In this section, we call up some important definitions and existing results, which will be needed in the proof of our main result. We denote strong and weak convergence of the sequence $\{x_n\}$ to a point x by '$\to$' and '$\rightharpoonup$', respectively.

Let E be a real Banach space and $1 < q \le 2 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. The modulus of smoothness of E is the function $\rho_E: \mathbb{R}^+ := [0, \infty) \to \mathbb{R}^+$ defined by

$$\rho_E(\tau) = \sup\left\{\frac{\|\bar{x} + \tau\bar{y}\| + \|\bar{x} - \tau\bar{y}\|}{2} - 1 : \bar{x} = 1 = \bar{y}\right\}.$$

The space E is called uniformly smooth if and only if $\frac{\rho_E(\tau)}{\tau} \to 0$ as $\tau \to 0$. Let $q > 1$, E is said to be q-uniformly smooth if there exists $\kappa_q > 0$ such that $\rho_E(\tau) \le \kappa_q \tau^q$ for all

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$\tau > 0$. The modulus of convexity of E is defined as

$$\beta_E(\epsilon) = \inf\left\{1 - \frac{\|\bar{x} + \bar{y}\|}{2} : \|\bar{x}\| = \|\bar{y}\| = 1; \quad \epsilon = \|\bar{x} - \bar{y}\|\right\}.$$

The Banach space E is called uniformly convex if and only if $\beta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. Now, suppose $p > 1$, then E is called p-uniformly convex if there exists a constant $C_p > 0$ such that $\beta_E(\epsilon) \geq C_p\epsilon^p$ for all $\epsilon \in (0, 2]$.

Remark 2.1. It is well known that every p-uniformly convex space is also strictly convex and reflexive. In addition, if a Banach space E is p-uniformly convex and uniformly smooth, then its dual space E^* is q-uniformly smooth and uniformly convex (see [11]).

Definition 2.2. see [12] *Let $p > 1$ be a real number, the generalized duality mapping $J_E^p : E \to 2^{E^*}$ is defined by*

$$J_E^p(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\},$$

where $\langle ., . \rangle$ denotes the duality pairing between elements of E and E^*. In particular, $J_E^p = J_E^2$ is called the normalized duality mapping. If E is p-uniformly convex and uniformly smooth, then E^* is q-uniformly smooth and uniformly convex. In this case, the generalized duality mapping J_E^p is one-to-one, single-valued and satisfies $J_E^p = (J_{E^*}^q)^{-1}$, where $J_{E^*}^q$ is the generalized duality mapping of E^*. Furthermore, if E is uniformly smooth, then the duality mapping J_E^p is norm-to-norm uniformly continuous on bounded subsets of E, and E is smooth if and only if J_E^p is single valued.

Let $f : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function, then the Fenchel conjugate of f denoted as $f^* : E^* \to (-\infty, +\infty]$ is defined as

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E,\ x^* \in E^*\}.$$

See [41] for more information about Fenchel conjugate.
Let the domain of f be denoted by $(\operatorname{dom} f) = \{x \in E : f(x) < +\infty\}$; hence, for any $x \in \operatorname{int}(\operatorname{dom} f)$ and $y \in E$, we define the right-hand derivative of f at x in the direction y by

$$f^0(x, y) = \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}.$$

The function f is said to be Gâteaux differentiable at x if $\lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}$ exists for any y. In this case, $f^0(x, y)$ coincides with $\bigtriangledown f(x)$ (the value of the gradient $\bigtriangledown f$ of f at x). The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable for any $x \in \operatorname{int}(\operatorname{dom} f)$. The function f is said to be Fréchet differentiable at x if its limit is attained uniformly in $||y|| = 1$. Moreover, f is said to be uniformly Fréchet differentiable on a subset C of E if the above limit is attained uniformly for $x \in C$ and $||y|| = 1$.
A function f is said to be Legendre if it satisfies the following conditions:

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(1) The interior of the domain of f, $\operatorname{int}(\operatorname{dom} f)$ is nonempty, f is Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} f)$ and $\operatorname{dom} \triangledown f = \operatorname{int}(\operatorname{dom} f)$.
(2) The interior of the domain of f^*, $\operatorname{int}(\operatorname{dom} f^*)$ is nonempty, f^* is Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} f^*)$ and $\operatorname{dom} \triangledown f^* = \operatorname{int}(\operatorname{dom} f)$.

Definition 2.3 *[6] Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $\Delta_f : E \times E \to [0, +\infty)$ defined by*

$$\Delta_f(x, y) := f(y) - f(x) - \langle \triangledown f(x), y - x\rangle$$

is called the Bregman distance with respect to f.

We highlight the following interesting properties of Bregman distance (see [6, 37]):

(i) $\Delta_f(x, x) = 0$, but $\Delta_f(x, y) = 0$ does not necessarily imply that $x = y$,
(ii) for $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int}(\operatorname{dom} f)$, we have

$$\Delta_f(x, y) + \Delta_f(y, z) - \Delta_f(x, z) = \langle \triangledown g(z) - \triangledown g(y), x - y\rangle, \tag{2.1}$$

(iii) for each $z \in E$, $\{x_i\}_{i=1}^N \subset E$ and $\{\alpha_n\}_{i=1}^N \subset (0, 1)$ with $\sum_{i=1}^N \alpha_i = 1$, we have

$$\Delta_f\left(z, \nabla g^*\left(\sum_{i=1}^N \alpha_i \nabla g(x_i)\right)\right) \leq \sum_{i=1}^N \alpha_i \Delta_f(z, x_i).$$

It is well-known that in general the Bregman distance Δ_f is not a metric because it fails to satisfy the symmetric and triangle inequality properties. Moreover, it is well known that the duality mapping J_E^p is the sub-differential of the functional $f_p(.) = \frac{1}{p}||.||^p$ for $p > 1$, see [10]. Then, the Bregman distance Δ_p is defined with respect to f_p as follows:

$$\begin{aligned}\Delta_p(x, y) &= \frac{1}{p}\|y\|^p - \frac{1}{p}\|x\|^p - \langle J_E^p x, y - x\rangle \\ &= \frac{1}{q}\|x\|^p - \langle J_E^p x, y\rangle + \frac{1}{p}\|y\|^p \\ &= \frac{1}{q}\|x\|^p - \frac{1}{q}\|y\|^p - \langle J_E^p x - J_E^p y, y\rangle. \end{aligned} \tag{2.2}$$

Definition 2.4. *Let C be a nonempty, closed and convex subset of a real Banach space E and let $T : C \to C$ be a nonlinear map. The mapping $I - T$ is said to be demiclosed*

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at zero if for any sequence $\{x_n\} \subset C$, *the following implication holds:* $x_n \rightharpoonup x$ *and* $(I-T)x_n \to 0 \implies x \in F(T)$.

Definition 2.5. [8, 29] *Let* $T : C \to \operatorname{int}(\operatorname{dom} f)$ *be a mapping. Then,*

(i) a point $p \in C$ *is called an asymptotic fixed point of* T *if* C *contains a sequence* $\{x_n\}$, *which converges weakly to* p *such that* $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. *We denote by* $\hat{F}(T)$ *the set of asymptotic fixed points of* T;

(ii) T *is called Bregman firmly nonexpansive if*

$$\langle \Delta_f(Tx) - \Delta_f(Ty), Tx - Ty\rangle \le \langle \Delta_f(x) - \Delta_f(y), Tx - Ty\rangle, \quad \forall x, y \in C.$$

(iii) T *is called Bregman strongly nonexpansive if* $\hat{F}(T) \neq \emptyset$ *and*

$$\Delta_f(p, Tx) \le \Delta_f(p, x), \qquad \forall p \in \hat{F}(T) \quad \text{and} \quad x \in C.$$

(iv) T *is said to be Bregman quasi-nonexpansive if*

$$F(T) \neq \emptyset \quad \text{and} \quad \Delta_f(p, Tx) \le \Delta_f(p, x), \quad \forall x \in C,\ p \in F(T).$$

Recall that the metric projection P_C from E onto C satisfies the following property:

$$\|x - P_C x\| \le \inf_{y\in C}\|x - y\|, \quad \forall x \in E.$$

It is well known that P_C is the unique minimizer of the norm distance. Moreover, P_C is characterized by the following property:

$$\langle J_E^p x - J_E^p(P_C x), y - P_C x\rangle \le 0, \quad \forall y \in C. \tag{2.3}$$

The Bregman projection from E onto C denoted by Π_C also satisfies the property

$$\Delta_p(x, \Pi_C(x)) = \inf_{y\in C}\Delta_p(x, y), \quad \forall x \in E. \tag{2.4}$$

Also, if C is a nonempty, closed and convex subset of a p-uniformly convex and uniformly smooth Banach space E and $x \in E$, then the following assertions hold (see [11]):

(i) $z = \Pi_C x$ if and only if

$$\langle J_E^p(x) - J_E^p(z), y - z\rangle \le 0, \quad \forall y \in C; \tag{2.5}$$

(ii)

$$\Delta_p(\Pi_C x, y) + \Delta_p(x, \Pi_C x) \le \Delta_p(x, y), \quad \forall y \in C. \tag{2.6}$$

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Lemma 2.6. *[43] Let C be a nonempty convex subset of a Banach space E. Let $g : C \to \mathbb{R}$ be a convex, subdifferentiable function on C. Then g attains its minimum at $x \in C$ if and only if $0 \in \partial g(x) + N_C(x)$, where $N_C(x)$ is the normal cone of C at x, that is*

$$N_C(x) := \{\hat{x} \in E^* : \langle x - \varphi, \hat{x}\rangle \geq 0, \quad \forall \varphi \in C\}.$$

Lemma 2.7. *[10] Let E be a Banach space and $x, y \in E$. If E is q-uniformly smooth, then there exists $C_q > 0$ such that*

$$\|x - y\|^q \leq \|x\|^q - q\langle J_q^E(x), y\rangle + C_q\|y\|^q.$$

Lemma 2.8. *[12] Let f and g be two convex functions on E, such that $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$, where f is continuous, then*

$$\partial(f + g)(x) = \partial f(x) + \partial g(x), \quad \forall x \in E.$$

Lemma 2.9. *[27] Let E be a real p-uniformly convex and uniformly smooth Banach space. Let $z, x_k \in E (k = 1, 2, \ldots, N)$ and $\alpha_k \in (0, 1)$ with $\sum_{k=1}^N \alpha_k = 1$. Then, we have*

$$\Delta_p\left(J_q^{E^*}\left(\sum_{k=1}^N \alpha_k J_p^E(x_k)\right), z\right) \leq \sum_{k=1}^N \alpha_k \Delta_p(x_k, z) - \alpha_i\alpha_j g_r^*\left(\|J_p^E(x_i) - J_p^E(x_j)\|\right),$$

for all $i, j \in \{1, 2, \ldots, N\}$ and $g_r^* : \mathbb{R}^+ \to \mathbb{R}^+$ being a strictly increasing function such that $g_r^*(0) = 0$.

Lemma 2.10. *[40] Let E be a real p-uniformly convex and uniformly smooth Banach space. Let $V_p : E^* \times E \to [0, +\infty)$ be defined by*

$$V_p(x, x^*) = \frac{1}{p}\|x\|^p - \langle x, x^*\rangle + \frac{1}{q}\|x^*\|^q, \quad \forall x \in E,\ x^* \in E^*.$$

Then the following assertions hold:

- *(i)* V_p *is nonnegative and convex in the first variable.*
- *(ii)* $\Delta_p\left(x, J_q^{E^*}(x^*)\right) = V_p(x, x^*), \quad \forall x \in E,\ x^* \in E^*.$
- *(iii)* $V_p(x, x^*) + \langle J_q^{E^*}(x^*) - x, y^*\rangle \leq V_p(x, x^* + y^*), \quad \forall x \in E,\ x^*, y^* \in E^*.$

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Lemma 2.11. *[11] Let E be a real p-uniformly convex and uniformly smooth Banach space. Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded sequences in E. Then the following assertions are equivalent:*

(i) $\lim_{n\to\infty} \Delta_p(x_n, y_n) = 0$;
(ii) $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

Lemma 2.12. *[46] Let $q \geq 1$ and $r > 0$ be two fixed real numbers. Then, a Banach space E is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g : \mathbb{R}^+ \to \mathbb{R}^*$, $g(0) = 0$ such that for all $x, y \in B_r$ and $0 \leq \alpha < 1$,*

$$\|\alpha x + (1-\alpha)y\|^q \leq \alpha\|x\|^q + (1-\alpha)\|y\|^q - W_q(\alpha)g(\|x-y\|),$$

where $W_q(\alpha) := \alpha^q(1-\alpha) + \alpha(1-\alpha)^q$ and $B_r := \{x \in E : \|x\| \leq r\}$.

Lemma 2.13. *[38] Let E be a real Banach space and let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{\Delta_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 2.14. *[26] Let $\{a_n\} \subset \mathbb{R}_+$, $\{\varepsilon_n\} \subset (0,1)$ be a sequence such that $\sum_{n=1}^{\infty} \varepsilon_n = \infty$ and $\{b_n\} \subset \mathbb{R}$. Assume that*

$$a_{n+1} \leq (1-\varepsilon_n)a_n + \varepsilon_n b_n, \quad \forall n \geq 0.$$

If $\limsup\limits_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition $\liminf\limits_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$, then $\lim\limits_{k\to\infty} a_n = 0$.

To solve the EP (1.6), the following assumptions are needed:

Assumption A:

(C1) f is pseudomonotone, that is, for all $x, y \in C$, $f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0$ and $f(x,x) = 0$, for all $x \in C$.

(C2) f satisfies the Bregman–Lipschitz type condition on C, that is, there exists two positive constants c_1 and c_2 such that

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\Delta_p(y,x) - c_2\Delta_p(y,z), \quad \forall x, y, z \in C,$$

where $p : E \to (-\infty, +\infty]$ is a Legendre function. The constants c_1 and c_2 are called Bregman–Lipschitz coefficients with respect to p.

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(C3) $f(x,.)$ is convex, lower semicontinuous and subdifferentiable on C for all $x \in C$.

(C4) f is jointly weakly continuous on $C \times C$ in the sense that if $x, y \in C$ and $\{x_n\}$ and $\{y_n\}$ converges weakly to x and y, respectively, then $f(x_n, y_n) \to f(x, y)$ as $n \to \infty$.

3. Proposed method

In this section, we present our method and discuss some of its features. We begin with the following assumptions under which our strong convergence result is obtained.

Assumption 3.1. *We assume that the following conditions hold:*

(1) (a) E_1, E_2 and E_3 are three p-uniformly convex and uniformly smooth real Banach spaces.

(b) C_i and Q_j are nonempty closed and convex subsets of E_1 and E_2, respectively, for $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, M$.

(c) $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear operators.

*(d) $f_i : C_i \times C_i \to \mathbb{R}$ and $g_j : Q_j \times Q_j \to \mathbb{R}$ are bifunctions satisfying conditions $C_1 - C_4$ of **Assumption A.***

(e) $D_s : E_1 \to E_1$ and $G_t : E_2 \to E_2$ are Bregman quasi-nonexpansive mappings such that $I - D_s$ and $I - G_t$ are demiclosed at zero for each $s = 1, 2, \dots, l$ and $t = 1, 2, \dots, m$.

(f) Assume that the solution set
$\Upsilon := \{\bar{x} \in \bigcap_{s=1}^{l} F(D_s) \cap \bigcap_{i=1}^{N} EP(C_i, f_i), \quad \bar{y} \in \bigcap_{t=1}^{m} F(G_t) \cap \bigcap_{j=1}^{M} EP(Q_j, g_j) : A\bar{x} = B\bar{y}\} \neq 0.$

(2) $\{\beta_n\}_{n=1}^{\infty}$, $\{\alpha_{n,s}\}_{s=0}^{l}$, $\{\eta_{n,t}\}_{t=0}^{m}$ are positive sequences satisfying the following conditions:

(a) $\{\beta_n\} \subset (0,1)$, $\lim\limits_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $\tau_0 > 0$, $\lambda_0 > 0$, $\kappa \in (0,1)$, $\epsilon \in (0,1)$.

(b) $\{\alpha_{n,s}\} \subset (0,1)$, $\sum_{s=0}^{l} \alpha_{n,s} = 1$ *and* $\liminf\limits_{n\to\infty} \alpha_{n,0}\alpha_{n,s} > 0$.

(c) $\{\eta_{n,t}\} \subset (0,1)$, $\sum_{t=0}^{m} \eta_{n,t} = 1$ *and* $\liminf\limits_{n\to\infty} \eta_{n,0}\eta_{n,t} > 0$.

We now present the proposed method of this paper.

Algorithm 3.2. *For fixed $\mu \in E_1$ and $\vartheta \in E_2$, choose an initial guess $(x_0, y_0) \in E_1 \times E_2$. Suppose that the nth iterate $(x_n, y_n) \subset E_1 \times E_2$ has been constructed; then we compute the $(n+1)$th iterate (x_{n+1}, y_{n+1}) via the iteration*

$$\begin{cases} s_n = J^q_{E_1^*}\left(J^p_{E_1}(x_n) - \rho_n A^* J^p_{E_3}(Ax_n - By_n)\right), \\ a^i_n = \arg\min\left\{f_i(s_n,\sigma) + \frac{1}{\tau_n}\Delta_p(\sigma, s_n): \quad \sigma \in C_i\right\}, \\ z^i_n = \arg\min\left\{f_i(a^i_n,\sigma) + \frac{1}{\tau_n}\Delta_p(\sigma, s_n): \quad \sigma \in C_i\right\}. \\ \textit{Obtain the farthest element of } z^i_n \textit{ from } s_n, \textit{ i.e.,} \\ i_n \in \arg\max\{\Delta_p(s_n, z^i_n) : i = 1,\dots,N\}. \\ \textit{Set } z^{i_n}_n = \bar{z}_n \\ u_n = J_q^{E_1^*}\left(\alpha_{n,0}J^p_{E_1}(\bar{z}_n) + \sum_{s=1}^{l}\alpha_{n,s}J^p_{E_1}(D_s\bar{z}_n)\right) \\ x_{n+1} = J^q_{E_1^*}\left(\beta_n J^p_{E_1}(\mu) + (1-\beta_n)J^p_{E_1}(u_n)\right), \\ t_n = J^q_{E_2^*}\left(J^p_{E_2}(y_n) + \rho_n B^* J^p_{E_3}(Ax_n - By_n)\right), \\ b^j_n = \arg\min\left\{g_j(t_n,\varphi) + \frac{1}{\lambda_n}\Delta_p(\varphi, t_n): \quad \varphi \in Q_j\right\}, \\ h^j_n = \arg\min\left\{g_j(b^j_n,\varphi) + \frac{1}{\lambda_n}\Delta_p(\varphi, t_n): \quad \varphi \in Q_j\right\}. \\ \textit{Obtain the farthest element of } h^j_n \textit{ from } t_n, \textit{ i.e.,} \\ j_n \in \arg\max\left\{\Delta_p(t_n, h^j_n) : j = 1,\dots,M\right\}. \\ \textit{Set } h^{j_n}_n = \bar{\theta}_n \\ v_n = J^q_{E_1^*}\left(\eta_{n,0}J^p_{E_2}(\bar{\theta}_n) + \sum_{t=1}^{m}\eta_{n,t}J^p_{E_2}(G_t\bar{\theta}_n)\right) \\ y_{n+1} = J^q_{E_2^*}\left(\beta_n J^p_{E_2}(\vartheta) + (1-\beta_n)J_p^{E_2}(v_n)\right), \end{cases} \tag{3.1}$$

where

$$\rho_n \in \left(\zeta, \left(\frac{q\|Ax_n - By_n\|^p}{C_q\|A^*J^p_{E_3}(Ax_n - By_n)\|^q + Q_q\|B^*J^p_{E_3}(Ax_n - By_n)\|^q} - \zeta\right)^{\frac{1}{q-1}}\right), \quad n \in \Omega \tag{3.2}$$

for small enough ζ; C_q and Q_q are constants of smoothness of E_1 and E_2, respectively. Otherwise, $\rho_n = \rho$ (ρ being any nonnegative value), where the set of indexes $\Omega = \{n : Ax_n - By_n \neq 0\}$.

$$\tau_{n+1} = \begin{cases} \min\left\{\tau_n, \min\limits_{1\le i\le N}\left\{\frac{\kappa(\Delta_p(a^i_n,s_n)+\Delta_p(z^i_n,a^i_n))}{f_i(s_n,z^i_n)-f_i(s_n,a^i_n)-f_i(a^i_n,z^i_n)}\right\}\right\}, & \textit{if } f_i(s_n,z^i_n) - f_i(s_n,a^i_n) \\ & \quad -f_i(a^i_n,z^i_n) > 0, \\ \tau_n, & \textit{otherwise.} \end{cases} \tag{3.3}$$

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and

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \min\limits_{1\le j\le M}\left\{\frac{\epsilon(\Delta_p(b_n^j,t_n)+\Delta_p(h_n^j,b_n^j))}{g_j(t_n,h_n^j)-g_j(t_n,b_n^j)-g_j(b_n^j,h_n^j)}\right\}\right\}, & \text{if } g_j(t_n,h_n^j)-g_j(t_n,b_n^j) \\ & \quad -g_j(b_n^j,h_n^j)>0, \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{3.4}$$

Remark 3.3.

(a) Algorithm 3.2 solves split equality EP consisting of two strongly convex optimization problems in parallel for $i = 1, 2, \ldots, N$, as well as another two strongly convex optimization problems in parallel for $j = 1, 2, \ldots, M$ under bounded linear operators.

(b) The step size $\{\rho_n\}$ given by Equation (3.2) is generated at each iteration by some simple computations. Thus, $\{\rho_n\}$ is easily implemented without the prior knowledge of the operator norms $\|A\|$ and $\|B\|$. Similarly, the step size $\{\tau_n\}$ given by Equation (3.3) and step size $\{\lambda_n\}$ given by Equation (3.4) do not depend on the prior estimates of the Lipschitz-like constants of the pseudomonotone bifunctions f_i, $i = 1, 2, \ldots, N$, and g_j, $j = 1, 2, \ldots, M$, unlike the step sizes used in [14, 22], which require finding the prior estimates of the Lipschitz-like constants of the pseudomonotone bifunctions, which is known to be computationally expensive.

(c) Moreover, our result in this paper extends the results in [22, 25] from the framework of Hilbert spaces to Banach spaces.

4. Convergence analysis

Lemma 4.1. *The sequences $\{\tau_n\}$ and $\{\lambda_n\}$ of step sizes generated by Algorithm 3.2 are well defined and bounded.*

Proof. Clearly, from Equations (3.3) and (3.4), we have $\tau_{n+1} \le \tau_n$ $\forall n \in \mathbb{N}$ and $\lambda_{n+1} \le \lambda_n$ $\forall n \in \mathbb{N}$. This implies that $\{\tau_n\}$ and $\{\lambda_n\}$ are monotonically decreasing sequences. Moreover, it follows from condition C_2 of **Assumption A** that

$$f_i(s_n, z_n^i) - f_i(s_n, a_n^i) - f_i(a_n^i, z_n^i) \le k_{1,i}\Delta_p(a_n^i, s_n) + k_{2,i}\Delta_p(z_n^i, a_n^i), \quad \forall i = 1, 2, \ldots, N.$$

Hence, we obtain for all $i = 1, 2, \ldots, N$

$$\begin{aligned} \frac{\kappa\left(\Delta_p(a_n^i, s_n) + \Delta_p(z_n^i, a_n^i)\right)}{f_i(s_n, z_n^i) - f_i(s_n, a_n^i) - f_i(a_n^i, z_n^i)} &\ge \frac{\kappa\left(\Delta_p(a_n^i, s_n) + \Delta_p(z_n^i, a_n^i)\right)}{k_{1,i}\Delta_p(a_n^i, s_n) + k_{2,i}\Delta_p(z_n^i, a_n^i)} \\ &\ge \frac{\kappa\left(\Delta_p(a_n^i, s_n) + \Delta_p(z_n^i, a_n^i)\right)}{\max\{k_{1,i}, k_{2,i}\}(\Delta_p(a_n^i, s_n) + \Delta_p(z_n^i, a_n^i))} \\ &\ge \frac{\kappa}{\max\{k_{1,i}, k_{2,i}\}}. \end{aligned}$$

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Similarly, we obtain

$$\frac{\epsilon\left(\Delta_p(b_n^j,t_n)+\Delta_p(h_n^j,b_n^j)\right)}{g_j(t_n,h_n^j)-g_j(t_n,b_n^j)-g_j(b_n^j,h_n^j)}\geq\frac{\epsilon}{\max\{c_{1,j},c_{2,j}\}},\quad\forall j=1,2,\ldots,M.$$

Hence, we conclude that $\{\tau_n\}$ has lower bound $\min\left\{\tau_0,\frac{\kappa}{\max\limits_{1\leq i\leq N}\{k_{1,i},k_{2,i}\}}\right\}>0$ and $\{\lambda_n\}$ has lower bound

$\min\left\{\lambda_0,\frac{\epsilon}{\max\limits_{1\leq j\leq M}\{c_{1,j},c_{2,j}\}}\right\}>0$. It then follows that $\lim\limits_{n\to\infty}\tau_n=\tau>0$ and $\lim\limits_{n\to\infty}\lambda_n=\lambda>0$. □

Lemma 4.2. *Let $C_i,\ i=1,2,\ldots,N$ and $Q_j, j=1,2,\ldots,M$ be nonempty, closed and convex subsets of E_1 and E_2, respectively. Suppose that $f_i: C_i\times C_i\to\mathbb{R},\ i=1,2,\ldots,N$ and $g_j: Q_j\times Q_j\to\mathbb{R},\ j=1,2,\ldots,M$ are bifunctions satisfying conditions C_1-C_4. Then, for all $(\bar{x},\bar{y})\in\Upsilon$, we have*

$$\Delta_p(\bar{x},z_n^i)\leq\Delta_p(\bar{x},s_n)-\left(1-\kappa\frac{\tau_n}{\tau_{n+1}}\right)\left(\Delta_p(a_n^i,s_n)+\Delta_p(z_n^i,a_n^i)\right),\quad\forall i=1,2,\ldots,N \tag{4.1}$$

and

$$\Delta_p(\bar{y},h_n^j)\leq\Delta_p(\bar{y},t_n)-\left(1-\epsilon\frac{\lambda_n}{\lambda_{n+1}}\right)\left(\Delta_p(b_n^j,t_n)+\Delta_p(h_n^j,b_n^j)\right),\quad\forall j=1,2,\ldots,M. \tag{4.2}$$

Proof. Since $z_n^i=\arg\min\left\{f_i(a_n^i,\sigma)+\frac{1}{\tau_n}\Delta_p(\sigma,s_n):\sigma\in C_i\right\}$, then from Lemma 2.6, we get

$$0\in\partial_2(\tau_nf_i(a_n^i,z_n^i)+\Delta_p(z_n^i,s_n))+N_{C_i}(z_n^i).$$

Then, there exists $\xi\in\partial_2f_i(a_n^i,z_n^i)$, $\bar{\xi}\in N_{C_i}(z_n^i)$, such that

$$\tau_n\xi+J_{E_1}^p(z_n^i)-J_{E_1}^p(s_n)+\bar{\xi}=0. \tag{4.3}$$

Also, by the definition of $\partial_2f_i(a_n^i,z_n^i)$, we obtain

$$f_i(a_n^i,\sigma)-f_i(a_n^i,z_n^i)\geq\langle\sigma-z_n^i,\xi\rangle,\quad\forall\sigma\in C_i.$$

If we replace σ with $\bar{x}$ in the inequality above, we have

$$f_i(a_n^i,\bar{x})-f_i(a_n^i,z_n^i)\geq\langle\bar{x}-z_n^i,\xi\rangle,\quad\forall\bar{x}\in\Upsilon. \tag{4.4}$$

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Using the definition of $N_{C_i}(z_n^i)$ together with Equation (4.3), we have

$$\langle \sigma - z_n^i, J_{E_1}^p(z_n^i) - J_{E_1}^p(s_n)\rangle \geq \tau_n \langle z_n^i - \sigma, \xi\rangle, \quad \forall \sigma \in C_i. \tag{4.5}$$

Again, if we let $\sigma = \bar{x}$ in Equation (4.5), we get

$$\langle \bar{x} - z_n^i, J_{E_1}^p(z_n^i) - J_{E_1}^p(s_n)\rangle \geq \tau_n \langle z_n^i - \bar{x}, \xi\rangle, \quad \forall \bar{x} \in \Upsilon. \tag{4.6}$$

The combination of Equations (4.4) and (4.6) gives

$$\begin{aligned}\langle \bar{x} - z_n^i, J_{E_1}^p(z_n^i) - J_{E_1}^p(s_n)\rangle &\geq \tau_n \langle f_i(a_n^i, z_n^i) - f_i(a_n^i, \bar{x})\rangle \\ &\geq \tau_n f_i(a_n^i, z_n^i),\end{aligned} \tag{4.7}$$

because $f_i(\bar{x}, a_n^i) \geq 0$ and f_i is pseudomonotone on C_i, $\forall i = 1, 2, \ldots, N$. Similarly, since $a_n^i = \arg\min\{f_i(s_n, \sigma) + \frac{1}{\tau_n}\Delta_p(\sigma, s_n) : \sigma \in C_i\}$, we obtain

$$\langle a_n^i - z_n^i, J_{E_1}^p(a_n^i) - J_{E_1}^p(s_n)\rangle \geq \tau_n \left[f_i(s_n, z_n^i) - f_i(s_n, a_n^i)\right]. \tag{4.8}$$

Using Equations (4.7) and (4.8) together, we get

$$\begin{aligned}&\langle \bar{x} - z_n^i, J_{E_1}^p(z_n^i) - J_{E_1}^p(s_n)\rangle + \langle a_n^i - z_n^i, J_{E_1}^p(a_n^i) - J_{E_1}^p(s_n)\rangle \\ &\qquad \geq \tau_n \left[f_i(s_n, z_n^i) - f_i(s_n, a_n^i) + f_i(a_n^i, z_n^i)\right].\end{aligned}$$

Applying Bregman three-point identity Equation (2.1), we obtain

$$\Delta_p(\bar{x}, z_n^i) \leq \Delta_p(\bar{x}, s_n) - \Delta_p(a_n^i, s_n) - \Delta_p(z_n^i, a_n^i) + \tau_n\{f_i(s_n, z_n^i) - f_i(s_n, a_n^i) - f(a_n^i, z_n^i)\}.$$

Furthermore, by the definition of τ_n, we obtain

$$\begin{aligned}\Delta_p(\bar{x}, z_n^i) &\leq \Delta_p(\bar{x}, s_n) - \Delta_p(a_n^i, s_n) - \Delta_p(z_n^i, a_n^i) \\ &\quad + \frac{\tau_n}{\tau_{n+1}}\tau_{n+1}\{f_i(s_n, z_n^i) - f_i(s_n, a_n^i) - f_i(a_n^i, z_n^i)\} \\ &\leq \Delta_p(\bar{x}, s_n) - \Delta_p(a_n^i, s_n) - \Delta_p(z_n^i, a_n^i) + \frac{\tau_n}{\tau_{n+1}}\kappa\left(\Delta_p(a_n^i, s_n) + \Delta_p(z_n^i, a_n^i)\right) \\ &= \Delta_p(\bar{x}, s_n) - \left(1 - \frac{\tau_n}{\tau_{n+1}}\kappa\right)\left(\Delta_p(a_n^i, s_n) + \Delta_p(z_n^i, a_n^i)\right).\end{aligned} \tag{4.9}$$

Following similar procedure, we obtain

$$\Delta_p(\bar{y}, h_n^j) \leq \Delta_p(\bar{y}, t_n) - \left(1 - \epsilon\frac{\lambda_n}{\lambda_{n+1}}\right)\left(\Delta_p(b_n^j, t_n) + \Delta_p(h_n^j, b_n^j)\right). \tag{4.10}$$

□

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Observe that since $\lim_{n\to\infty}\left(1-\frac{\tau_n}{\tau_{n+1}}\kappa\right)=1-\kappa>0$, then there exists $K\in\mathbb{N}$ such that

$$\left(1-\frac{\tau_n}{\tau_{n+1}}\kappa\right)>0, \quad \forall n\geq K.$$

Hence, from Equation (4.9), we get

$$\Delta_p(\bar{x}, z_n^i)\leq \Delta_p(\bar{x}, s_n), \quad \forall i=1,2\ldots,N,\ n\geq K. \tag{4.11}$$

Similarly, from Equation (4.10), we obtain

$$\Delta_p(\bar{y}, h_n^j)\leq \Delta_p(\bar{y}, t_n), \quad \forall n\geq L\in\mathbb{N}. \tag{4.12}$$

Lemma 4.3. *Suppose $\{x_n\}$ and $\{y_n\}$ are iterative sequences generated by Algorithm 3.2 under Assumption 3.1. Then, the sequences $\{x_n\}$ and $\{y_n\}$ are bounded.*

Proof. Let $(\bar{x},\bar{y})\in\Upsilon$. Since D_s is Bregman quasi-nonexpansive for each $s=1,2,\ldots,l$, we obtain from Equation (3.1) that

$$\begin{aligned}\Delta_p(\bar{x},u_n)&=\Delta_p\left(\bar{x}, J_q^{E_1^*}\left(\alpha_{n,0}J_{E_1}^p(\bar{z}_n)+\sum_{s=1}^{l}\alpha_{n,s}J_{E_1}^p(D_s\bar{z}_n)\right)\right)\\&\leq \alpha_{n,0}\Delta_p(\bar{x},\bar{z}_n)+\sum_{s=1}^{l}\alpha_{n,s}\Delta_p(\bar{x},D_s\bar{z}_n)\\&\leq \alpha_{n,0}\Delta_p(\bar{x},\bar{z}_n)+\sum_{s=1}^{l}\alpha_{n,s}\Delta_p(\bar{x},\bar{z}_n)\\&=\Delta_p(\bar{x},\bar{z}_n).\end{aligned} \tag{4.13}$$

Similarly, we obtain

$$\Delta_p(\bar{y}, v_n)\leq \Delta_p(\bar{y},\bar{\theta}_n). \tag{4.14}$$

Furthermore, from Equation (3.1), Lemma 2.7 and Lemma 2.10, we obtain

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$$\begin{aligned}
\Delta_p(\bar{x}, s_n) &= \Delta_p\left(\bar{x}, J^q_{E_1^*}\left(J^p_{E_1}(x_n) - \rho_n A^* J^p_{E_3}(Ax_n - By_n)\right)\right) \\
&= V_p\left(\bar{x}, J^p_{E_1}(x_n) - \rho_n A^* J^p_{E_3}(Ax_n - By_n)\right) \\
&= \frac{1}{p}\|\bar{x}\|^p - \langle \bar{x}, J^p_{E_1}(x_n)\rangle + \rho_n\langle \bar{x}, A^* J^p_{E_3}(Ax_n - By_n)\rangle \\
&+ \frac{1}{q}\|J^p_{E_1}(x_n) - \rho_n A^* J^p_{E_3}(Ax_n - By_n)\|^q \\
&\leq \frac{1}{p}\|\bar{x}\|^p - \langle \bar{x}, J^p_{E_1}(x_n)\rangle + \rho_n\langle A\bar{x}, J^p_{E_3}(Ax_n - By_n)\rangle \\
&+ \frac{1}{q}\|J^p_{E_1}(x_n)\|^q - \rho_n\langle J^p_{E_3}(Ax_n - By_n), Ax_n\rangle + \frac{C_q}{q}\rho_n^q\|A^* J^p_{E_3}(Ax_n - By_n)\|^q \\
&= \frac{1}{p}\|\bar{x}\|^p - \langle \bar{x}, J^p_{E_1}(x_n)\rangle + \frac{1}{q}\|J^p_{E_1}(x_n)\|^q - \rho_n\langle J^p_{E_3}(Ax_n - By_n), Ax_n - A\bar{x}\rangle \\
&+ \frac{C_q}{q}\rho_n^q\|A^* J_p^{E_3}(Ax_n - By_n)\|^q \\
&= \Delta_p(\bar{x}, x_n) - \rho_n\langle J^p_{E_3}(Ax_n - By_n), Ax_n - A\bar{x}\rangle \\
&+ \frac{C_q}{q}\rho_n^q\|A^* J_p^{E_3}(Ax_n - By_n)\|^q. \qquad (4.15)
\end{aligned}$$

Similarly, we have

$$\Delta_p(\bar{y}, t_n) \leq \Delta_p(\bar{y}, y_n) - \rho_n\langle J^p_{E_3}(Ax_n - By_n), B\bar{y} - By_n\rangle + \frac{Q_q}{q}\rho_n^q\|B^* J^p_{E_3}(Ax_n - By_n)\|^q. \qquad (4.16)$$

Combining Equations (4.15) and (4.16) and noting that $A\bar{x} = B\bar{y}$, we have

$$\begin{aligned}
\Delta_p(\bar{x}, s_n) + \Delta_p(\bar{y}, t_n) &\leq \Delta_p(\bar{x}, x_n) + \Delta_p(\bar{y}, y_n) - \rho_n\Bigg[\|Ax_n - By_n\|^p - \frac{\rho_n^{q-1}}{q} \\
&\times\left(C_q\|A^* J^p_{E_3}(Ax_n - By_n)\|^q + Q_q\|B^* J^p_{E_3}(Ax_n - By_n)\|^q\right)\Bigg]. \qquad (4.17)
\end{aligned}$$

Hence,

$$\Delta_p(\bar{x}, s_n) + \Delta_p(\bar{y}, t_n) \leq \Delta_p(\bar{x}, x_n) + \Delta_p(\bar{y}, y_n). \qquad (4.18)$$

Also, from Equation (3.1) and applying Equation (4.11), we obtain

$$\begin{aligned}
\Delta_p(\bar{x}, x_{n+1}) &= \Delta_p\left(\bar{x}, J^q_{E_1^*}\left(\beta_n J^p_{E_1}(\mu) + (1-\beta_n)J^p_{E_1}(u_n)\right)\right) \\
&\leq \beta_n\Delta_p(\bar{x}, \mu) + (1-\beta_n)\Delta_p(\bar{x}, u_n) \\
&\leq \beta_n\Delta_p(\bar{x}, \mu) + (1-\beta_n)\Delta_p(\bar{x}, \bar{z}_n) \\
&\leq \beta_n\Delta_p(\bar{x}, \mu) + (1-\beta_n)\Delta_p(\bar{x}, s_n). \qquad (4.19)
\end{aligned}$$

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In like manner, we have

$$\Delta_p(\bar{y}, y_{n+1}) \leq \beta_n \Delta_p(\bar{y}, \vartheta) + (1-\beta_n)\Delta_p(\bar{y}, t_n). \tag{4.20}$$

It follows from Equations (4.18), (4.19) and (4.20) that

$$\begin{aligned}
\Delta_p(\bar{x}, x_{n+1}) + \Delta_p(\bar{y}, y_{n+1}) &\leq \beta_n \left(\Delta_p(\bar{x}, \mu) + \Delta_p(\bar{y}, \vartheta)\right) + (1-\beta_n)\left(\Delta_p(\bar{x}, s_n) + \Delta_p(\bar{y}, t_n)\right) \\
&\leq \beta_n \left(\Delta_p(\bar{x}, \mu) + \Delta_p(\bar{y}, \vartheta)\right) + (1-\beta_n)\left(\Delta_p(\bar{x}, x_n) + \Delta_p(\bar{y}, y_n)\right) \\
&\leq \max\{\Delta_p(\bar{x}, \mu) + \Delta_p(\bar{y}, \vartheta), \Delta_p(\bar{x}, x_n) + \Delta_p(\bar{y}, y_n)\} \\
&\ \vdots \\
&\leq \max\{\Delta_p(\bar{x}, \mu) + \Delta_p(\bar{y}, \vartheta), \Delta_p(\bar{x}, x_{\bar{N}}) + \Delta_p(\bar{y}, y_{\bar{N}})\}, \\
\bar{N} &= \max\{K, L\}.
\end{aligned} \tag{4.21}$$

Therefore, $\{\Delta_p(\bar{x}, x_n) + \Delta_p(\bar{y}, y_n)\}$ is bounded, and consequently $\{\Delta_p(\bar{x}, x_n)\}$ and $\{\Delta_p(\bar{y}, y_n)\}$ are bounded. Hence, by Lemma 2.13, the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. Therefore, $\{s_n\}$, $\{a_n^i\}$, $\{z_n^i\}$, $\{u_n\}$, $\{t_n\}$, $\{b_n^j\}$, $\{h_n^j\}$ and $\{v_n\}$ are all bounded. □

Lemma 4.4. *Assume that* $r = \sup\{\|J_{E_1}^p(\bar{z}_n)\|, \ \|J_{E_1}^p(D_s\bar{z}_n)\|\}$ *and let* $(\bar{x}, \bar{y}) \in \Upsilon$. *Then, the following inequality holds:*

$$\begin{aligned}
\Delta_p(\bar{x}, x_{n+1}) + \Delta_p(\bar{y}, y_{n+1}) &\leq \beta_n \left[\Delta_p(\bar{x}, \mu) + \Delta_p(\bar{y}, \vartheta)\right] + (1-\beta_n)[\Delta_p(\bar{x}, x_n) + \Delta_p(\bar{y}, y_n)] \\
&\quad - (1-\beta_n)\left(\frac{W_q(\alpha_{n,s})}{q} g\left(\|J_{E_1}^p(\bar{z}_n) - J_{E_1}^p(D_s\bar{z}_n)\|\right)\right. \\
&\quad \left. + \frac{W_q(\eta_{n,t})}{q} g\left(\|J_{E_2}^p(\bar{\theta}_n) - J_{E_2}^p(G_t\bar{\theta}_n)\|\right)\right),
\end{aligned} \tag{4.22}$$

where $W_q(\alpha_{n,s}) = (\alpha_{n,0})^q \sum_{s=1}^{l} \alpha_{n,s} + \alpha_{n,0}(\sum_{s=1}^{l} \alpha_{n,s})^q$ *and* $W_q(\eta_{n,t}) = (\eta_{n,0})^q \sum_{t=1}^{m} \eta_{n,t} + \eta_{n,0}(\sum_{t=1}^{m} \eta_{n,t})^q$.

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Proof. Let $(\bar{x},\bar{y}) \in \Upsilon$. Then, from Equation (3.1), Lemma 2.10 and Lemma 2.12, we obtain

$$\begin{aligned}
\Delta_p(\bar{x},u_n) &= \Delta_p\left(\bar{x}, J^q_{E_1^*}\left(\alpha_{n,0}J^p_{E_1}(\bar{z}_n) + \sum_{s=1}^{l}\alpha_{n,s}J^p_{E_1}(D_s\bar{z}_n)\right)\right) \\
&= V_p\left(\bar{x}, \alpha_{n,0}J^p_{E_1}(\bar{z}_n) + \sum_{s=1}^{l}\alpha_{n,s}J^p_{E_1}(D_s\bar{z}_n)\right) \\
&= \frac{1}{p}\|\bar{x}\|^p - \alpha_{n,0}\langle \bar{x}, J^p_{E_1}(\bar{z}_n)\rangle - \sum_{s=1}^{l}\alpha_{n,s}\langle \bar{x}, J^p_{E_1}(D_s\bar{z}_n)\rangle \\
&+ \frac{1}{q}\|\alpha_{n,0}J^p_{E_1}(\bar{z}_n) + \sum_{s=1}^{l}\alpha_{n,s}J^p_{E_1}(D_s\bar{z}_n)\|^q \\
&\le \frac{1}{p}\|\bar{x}\|^p - \alpha_{n,0}\langle \bar{x}, J^p_{E_1}(\bar{z}_n)\rangle - \sum_{s=1}^{l}\alpha_{n,s}\langle \bar{x}, J^p_{E_1}(D_s\bar{z}_n)\rangle \\
&+ \frac{1}{q}\alpha_{n,0}\|J^p_{E_1}(\bar{z}_n)\|^p + \frac{1}{q}\sum_{s=1}^{l}\alpha_{n,s}\|J^p_{E_1}(D_s\bar{z}_n)\|^P \\
&- \frac{W_q(\alpha_{n,s})}{q}g\left(\|J^p_{E_1}(\bar{z}_n) - J^p_{E_1}(D_s\bar{z}_n)\|\right) \\
&= \frac{1}{p}\alpha_{n,0}\|\bar{x}\|^p + \sum_{s=1}^{l}\alpha_{n,s}\frac{1}{p}\|\bar{x}\|^p - \alpha_{n,0}\langle \bar{x}, J^p_{E_1}(\bar{z}_n)\rangle - \sum_{s=1}^{l}\alpha_{n,s}\langle \bar{x}, J^p_{E_1}(D_s\bar{z}_n)\rangle \\
&+ \frac{1}{q}\alpha_{n,0}\|J^p_{E_1}(\bar{z}_n)\|^p + \frac{1}{q}\sum_{s=1}^{l}\alpha_{n,s}\|J^p_{E_1}(D_s\bar{z}_n)\|^P \\
&- \frac{W_q(\alpha_{n,s})}{q}g\left(\|J^p_{E_1}(\bar{z}_n) - J^p_{E_1}(D_s\bar{z}_n)\|\right) \\
&= \alpha_{n,0}\left\{\frac{1}{p}\|\bar{x}\|^p - \langle \bar{x}, J^p_{E_1}(\bar{z}_n)\rangle + \frac{1}{q}\|J^p_{E_1}(\bar{z}_n)\|^p\right\} \\
&+ \sum_{s=1}^{l}\alpha_{n,s}\left\{\frac{1}{p}\|\bar{x}\|^p - \langle \bar{x}, J^p_{E_1}(D_s\bar{z}_n)\rangle + \frac{1}{q}\|J^p_{E_1}(D_s\bar{z}_n)\|^p\right\} \\
&- \frac{W_q(\alpha_{n,s})}{q}g\left(\|J^p_{E_1}(\bar{z}_n) - J^p_{E_1}(D_s\bar{z}_n)\|\right) \\
&= \alpha_{n,0}\Delta_p(\bar{x},\bar{z}_n) + \sum_{s=1}^{l}\alpha_{n,s}\Delta_p(\bar{x},D_s\bar{z}_n) \\
&- \frac{W_q(\alpha_{n,s})}{q}g\left(\|J^p_{E_1}(\bar{z}_n) - J^p_{E_1}(D_s\bar{z}_n)\|\right).
\end{aligned}$$

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By the Bregman quasi-nonexpansivity of D_s for $s = 1, 2, \ldots, l$, we get

$$\begin{aligned}\Delta_p(\bar{x}, u_n) &\leq \alpha_{n,0}\Delta_p(\bar{x}, \bar{z}_n) + \sum_{s=1}^{l} \alpha_{n,s}\Delta_p(\bar{x}, \bar{z}_n) - \frac{W_q(\alpha_{n,s})}{q} g\left(\|J^p_{E_1}(\bar{z}_n) - J^p_{E_1}(D_s\bar{z}_n)\|\right) \\ &= \Delta_p(\bar{x}, \bar{z}_n) - \frac{W_q(\alpha_{n,s})}{q} g\left(\|J^p_{E_1}(\bar{z}_n) - J^p_{E_1}(D_s\bar{z}_n)\|\right). \end{aligned} \tag{4.23}$$

By the definition of x_{n+1} and applying Equations (4.11) and (4.13) from Equation (4.23), we obtain

$$\begin{aligned}\Delta_p(\bar{x}, x_{n+1}) &= \Delta_p\left(\bar{x}, J^q_{E_1^*}\left(\beta_n J^p_{E_1}(\mu) + (1-\beta_n)J^p_{E_1}(u_n)\right)\right) \\ &\leq \beta_n\Delta_p(\bar{x}, \mu) + (1-\beta_n)\Delta_p(\bar{x}, u_n) \leq \beta_n\Delta_p(\bar{x}, \mu) + (1-\beta_n) \\ &\quad \times \left(\Delta_p(\bar{x}, \bar{z}_n) - \frac{W_q(\alpha_{n,s})}{q} g\left(\|J^p_{E_1}(\bar{z}_n) - J^p_{E_1}(D_s\bar{z}_n)\|\right)\right) \\ &\leq \beta_n\Delta_p(\bar{x}, \mu) + (1-\beta_n)\Delta_p(\bar{x}, s_n) - (1-\beta_n)\frac{W_q(\alpha_{n,s})}{q} g \\ &\quad \times \left(\|J^p_{E_1}(\bar{z}_n) - J^p_{E_1}(D_s\bar{z}_n)\|\right). \end{aligned} \tag{4.24}$$

Following similar argument, we have

$$\begin{aligned}\Delta_p(\bar{y}, y_{n+1}) &\leq \beta_n\Delta_p(\bar{y}, \vartheta) + (1-\beta_n)\Delta_p(\bar{y}, t_n) - (1-\beta_n)\frac{W_q(\eta_{n,t})}{q} g \\ &\quad \times \left(\|J^p_{E_2}(\bar{\theta}_n) - J^p_{E_2}(G_t\bar{\theta}_n)\|\right). \end{aligned} \tag{4.25}$$

By adding Equations (4.24) and (4.25) and applying Equation (4.18), we get

$$\begin{aligned}\Delta_p(\bar{x}, x_{n+1}) + \Delta_p(\bar{y}, y_{n+1}) &\leq \beta_n\left[\Delta_p(\bar{x}, \mu) + \Delta_p(\bar{y}, \vartheta)\right] + (1-\beta_n)[\Delta_p(\bar{x}, s_n) + \Delta_p(\bar{y}, t_n)] \\ &\quad - (1-\beta_n)\left(\frac{W_q(\alpha_{n,s})}{q} g\left(\|J^p_{E_1}(\bar{z}_n) - J^p_{E_1}(D_s\bar{z}_n)\|\right)\right. \\ &\quad \left. + \frac{W_q(\eta_{n,t})}{q} g\left(\|J^p_{E_2}(\bar{\theta}_n) - J^p_{E_2}(G_t\bar{\theta}_n)\|\right)\right) \\ &\leq \beta_n\left[\Delta_p(\bar{x}, \mu) + \Delta_p(\bar{y}, \vartheta)\right] + (1-\beta_n)[\Delta_p(\bar{x}, x_n) + \Delta_p(\bar{y}, y_n)] \\ &\quad - (1-\beta_n)\left(\frac{W_q(\alpha_{n,s})}{q} g\left(\|J^p_{E_1}(\bar{z}_n) - J^p_{E_1}(D_s\bar{z}_n)\|\right)\right. \\ &\quad \left. + \frac{W_q(\eta_{n,t})}{q} g\left(\|J^p_{E_2}(\bar{\theta}_n) - J^p_{E_2}(G_t\bar{\theta}_n)\|\right)\right), \end{aligned}$$

which is the required inequality. □

We now present the main theorem for our proposed algorithm as follows.

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Theorem 4.5 *Suppose* $\{(x_n, y_n)\}$ *is a sequence generated by Algorithm 3.2 under Assumption 3.1. Then* $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y}) \in \Upsilon$*, where* $\bar{x} = \Pi_\Upsilon(\mu)$ *and* $\bar{y} = \Pi_\Upsilon(\vartheta)$.

Proof. Let $(\bar{x}, \bar{y}) = (\Pi_\Upsilon(\mu), \Pi_\Upsilon(\vartheta))$. It follows from Algorithm 3.2 and by applying Lemma 2.10 (iii) that

$$\begin{aligned}\Delta_p(\bar{x}, x_{n+1}) &= \Delta_p\left(\bar{x}, J^q_{E_1^*}\left(\beta_n J^p_{E_1}(\mu) + (1-\beta_n)J^p_{E_1}(u_n)\right)\right)\\ &= V_p\left(\bar{x}, \beta_n J^p_{E_1}(\mu) + (1-\beta_n)J^p_{E_1}(u_n)\right)\\ &\le V_p\left(\bar{x}, \beta_n J^p_{E_1}(\mu) + (1-\beta_n)J^p_{E_1}(u_n) - \beta_n\left(J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x})\right)\right)\\ &\quad + \beta_n\langle J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x}), x_{n+1} - \bar{x}\rangle\\ &= V_p\left(\bar{x}, \beta_n J^p_{E_1}(\bar{x}) + (1-\beta_n)J^p_{E_1}(u_n)\right) + \beta_n\langle J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x}), x_{n+1} - \bar{x}\rangle\\ &\le \beta_n\Delta_p(\bar{x}, \bar{x}) + (1-\beta_n)\Delta_p(\bar{x}, u_n) + \beta_n\langle J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x}), x_{n+1} - \bar{x}\rangle\\ &\le (1-\beta_n)\Delta_p(\bar{x}, s_n) + \beta_n\langle J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x}), x_{n+1} - \bar{x}\rangle. \end{aligned} \tag{4.26}$$

In the same vein, we have

$$\Delta_p(\bar{y}, y_{n+1}) \le (1-\beta_n)\Delta_p(\bar{y}, t_n) + \beta_n\langle J^p_{E_2}(\vartheta) - J^p_{E_2}(\bar{y}), y_{n+1} - \bar{y}\rangle. \tag{4.27}$$

Hence, by adding Equations (4.26) and (4.27) and applying Equation (4.18), we get

$$\begin{aligned}\Delta_p(\bar{x}, x_{n+1}) + \Delta_p(\bar{y}, y_{n+1}) &\le (1-\beta_n)[\Delta_p(\bar{x}, s_n) + \Delta_p(\bar{y}, t_n)]\\ &\quad + \beta_n\left(\langle J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x}), x_{n+1} - \bar{x}\rangle + \langle J^p_{E_2}(\vartheta) - J^p_{E_2}(\bar{y}), y_{n+1} - \bar{y}\rangle\right)\\ &\le (1-\beta_n)[\Delta_p(\bar{x}, x_n) + \Delta_p(\bar{y}, y_n)]\\ &\quad + \beta_n\left(\langle J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x}), x_{n+1} - \bar{x}\rangle + \langle J^p_{E_2}(\vartheta) - J^p_{E_2}(\bar{y}), y_{n+1} - \bar{y}\rangle\right)\\ &= (1-\beta_n)[\Delta_p(\bar{x}, x_n) + \Delta_p(\bar{y}, y_n)] + \beta_n\chi_n, \quad \forall n \ge 1, \end{aligned} \tag{4.28}$$

where $\chi_n := \left(\langle J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x}), x_{n+1} - \bar{x}\rangle + \langle J^p_{E_2}(\vartheta) - J^p_{E_2}(\bar{y}), y_{n+1} - \bar{y}\rangle\right)$.
In order to show that $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$ by Lemma 2.14, we only need to show that $\limsup_{k\to\infty}\chi_{n_k} \le 0$ for every subsequence $\{\Delta_p(x_{n_k}, \bar{x})\}$ of $\{\Delta_p(x_n, \bar{x})\}$ and $\{\Delta_p(y_{n_k}, \bar{y})\}$ of $\{\Delta_p(y_n, \bar{y})\}$ satisfy the inequality

$$\liminf_{k\to\infty}\left(\left[\Delta_p(\bar{x}, x_{n_k+1}) + \Delta_p(\bar{y}, y_{n_k+1})\right] - \left[\Delta_p(\bar{x}, x_{n_k}) + \Delta_p(\bar{y}, y_{n_k})\right]\right) \ge 0. \tag{4.29}$$

Now, from Algorithm 3.2 and Lemma 4.2, we obtain

$$\begin{aligned}\Delta_p(\bar{x}, x_{n+1}) &= \Delta_p\left(\bar{x}, J^q_{E_1^*}(\beta_n J^p_{E_1}(\mu) + (1-\beta_n)J^p_{E_1}(u_n))\right)\\ &\leq \beta_n\Delta_p(\mu, \bar{x}) + (1-\beta_n)\Delta_p(\bar{x}, u_n)\\ &\leq \beta_n\Delta_p(\mu, \bar{x}) + (1-\beta_n)\Delta_p(\bar{x}, \bar{z}_n)\\ &\leq \beta_n\Delta_p(\mu, \bar{x}) + (1-\beta_n)\Delta_p(\bar{x}, s_n) - (1-\beta_n)\\ &\quad \times\left(1-\frac{\tau_n}{\tau_{n+1}}\kappa\right)\left(\Delta_p(a_n^{in}, s_n) + \Delta_p(z_n^{in}, a_n^{in})\right). \end{aligned} \tag{4.30}$$

In the same vein, we obtain

$$\begin{aligned}\Delta_p(\bar{y}, y_{n+1}) &\leq \beta_n\Delta_p(\vartheta, \bar{y}) + (1-\beta_n)\Delta_p(\bar{y}, t_n) - (1-\beta_n)\\ &\quad \times\left(1-\frac{\lambda_n}{\lambda_{n+1}}\eta\right)\left(\Delta_p(b_n^{jn}, t_n) + \Delta_p(h_n^{jn}, b_n^{jn})\right). \end{aligned} \tag{4.31}$$

Adding Equations (4.30) and (4.31) together, we obtain

$$\begin{aligned}&\Delta_p(\bar{x}, x_{n+1}) + \Delta_p(\bar{y}, y_{n+1}) \leq \beta_n[\Delta_p(\mu, \bar{x}) + \Delta_p(\vartheta, \bar{y})] + (1-\beta_n)[\Delta_p(\bar{x}, s_n) + \Delta_p(\bar{y}, t_n)]\\ &\quad - (1-\beta_n)\left(1-\frac{\tau_n}{\tau_{n+1}}\kappa\right)\left(\Delta_p(a_n^{in}, s_n) + \Delta_p(z_n^{in}, a_n^{in})\right)\\ &\quad - (1-\beta_n)\left(1-\frac{\lambda_n}{\lambda_{n+1}}\eta\right)\left(\Delta_p(b_n^{jn}, t_n) + \Delta_p(h_n^{jn}, b_n^{jn})\right). \end{aligned} \tag{4.32}$$

Applying Equation (4.17) in Equation (4.32), we obtain

$$\begin{aligned}&\Delta_p(\bar{x}, x_{n+1}) + \Delta_p(\bar{y}, y_{n+1}) \leq \beta_n\left[\Delta_p(\mu, \bar{x}) + \Delta_p(\vartheta, \bar{y})\right] + (1-\beta_n)\left[\Delta_p(\bar{x}, x_n) + \Delta_p(\bar{y}, y_n)\right]\\ &\quad - (1-\beta_n)\rho_n\left[\|Ax_n - By_n\|^p - \frac{\rho_n^{q-1}}{q}\left(C_q\|A^*J^p_{E_3}(Ax_n - By_n)\|^q\right.\right.\\ &\quad \left.\left. + Q_q\|B^*J^p_{E_3}(Ax_n - By_n)\|^q\right)\right] - (1-\beta_n)\left(1-\frac{\tau_n}{\tau_{n+1}}\kappa\right)\left(\Delta_p(a_n^{in}, s_n) + \Delta_p(z_n^{in}, a_n^{in})\right)\\ &\quad - (1-\beta_n)\left(1-\frac{\lambda_n}{\lambda_{n+1}}\eta\right)\left(\Delta_p(b_n^{jn}, t_n) + \Delta_p(h_n^{jn}, b_n^{jn})\right). \end{aligned} \tag{4.33}$$

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By Equation (4.29), Assumption 3.1(2)(a) and (4.33), we obtain

$$\begin{aligned}
&\limsup_{k\to\infty}\left((1-\beta_{n_k})\rho_{n_k}\left[\|Ax_{n_k}-By_{n_k}\|^p-\frac{\rho_{n_k}^{q-1}}{q}\left(C_q\|A^*J_p^{E3}(Ax_{n_k}-By_{n_k})\|^q\right.\right.\right.\\
&\left.\left.\left.+Q_q\|B^*J_p^{E3}(Ax_{n_k}-By_{n_k})\|^q\right)\right]\right)\\
&\leq \limsup_{k\to\infty}\left(\beta_{n_k}\left[\Delta_p(\mu,\bar{x})+\Delta_p(\vartheta,\bar{y})\right]+(1-\beta_{n_k})\left[\Delta_p(\bar{x},x_{n_k})+\Delta_p(\bar{y},y_{n_k})\right]\right.\\
&\left.-\left[\Delta_p(\bar{x},x_{n_k+1})+\Delta_p(\bar{y},y_{n_k+1})\right]\right)\\
&=-\liminf_{k\to\infty}\left(\left[\Delta_p(\bar{x},x_{n_k+1})+\Delta_p(\bar{y},y_{n_k+1})\right]-\left[\Delta_p(\bar{x},x_{n_k})+\Delta_p(\bar{y},y_{n_k})\right]\right)\\
&\leq 0. \qquad (4.34)
\end{aligned}$$

In the same vein as in Equation (4.34), we get using Equation (4.29), Assumption 3.1(2)(a) and Equation (4.33) that

$$\begin{aligned}
&\limsup_{k\to\infty}\left((1-\beta_{n_k})\left(1-\frac{\tau_{n_k}}{\tau_{n_k+1}}\kappa\right)\left(\Delta_p(a_{n_k}^{in_k},s_{n_k})+\Delta_p(z_{n_k}^{in_k},a_{n_k}^{in_k})\right)\right.\\
&\quad\left.+\left(1-\frac{\lambda_{n_k}}{\lambda_{n_k+1}}\eta\right)\left(\Delta_p(b_{n_k}^{jn_k},t_{n_k})+\Delta_p(h_{n_k}^{jn_k},b_{n_k}^{jn_k})\right)\right)\\
&\quad\leq \limsup_{k\to\infty}\left(\beta_{n_k}\left[\Delta_p(\mu,\bar{x})+\Delta_p(\vartheta,\bar{y})\right]+(1-\beta_{n_k})\left[\Delta_p(\bar{x},x_{n_k})+\Delta_p(\bar{y},y_{n_k})\right]\right.\\
&\quad\left.-\left[\Delta_p(\bar{x},x_{n_k+1})+\Delta_p(\bar{y},y_{n_k+1})\right]\right)\\
&\quad=-\liminf_{k\to\infty}\left(\left[\Delta_p(\bar{x},x_{n_k+1})+\Delta_p(\bar{y},y_{n_k+1})\right]-\left[\Delta_p(\bar{x},x_{n_k})+\Delta_p(\bar{y},y_{n_k})\right]\right)\\
&\quad\leq 0. \qquad (4.35)
\end{aligned}$$

Now, suppose we let $\varrho_{n_k}=C_q\|A^*J_{E_3}^p(Ax_{n_k}-By_{n_k})\|^q+Q_q\|B^*J_{E_3}^p(Ax_{n_k}-By_{n_k})\|^q$. Using the condition we placed on our step size ρ_{n_k}, we have that

$$\rho_{n_k}^{q-1}<\frac{q\|Ax_{n_k}-By_{n_k}\|^p}{\varrho_{n_k}}-\zeta,$$

it follows that

$$\rho_{n_k}^{q-1}\varrho_{n_k}<q\|Ax_{n_k}-By_{n_k}\|^p-\zeta\varrho_{n_k}, \qquad (4.36)$$

Hence, by Equations (4.34) and (4.36), we have

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$$\frac{\zeta \varrho_{n_k}}{q} < \left(\|Ax_{n_k} - By_{n_k}\|^p - \frac{\rho_{n_k}^{q-1}}{q} \varrho_{n_k} \right) \to 0, \quad \text{as } k \to \infty.$$

Thus, $C_q\|A^* J_{E_3}^p (Ax_{n_k} - By_{n_k})\|^q + Q_q\|B^* J_{E_3}^p (Ax_{n_k} - By_{n_k})\|^q \to 0 \quad \text{as } k \to \infty,$ which implies that

$$\lim_{k\to\infty} \|A^* J_{E_3}^p (Ax_{n_k} - By_{n_k})\|^q = 0 \tag{4.37}$$

and

$$\lim_{k\to\infty} \|B^* J_{E_3}^p (Ax_{n_k} - By_{n_k})\|^q = 0. \tag{4.38}$$

Additionally, we obtain from Equation (4.34) that

$$\begin{aligned}
&\limsup_{k\to\infty} \left((1-\beta_{n_k})\rho_{n_k} \left[\|Ax_{n_k} - By_{n_k}\|^p \right] \right) \\
&\leq \limsup_{k\to\infty} \left(\beta_{n_k} \left[\Delta_p(\mu, \bar{x}) + \Delta_p(\vartheta, \bar{y}) \right] + (1-\beta_{n_k}) \left[\Delta_p(\bar{x}, x_{n_k}) + \Delta_p(\bar{y}, y_{n_k}) \right] \right. \\
&\left. - \left[\Delta_p(\bar{x}, x_{n_k+1}) + \Delta_p(\bar{y}, y_{n_k+1}) \right] \right) + \limsup_{k\to\infty} (1-\beta_{n_k}) \frac{\rho_{n_k}^q}{q} \varrho_{n_k} \\
&= -\liminf_{k\to\infty} \left(\left[\Delta_p(\bar{x}, x_{n_k+1}) + \Delta_p(\bar{y}, y_{n_k+1}) \right] - \left[\Delta_p(\bar{x}, x_{n_k}) + \Delta_p(\bar{y}, y_{n_k}) \right] \right) \\
&\leq 0.
\end{aligned} \tag{4.39}$$

Thus, we conclude from Equations (4.34), (4.35) and (4.39) that

$$\lim_{k\to\infty} \|Ax_{n_k} - By_{n_k}\| = 0, \tag{4.40}$$

$$\begin{cases}
\lim_{k\to\infty} \Delta_p(a_{n_k}^{in_k}, s_{n_k}) = 0, \\
\lim_{k\to\infty} \Delta_p(z_{n_k}^{in_k}, a_{n_k}^{in_k}) = 0, \\
\lim_{k\to\infty} \Delta_p(b_{n_k}^{jn_k}, t_{n_k}) = 0, \\
\lim_{k\to\infty} \Delta_p(h_{n_k}^{jn_k}, b_{n_k}^{jn_k}) = 0.
\end{cases} \tag{4.41}$$

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Therefore, by Lemma 2.11, we obtain

$$\begin{cases} \lim_{k\to\infty} \|a_{n_k}^{in_k} - s_{n_k}\| = 0, \\ \lim_{k\to\infty} \|z_{n_k}^{in_k} - a_{n_k}^{in_k}\| = 0, \\ \lim_{k\to\infty} \|b_{n_k}^{jn_k} - t_{n_k}\| = 0, \\ \lim_{k\to\infty} \|h_{n_k}^{jn_k} - b_{n_k}^{jn_k}\| = 0. \end{cases} \tag{4.42}$$

Observe that by Equation (4.42) and Lemma 2.11, we have

$$\lim_{k\to\infty} \|s_{n_k} - z_{n_k}^{in_k}\| = 0, \qquad \lim_{k\to\infty} \Delta_p(s_{n_k}, z_{n_k}^{in_k}) = 0. \tag{4.43}$$

In like manner, we have

$$\lim_{k\to\infty} \|t_{n_k} - h_{n_k}^{jn_k}\| = 0, \qquad \lim_{k\to\infty} \Delta_p(t_{n_k}, h_{n_k}^{jn_k}) = 0. \tag{4.44}$$

By the definitions of i_n and j_n, it follows that

$$\lim_{k\to\infty} \Delta_p(s_{n_k}, z_{n_k}^{i}) = 0,\ i = 1, 2, \ldots, N \quad \text{and} \quad \lim_{k\to\infty} \Delta_p(t_{n_k}, h_{n_k}^{j}) = 0,\ j = 1, 2, \ldots, M. \tag{4.45}$$

Consequently, we have

$$\lim_{k\to\infty} \|s_{n_k} - z_{n_k}^{i}\| = 0,\ i = 1, 2, \ldots, N \quad \text{and} \quad \lim_{k\to\infty} \|t_{n_k}, h_{n_k}^{j}\| = 0,\ j = 1, 2, \ldots, M. \tag{4.46}$$

From Equation (4.1) and by applying the three-point identity (2.1) and (4.46), we have

$$\begin{aligned} \left(1 - \kappa\frac{\tau_{n_k}}{\tau_{n_k+1}}\right) \Delta_p(a_{n_k}^{i}, s_{n_k}) &\leq \Delta_p(\bar{x}, s_n) - \Delta_p(\bar{x}, z_{n_k}^{i}) \\ &\leq \Delta_p(\bar{x}, s_{n_k}) - \Delta_p(\bar{x}, z_{n_k}^{i}) + \Delta_p(s_{n_k}, z_{n_k}^{i}) \\ &= \langle \bar{x} - s_{n_k}, J_{E_1}^p(z_{n_k}^{i}) - J_{E_1}^p(s_{n_k})\rangle \to 0, \quad k \to \infty. \end{aligned}$$

Hence, we have

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$$\Delta_p(a^i_{n_k}, s_{n_k}) \to 0, \quad k \to \infty, \quad i = 1, 2, \ldots, N.$$

Consequently, we obtain

$$\|a^i_{n_k} - s_{n_k}\| \to 0, \quad k \to \infty, \quad i = 1, 2, \ldots, N. \tag{4.47}$$

Following similar procedure, we have

$$\|b^j_{n_k} - t_{n_k}\| \to 0, \quad k \to \infty, \quad j = 1, 2, \ldots, M. \tag{4.48}$$

Furthermore, using Equations (4.22) and (4.29), we have

$$\begin{aligned}
&\limsup_{k\to\infty}(1-\beta_{n_k})\left(\frac{W_q(\alpha_{n_k,s})}{q} g\left(\|J^p_{E_1}(\bar{z}_{n_k}) - J^p_{E_1}(D_s\bar{z}_{n_k})\|\right)\right. \\
&\qquad \left. + \frac{W_q(\eta_{n_k,t})}{q} g\left(\|J^p_{E_2}(\bar{\theta}_{n_k}) - J^p_{E_2}(G_t\bar{\theta}_{n_k})\|\right)\right) \\
&\qquad \le \limsup_{k\to\infty}\left(\beta_{n_k}\left[\Delta_p(\mu,\bar{x}) + \Delta_p(\vartheta,\bar{y})\right] + (1-\beta_{n_k})\left[\Delta_p(\bar{x},x_{n_k}) + \Delta_p(\bar{y},y_{n_k})\right]\right. \\
&\qquad \left. - \left[\Delta_p(\bar{x},x_{n_k+1}) + \Delta_p(\bar{y},y_{n_k+1})\right]\right) \\
&\qquad = -\liminf_{k\to\infty}\left(\left[\Delta_p(\bar{x},x_{n_k+1}) + \Delta_p(\bar{y},y_{n_k+1})\right] - \left[\Delta_p(\bar{x},x_{n_k}) + \Delta_p(\bar{y},y_{n_k})\right]\right) \\
&\qquad \le 0.
\end{aligned} \tag{4.49}$$

Thus,

$$\begin{aligned}
\lim_{k\to\infty}&\left(\frac{W_q(\alpha_{n_k,s})}{q} g\left(\|J^p_{E_1}(\bar{z}_{n_k}) - J^p_{E_1}(D_s\bar{z}_{n_k})\|\right)\right. \\
&\left. + \frac{W_q(\eta_{n_k,t})}{q} g\left(\|J^p_{E_2}(\bar{\theta}_{n_k}) - J^p_{E_2}(G_t\bar{\theta}_{n_k})\|\right)\right) = 0.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\lim_{k\to\infty} g\left(\|J^p_{E_1}(\bar{z}_{n_k}) - J^p_{E_1}(D_s\bar{z}_{n_k})\|\right) = 0, \ s = 1, 2, \ldots, l, \\
&\lim_{k\to\infty} g\left(\|J^p_{E_2}(\bar{\theta}_{n_k}) - J^p_{E_2}(G_t\bar{\theta}_{n_k})\|\right) = 0, \ t = 1, 2, \ldots, m.
\end{aligned}$$

By the property of g, and sine $J^q_{E^*_1}$ and $J^q_{E^*_2}$ are norm-to-norm uniformly continuous on bounded subsets of E_1 and E_2, respectively, then we obtain

$$\lim_{k\to\infty}\|D_s\bar{z}_{n_k} - \bar{z}_{n_k}\| = 0, \quad \forall s = 1, 2, \ldots, l \tag{4.50}$$

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and

$$\lim_{k\to\infty} \|G_t\bar{\theta}_{n_k} - \bar{\theta}_{n_k}\| = 0, \quad \forall t = 1, 2, \ldots, m. \tag{4.51}$$

Observe that from Equation (3.1) and by Equation (4.40), we obtain

$$\begin{aligned}\|J^p_{E_1}(s_{n_k}) - J^p_{E_1}(x_{n_k})\| &= \|J^p_{E_1}(x_{n_k}) - \rho_{n_k} A^* J^p_{E_3}(Ax_{n_k} - By_{n_k}) - J^p_{E_1}(x_{n_k})\| \\ &= \rho_{n_k}\|A^* J^p_{E_3}(Ax_{n_k} - By_{n_k})\| \to 0 \quad \text{as } k \to \infty. \end{aligned} \tag{4.52}$$

Also, because E_1 is uniformly smooth, $J^q_{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1, then we have

$$\lim_{k\to\infty} \|s_{n_k} - x_{n_k}\| = 0. \tag{4.53}$$

In the same vein, we get

$$\lim_{k\to\infty} \|t_{n_k} - y_{n_k}\| = 0. \tag{4.54}$$

Moreover, it is easy to see from Equations (4.46) and (4.53)

$$\lim_{k\to\infty} \|z^i_{n_k} - x_{n_k}\| \le \lim_{k\to\infty} \|z^i_{n_k} - s_{n_k}\| + \lim_{k\to\infty} \|s_{n_k} - x_{n_k}\| = 0, \quad \forall i = 1, 2, \ldots, N. \tag{4.55}$$

In the same way, we obtain from Equations (4.46) and (4.54) that

$$\|h^j_{n_k} - y_{n_k}\| \le \|h^j_{n_k} - t_{n_k}\| + \|t_{n_k} - y_{n_k}\| \to 0 \quad \text{as } k \to \infty, \ \forall j = 1, 2, \ldots, M. \tag{4.56}$$

Moreover, we obtain from Equations (3.1) and (4.50) that

$$\begin{aligned}\lim_{k\to\infty}\|J^p_{E_1}(u_{n_k}) - J^p_{E_1}(\bar{z}_{n_k})\| &= \|\alpha_{n_k,0}J^p_{E_1}(\bar{z}_{n_k}) + \sum_{s=1}^{l} \alpha_{n_k,s} J^p_{E_1}(D_s\bar{z}_{n_k}) - J^p_{E_1}(\bar{z}_{n_k})\| \\ &\le \alpha_{n_k,0}\|J^p_{E_1}(\bar{z}_{n_k}) - J(\bar{z}_{n_k})\| + \sum_{s=1}^{l} \alpha_{n_k,s}\|J^p_{E_1}(D_s\bar{z}_{n_k}) - J^p_{E_1}(\bar{z}_{n_k})\|,\end{aligned}$$

which implies that

$$\lim_{k\to\infty} \|J^p_{E_1}(u_{n_k}) - J^p_{E_1}\bar{z}_{n_k}\| = 0.$$

By the uniform continuity of $J^q_{E_1^*}$ on bounded subsets of E_1^*, we have

$$\lim_{k\to\infty} \|u_{n_k} - \bar{z}_{n_k}\| = 0. \tag{4.57}$$

Hence, from Equations (4.55) and (4.57), we obtain

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$$\lim_{k\to\infty} \|u_{n_k} - z^i_{n_k}\| = 0, \quad \forall i = 1, 2, \ldots, N. \tag{4.58}$$

Similarly, we obtain

$$\lim_{k\to\infty} \|v_{n_k} - h^j_{n_k}\| = 0, \quad \forall j = 1, 2, \ldots, M. \tag{4.59}$$

It is easy to see from Equations (4.55) and (4.58) that

$$\|u_{n_k} - x_{n_k}\| \le \|u_{n_k} - z^i_{n_k}\| + \|z^i_{n_k} - x_{n_k}\| \to 0 \quad \text{as } k \to \infty. \tag{4.60}$$

Similarly, we obtain from Equations (4.56) and (4.59) that

$$\|v_{n_k} - y_{n_k}\| \le \|v_{n_k} - h^j_{n_k}\| + \|h^j_{n_k} - y_{n_k}\| \to 0 \quad \text{as } k \to \infty. \tag{4.61}$$

Furthermore, from Equation (3.1) and the fact that $\lim_{k\to\infty} \beta_{n_k} = 0$, we obtain

$$\lim_{k\to\infty} \|J^p_{E_1}(x_{n_k+1}) - J^p_{E_1}(u_{n_k})\| = 0.$$

In the same way, we get

$$\lim_{k\to\infty} \|J^p_{E_2}(y_{n_k+1}) - J^p_{E_2}(v_{n_k})\| = 0.$$

Since $J^p_{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1, we obtain

$$\lim_{k\to\infty} \|x_{n_k+1} - u_{n_k}\| = 0. \tag{4.62}$$

Similarly, we get

$$\lim_{k\to\infty} \|y_{n_k+1} - v_{n_k}\| = 0. \tag{4.63}$$

Hence, from Equations (4.60) and (4.62), we obtain

$$\lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| = 0. \tag{4.64}$$

In the same vein, from Equations (4.61) and (4.63), we get

$$\lim_{k\to\infty} \|y_{n_k+1} - y_{n_k}\| = 0. \tag{4.65}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, then $w_\omega(x_n)$ and $w_\omega(y_n)$ are nonempty. Now, let $(x^*, y^*) \in w_\omega(x_n, y_n)$ be arbitrary elements. Then, there exists subsequences $\{x_{n_k}\}$ of

$\{x_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ that converge weakly to $x^* \in E_1$ and $y^* \in E_2$, respectively. Also, from Equations (4.55) and (4.56), $\{z^i_{n_k}\}$ converges weakly to $x^* \in C_i$ for each $i = 1, 2, \ldots, N$ and $\{h^j_{n_k}\}$ converges weakly to $y^* \in Q_j$ for each $j = 1, 2, \ldots, M$. Using Equations (4.50) and (4.51) and by the demiclosedness of $I - D_s$ and $I - G_t$, we obtain

$$x^* \in F(D_s),\ \forall s = 1, 2, \ldots, l \quad \text{and} \quad y^* \in F(G_t),\ \forall t = 1, 2, \ldots, m, \tag{4.66}$$

which implies that

$$x^* \in \bigcap_{s=1}^{l} F(D_s) \quad \text{and} \quad y^* \in \bigcap_{t=1}^{m} F(G_t). \tag{4.67}$$

Next, recall that

$$a^i_{n_k} = \arg\min_{\sigma \in C_i} \{f_i(s_{n_k}, \sigma) + \frac{1}{\tau_n} \Delta_p(\sigma, s_{n_k})\}.$$

Using Lemma 2.6 and applying condition (C4), we get

$$0 \in \partial_2(\tau_{n_k} f_i(s_{n_k}, a^i_{n_k}) + \Delta_p(a^i_{n_k}, s_{n_k})) + N_{C_i}(a^i_{n_k}).$$

Hence, there exists $\varsigma^i_{n_k} \in \partial_2 f_i(s_{n_k}, a^i_{n_k})$ and $\bar{\varsigma}^i_{n_k} \in N_{C_i}(a^i_{n_k})$ such that

$$\tau_{n_k} \varsigma^i_{n_k} + J^p_{E_1}(a^i_{n_k}) - J^p_{E_1}(s_{n_k}) + \bar{\varsigma}^i_{n_k} = 0. \tag{4.68}$$

Since $\bar{\varsigma}^i_{n_k} \in N_{C_i}(a^i_{n_k})$, $\langle \omega - a^i_{n_k}, \bar{\varsigma}^i_{n_k} \rangle \leq 0$ for all $\omega \in C_i$, then this together with Equation (4.68) gives

$$\tau_{n_k} \langle \omega - a^i_{n_k}, \varsigma^i_{n_k} \rangle \geq \langle a^i_{n_k} - \omega, J^p_{E_1}(a^i_{n_k}) - J^p_{E_1}(s_{n_k}) \rangle, \quad \forall \omega \in C_i. \tag{4.69}$$

Again, since $\varsigma^i_{n_k} \in \partial_2 f_i(s_{n_k}, a^i_{n_k})$, we obtain

$$f_i(s_{n_k}, \omega) - f_i(s_{n_k}, a^i_{n_k}) \geq \langle w - a^i_{n_k}, \varsigma^i_{n_k} \rangle \quad \forall \omega \in C_i. \tag{4.70}$$

Combining Equations (4.69) and (4.70), we obtain

$$\tau_{n_k} \left[f_i(s_{n_k}, \omega) - f_i(s_{n_k}, a^i_{n_k}) \right] \geq \langle a^i_{n_k} - \omega, J^p_{E_1}(a^i_{n_k}) - J^p_{E_1}(s_{n_k}) \rangle, \quad \forall \omega \in C_i,$$

which implies that

$$\begin{aligned} \tau_{n_k} \left[f_i(s_{n_k}, a^i_{n_k}) - f_i(s_{n_k}, \omega) \right] &\leq \langle J^p_{E_1}(s_{n_k}) - J^p_{E_1}(a^i_{n_k}), a^i_{n_k} - \omega \rangle \\ &\leq \| J^p_{E_1}(s_{n_k}) - J^p_{E_1}(a^i_{n_k}) \| \| a^i_{n_k} - \omega \|. \end{aligned} \tag{4.71}$$

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Since $J^p_{E_1}$ is uniformly continuous, applying Equation (4.47) to Equation (4.71) and using Equation (4.53) together with the fact that $x_{n_k} \rightharpoonup x^*$, we get

$$-f_i(x^*, \omega) \leq 0, \quad \forall \omega \in C_i,\ i = 1, 2, \ldots, N,$$

which implies that

$$f_i(x^*, \omega) \geq 0, \quad \forall \omega \in C_i,\ i = 1, 2, \ldots, N.$$

Hence, we have

$$x^* \in \bigcap_{i=1}^{N} \mathrm{EP}(C_i, f_i).$$

Similarly, we obtain

$$g_j(y^*, z) \geq 0, \quad \forall z \in Q_j,\ j = 1, 2, \ldots, M,$$

which implies that

$$y^* \in \bigcap_{j=1}^{M} \mathrm{EP}(Q_j, g_j).$$

Next, recall that $\{x_{n_k}\}$ and $\{y_{n_k}\}$ converges to x^* and y^*, respectively, where $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear operators. Then, by Equation (4.40) and the weakly lower semi-continuity of the norm, we have

$$\|Ax^* - By^*\| \leq \liminf_{k \to \infty} \|Ax_{n_k} - By_{n_k}\| = 0,$$

which implies that

$$Ax^* = By^*$$

Since $(x^*, y^*) \in w_\omega(x_n, y_n)$ is an arbitrary element, then it follows that

$$w_\omega(x_n, y_n) \subset \Upsilon.$$

Next, by the boundedness of $\{x_{n_k}\}$ and $\{y_{n_k}\}$, there exist subsequences $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ and $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup \hat{x} \in E_1$ and $y_{n_{k_j}} \rightharpoonup \hat{y} \in E_2$ and

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$$\lim_{j\to\infty}\left(\langle J^p_{E_1}(\mu)-J^p_{E_1}(\bar{x}),x_{n_{k_j}}-\bar{x}\rangle+\langle J^p_{E_2}(\vartheta)-J^p_{E_2}(\bar{y}),y_{n_{k_j}}-\bar{y}\rangle\right)$$
$$=\limsup_{k\to\infty}\left(\langle J^p_{E_1}(\mu)-J^p_{E_1}(\bar{x}),x_{n_k}-\bar{x}\rangle+\langle J^p_{E_2}(\vartheta)-J^p_{E_2}(\bar{y}),y_{n_k}-\bar{y}\rangle\right)$$

Since $\bar{x}=\Pi_\Upsilon(\mu)$ and $\bar{y}=\Pi_\Upsilon(\vartheta)$, then by Equation (2.5), (4.64) and (4.65), we have

$$\begin{aligned}\limsup_{k\to\infty}&\left(\langle J^p_{E_1}(\mu)-J^p_{E_1}(\bar{x}),x_{n_{k+1}}-\bar{x}\rangle+\langle J^p_{E_2}(\vartheta)-J^p_{E_2}(\bar{y}),y_{n_{k+1}}-\bar{y}\rangle\right)\\
&=\limsup_{k\to\infty}\left(\langle J^p_{E_1}(\mu)-J^p_{E_1}(\bar{x}),x_{n_k}-\bar{x}\rangle+\langle J^p_{E_1}(\mu)-J^p_{E_1}(\bar{x}),x_{n_{k+1}-x_{n_k}}\rangle\right)\\
&+\limsup_{k\to\infty}\left(\langle J^p_{E_2}(\vartheta)-J^p_{E_2}(\bar{y}),y_{n_k}-\bar{y}\rangle+\langle J^p_{E_2}(\vartheta)-J^p_{E_2}(\bar{y}),y_{n_{k+1}-y_{n_k}}\rangle\right)\\
&=\lim_{j\to\infty}\left(\langle J^p_{E_1}(u)-J^p_{E_1}(\bar{x}),x_{n_{k_j}}-\bar{x}\rangle+\langle J^p_{E_2}(\vartheta)-J^p_{E_2}(\bar{y}),y_{n_{k_j}}-\bar{y}\rangle\right)\\
&=\langle J^p_{E_1}(\mu)-J^p_{E_1}(\bar{x}),\hat{x}-\bar{x}\rangle+\langle J^p_{E_2}(\vartheta)-J^p_{E_2}(\bar{y}),\hat{y}-\bar{y}\rangle\\
&\leq 0.\end{aligned}\tag{4.72}$$

Hence, by Equation (4.72), we have $\limsup_{k\to\infty}\chi_{n_k}\leq 0$. Therefore, by applying Lemma 2.14 to Equation (4.28), it follows that $\{(x_n,y_n)\}$ converges strongly to $(\bar{x},\bar{y})\in\Upsilon$ as required. □

Some corollaries

The following consequent result can easily be obtained from Theorem 4.5 by setting $l=m=N=M=1$.

Corollary. *Let E_1, E_2 and E_3 be three p-uniformly convex Banach space and C,Q be nonempty, closed and convex subsets of E_1 and E_2, respectively. Suppose $f:C\times C\to\mathbb{R}$ and $g:Q\times Q\to\mathbb{R}$ be bifunctions satisfying (C1)–(C4) of* ***Assumption A****. Let $A:E_1\to E_3$ and $B:E_2\to E_3$ be bounded linear operators and let $D:E_1\to E_1$ and $G:E_2\to E_2$ be Bregman quasi-nonexpansive mappings such that $I-D$ and $I-G$ are demiclosed at zero and $\Upsilon:=\{\bar{x}\in F(D)\cap EP(C,f),\ \bar{y}\in F(G)\cap EP(Q,g):A\bar{x}=B\bar{y}\}\neq\emptyset$. Suppose other conditions of Theorem 4.5 hold. For fixed $\mu\in E_1$ and $\vartheta\in E_2$ and initial point*

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$(x_0, y_0) \in E_1 \times E_2$, let $\{(x_n, y_n)\}$ be a sequence generated as follows:

$$\begin{cases} s_n = J^q_{E_1^*}\left(J^p_{E_1}(x_n) - \rho_n A^* J^p_{E_3}(Ax_n - By_n)\right), \\ y_n = \arg\min\{f(s_n, \sigma) + \frac{1}{\tau_n}\Delta_p(\sigma, s_n) : \quad \sigma \in C\}, \\ z_n = \arg\min\{f(y_n, \sigma) + \frac{1}{\tau_n}\Delta_p(\sigma, s_n) : \quad \sigma \in C\}. \\ u_n = J_q^{E_1^*}\left(\alpha_{n,0} J^p_{E_1}(z_n) + \alpha_{n,1} J^p_{E_1}(Dz_n)\right) \\ x_{n+1} = J^q_{E_1^*}\left(\beta_n J^p_{E_1}(\mu) + (1-\beta_n) J^p_{E_1}(u_n)\right), \\ t_n = J^q_{E_2^*}\left(J^p_{E_2}(y_n) + \rho_n B^* J^p_{E_3}(Ax_n - By_n)\right), \\ b_n = \arg\min\{g(t_n, \varphi) + \frac{1}{\lambda_n}\Delta_p(\varphi, t_n) : \quad \varphi \in Q\}, \\ h_n = \arg\min\{g(b_n, \varphi) + \frac{1}{\lambda_n}\Delta_p(\varphi, t_n) : \quad \varphi \in Q\}. \\ v_n = J^q_{E_1^*}\left(\eta_{n,0} J^p_{E_2}(h_n) + \eta_{n,1} J^p_{E_2}(Gh_n)\right) \\ y_{n+1} = J^q_{E_2^*}\left(\beta_n J^p_{E_2}(\vartheta) + (1-\beta_n) J_p^{E_2}(v_n)\right), \end{cases} \tag{4.73}$$

where $\rho_n \in \left(\zeta, \left(\frac{q\|Ax_n - By_n\|^p}{C_q\|A^* J^p_{E_3}(Ax_n - By_n)\|^q + Q_q\|B^* J^p_{E_3}(Ax_n - By_n)\|^q} - \zeta\right)^{\frac{1}{q-1}}\right)$, *$n \in \Omega$, for small enough ζ; C_q and Q_q are constants of smoothness of E_1 and E_2, respectively. Otherwise, $\rho_n = \rho$ (ρ being any nonnegative value), where the set of indexes $\Omega = \{n : Ax_n - By_n \neq 0\}$.*

$$\tau_{n+1} = \begin{cases} \min\left\{\tau_n, \min\left\{\frac{\kappa(\Delta_p(y_n, s_n) + \Delta_p(z_n, y_n))}{f(s_n, z_n) - f(s_n, y_n) - f(y_n, z_n)}\right\}\right\}, & \text{if } f(s_n, z_n) - f(s_n, y_n) \\ & \quad -f(y_n, z_n) > 0, \\ \tau_n, & \text{otherwise.} \end{cases} \tag{4.74}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \min\left\{\frac{\epsilon(\Delta_p(b_n, t_n) + \Delta_p(h_n, b_n))}{g(t_n, h_n) - g(t_n, b_n) - g(b_n, h_n)}\right\}\right\}, & \text{if } g(t_n, h_n) - g(t_n, b_n) \\ & \quad -g(b_n, h_n) > 0, \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{4.75}$$

Then, the sequence $\{(x_n, y_n)\}$ generated by Equation (4.73) converges strongly to $(\bar{x}, \bar{y}) \in \Upsilon$.

Let $E_r = H_r$, $r = 1, 2, 3$ be real Hilbert spaces, then we obtain the following consequent result for approximating a common solution of multiple sets split equality pseudomonotone EP and common fixed point problems of quasi-nonexpansive mappings in real Hilbert spaces.

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Corollary 4.7. *Let H_1, H_2 and H_3 be three real Hilbert spaces, and let C_i and Q_j be nonempty closed and convex subsets of H_1 and H_2, respectively, for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$. Suppose $f_i : C_i \times C_i \to \mathbb{R}$ and $g_j : Q_j \times Q_j \to \mathbb{R}$ are bifunctions satisfying (C1)–(C4) of **Assumption A**. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators. Let $D_s : H_1 \to H_1$ and $G_t : H_2 \to H_2$ be quasi-nonexpansive mappings such that $\Upsilon := \{\bar{x} \in \bigcap_{s=1}^{l} F(D_s) \cap \bigcap_{i=1}^{N} EP(C_i, f_i),\ \bar{y} \in \bigcap_{t=1}^{m} F(G_t) \cap \bigcap_{j=1}^{M} EP(Q_j, g_j) : A\bar{x} = B\bar{y}\} \neq \emptyset$. Suppose other conditions of Theorem 4.5 hold. For fixed $\mu \in H_1$ and $\vartheta \in H_2$ and initial point $(x_0, y_0) \in H_1 \times H_2$, let $\{(x_n, y_n)\}$ be a sequence generated as follows:*

$$\begin{cases} s_n = (x_n - \rho_n A^*(Ax_n - By_n)), \\ a_n^i = \arg\min \left\{ \tau_n f_i(s_n, \sigma) + \frac{1}{2}\|\sigma - s_n\|^2 : \quad \sigma \in C_i \right\}, \\ z_n^i = \arg\min \left\{ \tau_n f_i(a_n^i, \sigma) + \frac{1}{2}\|\sigma - s_n\|^2 : \quad \sigma \in C_i \right\}. \\ \text{Obtain the farthest element of } z_n^i \text{ from } s_n, \text{ i.e.,} \\ i_n \in \arg\max \left\{ \frac{1}{2}\|s_n - z_n^i\|^2 : i = 1, \ldots, N \right\}. \\ \text{Set } z_n^{i_n} = \bar{z}_n \\ u_n = \alpha_{n,0}\bar{z}_n + \sum_{s=1}^{l} \alpha_{n,s}(D_s \bar{z}_n) \\ x_{n+1} = \beta_n(\mu) + (1 - \beta_n)(u_n), \\ t_n = (y_n + \rho_n B^*(Ax_n - By_n)), \\ b_n^j = \arg\min \left\{ \lambda_n g_j(t_n, \varphi) + \frac{1}{2}\|\varphi - t_n\|^2 : \quad \varphi \in Q_j \right\}, \\ h_n^j = \arg\min \left\{ \lambda_n g_j(b_n^j, \varphi) + \frac{1}{2}\|\varphi - t_n\|^2 : \quad \varphi \in Q_j \right\}. \\ \text{Obtain the farthest element of } h_n^j \text{ from } t_n, \text{ i.e.,} \\ j_n \in \arg\max \left\{ \frac{1}{2}\|t_n, h_n^j\|^2 : j = 1, \ldots, M \right\}. \\ \text{Set } h_n^{j_n} = \bar{\theta}_n \\ v_n = \eta_{n,0}\bar{\theta}_n + \sum_{t=1}^{m} \eta_{n,t}(G_t \bar{\theta}_n) \\ y_{n+1} = \beta_n(\vartheta) + (1 - \beta_n)(v_n), \end{cases} \tag{4.76}$$

where $\rho_n \in \left(\zeta, \left(\frac{2\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2} - \zeta\right)\right)$, $n \in \Omega$, for small enough ζ. Otherwise, $\rho_n = \rho$ (ρ being any nonnegative value), where the set of indexes $\Omega = \{n : Ax_n - By_n \neq 0\}$.

$$\tau_{n+1} = \begin{cases} \min \left\{ \tau_n, \min\limits_{1 \le i \le N} \left\{ \frac{\kappa}{2} \frac{\|a_n^i - s_n\|^2 + \|z_n^i - a_n^i\|^2}{f_i(s_n, z_n^i) - f_i(s_n, a_n^i) - f_i(a_n^i, z_n^i)} \right\} \right\}, & \text{if } f_i(s_n, z_n^i) - f_i(s_n, a_n^i) \\ & \quad - f_i(a_n^i, z_n^i) > 0, \\ \tau_n, & \text{otherwise} \end{cases} \tag{4.77}$$

and

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$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \min\limits_{1\le j\le M}\left\{\frac{\epsilon}{2}\frac{\|b_n^j - t_n\|^2 + \|h_n^j - b_n^j\|^2}{g_j(t_n,h_n^j) - g_j(t_n,b_n^j) - g_j(b_n^j,h_n^j)}\right\}\right\}, & \text{if } g_j(t_n,h_n^j) - g_j(t_n,b_n^j) \\ & -g_j(b_n^j,h_n^j) > 0, \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{4.78}$$

Then the sequence $\{(x_n, y_n)\}$ *generated by Equation (4.76) converges strongly to* $(\bar{x}, \bar{y}) \in \Upsilon$.

5. Application

5.1. Multiple set split equality variational inequality problem

In this section, we apply our result to study the multiple set split equality variational inequality problem (MSSEVIP).
Let $U : C \to E^*$ be a nonlinear mapping. The classical VIP is formulated as locating a point

$$x^* \in C \quad \text{such that } \langle \bar{x} - x^*, U(x^*)\rangle \ge 0, \ \forall \bar{x} \in C. \tag{5.1}$$

The solution set of VIP (5.1) is denoted by $\mathrm{VI}(C, U)$. Variational inequalities have been found very applicable in several real-world problems such as optimization problems, minimax theorems, differential equations and in certain applications to economic theory and mechanics. For more details on variational inequalities, see [2, 35, 36, 44] and the references therein.
Now we consider the MSSEVIP defined as follows:

$$\begin{aligned} &\text{find } \bar{x} \in C_i \quad \text{such that } \langle U_i\bar{x}, x - \bar{x}\rangle \ge 0, \ \forall x \in C_i, \ i = 1, 2, \ldots, N \\ &\text{and } \bar{y} \in Q_j \quad \text{such that } \langle V_j\bar{y}, y - \bar{y}\rangle \ge 0, \ \forall y \in Q_j, \ j = 1, 2, \ldots, M \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{such that } A\bar{x} = B\bar{y}, \end{aligned} \tag{5.2}$$

where $U_i : E_1 \to E_1$ and $V_j : E_2 \to E_2$ are two nonlinear mappings, and $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are two bounded linear operators. When viewed separately, Equation (5.2) consists of two classical multiple sets variational inequality problem (MSVIP) whose solution sets are denoted by $\mathrm{VI}(C_i, U_i)$ and $\mathrm{VI}(Q_j, V_j)$, respectively.
Let $U : C \to E^*$ be a nonlinear mapping. Then, U is said to be

(D1) pseudomonotone; if for any $x, y \in C$, we have

$$\langle Ux, y - x\rangle \ge 0 \Longrightarrow \langle Uy, y - x\rangle \ge 0,$$

(D2) K-Lipschitz continuous, if there exists a constant $K > 0$ such that

$$\|Ux - Uy\| \le K\|x - y\|, \quad \forall x, y \in C,$$

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(D3) sequentially weakly continuous, if for any sequence $\{x_n\} \subset C$, we have $x_n \rightharpoonup x \in C$ implying that $Ux_n \rightharpoonup Ux \in E^*$.

We need the following lemma to establish our next result.

Lemma 5.1. *[14] Let C be a nonempty, closed convex subset of a reflexive, smooth and strictly convex Banach space E, $U : C \to E^*$ be a nonlinear mapping. Then*

$$\Pi_C \left(J^q_{E^*}[J^p_E(x) - \lambda U(y)]\right) = \arg\min_{\omega \in C} \{\lambda\langle \omega - y, U(y)\rangle + \Delta_p(\omega, x)\} \tag{5.3}$$

for all $x \in E$, $y \in C$ and $\lambda \in (0, +\infty)$.

Setting $f_i(x, y) = \langle U_i x, y - x\rangle$, $\forall x, y \in C_i$, $i = 1, 2, \ldots, N$ and $g_j(x, y) = \langle V_j x, y - x\rangle$, $\forall x, y \in Q_j$, $j = 1, 2, \ldots, M$ in Algorithm (3.2), then the bifunctions f_i and g_j satisfy conditions (C1)–(C4) of **Assumption A** (see [14]).

Hence, by applying Theorem 4.5 and Lemma 5.1, we obtain the following consequent result for approximating a common solution of MSSEVIP and common fixed point problem for finite families of Bregman quasi-nonexpansive mappings in p-uniformly convex real Banach spaces, which are also uniformly smooth.

Theorem 5.2. *Let E_1, E_2 and E_3 be three p-uniformly convex and uniformly smooth real Banach spaces. Let C_i, $i = 1, 2, \ldots, N$ and Q_j, $j = 1, 2, \ldots, M$ be nonempty, closed and convex subsets of E_1 and E_2, respectively. Let $U_i : C_i \to E^*$ and $V_j : Q_j \to E^*$ be two nonlinear mappings satisfying conditions (D1)–(D3) above. Let $D_s : E_1 \to E_1$, $s = 1, 2, \ldots, l$ and $G_t : E_2 \to E_2$, $t = 1, 2, \ldots, m$ be two finite families of Bregman quasi-nonexpansive mappings such that $I - D_s$ and $I - G_t$ are demiclosed at zero for each s and t, respectively. Suppose that Assumption 3.12(a)–2(c) holds and the solution set $\Upsilon := \{\bar{x} \in F(D_s) \cap VI(C_i, U_i),\ \bar{y} \in F(G_t) \cap VI(Q_j, V_j) : A\bar{x} = B\bar{y}\} \neq \emptyset$. Then, the sequence $\{x_n, y_n\}$ generated by Algorithm (5.3) below converges strongly to $(\bar{x}, \bar{y}) \in \Upsilon$, where $\bar{x} = \Pi_\Upsilon(\mu)$ and $\bar{y} = \Pi_\Upsilon(\vartheta)$.*

Algorithm 5.3. *For fixed $\mu \in E_1$ and $\vartheta \in E_2$, choose an initial guess $(x_0, y_0) \in E_1 \times E_2$. Suppose that the nth iterate $(x_n, y_n) \subset E_1 \times E_2$ has been constructed, then we compute the $(n+1)$th iterate (x_{n+1}, y_{n+1}) via the iteration*

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$$\begin{cases} s_n = J^q_{E_1^*}\left(J^p_{E_1}(x_n) - \rho_n A^* J^p_{E_3}(Ax_n - By_n)\right), \\ a^i_n = \Pi_{C_i}\left[J^p_{E_1^*}(J^p_{E_1}(s_n) - \tau_n U_i(s_n))\right], \quad i = 1, 2, \ldots, N \\ z^i_n = \Pi_{C_i}\left[J^p_{E_1^*}(J^p_{E_1}(a^i_n) - \tau_n U_i(s_n))\right], \quad i = 1, 2, \ldots, N \\ i_n \in \arg\max\left\{\Delta_p(s_n, z^i_n) : i = 1, \ldots, N\right\}, \quad z^{i_n}_n = \bar{z}_n \\ u_n = J^{E_1^*}_q\left(\alpha_{n,0} J^p_{E_1}(\bar{z}_n) + \sum_{i=1}^N \alpha_{n,i} J^p_{E_1}(D_s \bar{z}_n)\right) \\ x_{n+1} = J^q_{E_1^*}\left(\beta_n J^p_{E_1}(\mu) + (1-\beta_n) J^p_{E_1}(u_n)\right), \\ t_n = J^q_{E_2^*}\left(J^p_{E_2}(y_n) + \rho_n B^* J^p_{E_3}(Ax_n - By_n)\right), \\ b^j_n = \Pi_{Q_j}\left[J^{E_2^*}_p(J^{E_2}_p(t_n) - \lambda_n V_j(t_n))\right] \quad j = 1, 2, \ldots, M \\ h^j_n = \Pi_{Q_j}\left[J^{E_2^*}_p(J^{E_2}_p(t_n) - \lambda_n V_j(b^j_n))\right] \quad j = 1, 2, \ldots, M \\ j_n \in \arg\max\left\{\Delta_p(t_n, h^j_n) : j = 1, \ldots, M\right\}, \quad h^{j_n}_n = \bar{\theta}_n \\ v_n = J^q_{E_1^*}\left(\eta_{n,0} J^p_{E_2}(\bar{\theta}_n) + \sum_{t=1}^m \eta_{n,t} J^p_{E_2}(G_t \bar{\theta}_n)\right) \\ y_{n+1} = J^q_{E_2^*}\left(\beta_n J^p_{E_2}(\vartheta) + (1-\beta_n) J^{E_2}_p(v_n)\right). \end{cases}$$

where $\rho_n \in \left(\zeta, \left(\frac{q\|Ax_n - By_n\|^p}{C_q\|A^* J^{E_3}_p(Ax_n - By_n)\|^q + Q_q\|B^* J^{E_3}_p(Ax_n - By_n)\|^q} - \zeta\right)^{\frac{1}{q-1}}\right)$ $n \in \Omega$, *for small enough* ζ, C_q *and* Q_q *are constants of smoothness of* E_1 *and* E_2, *respectively. Otherwise,* $\rho_n = \rho$ *(*ρ *being any nonnegative value), where the set of indexes* $\Omega = \{n : Ax_n - By_n \neq 0\}$.

$$\tau_{n+1} = \begin{cases} \min\left\{\tau_n, \min_{1\le i\le N}\left\{\frac{\kappa(\Delta_p(a^i_n, s_n) + \Delta_p(z^i_n, a^i_n))}{\langle U_i s_n - U_i a^i_n, z^i_n - a^i_n\rangle}\right\}\right\}, & \text{if } \langle U_i s_n - U_i a^i_n, z^i_n - a^i_n\rangle > 0, \\ \tau_n, & \text{otherwise} \end{cases}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \min_{1\le j\le M}\left\{\frac{\epsilon(\Delta_p(b^j_n, t_n) + \Delta_p(h^j_n, b^j_n))}{\langle V_j t_n - V_j b^j_n, h^j_n - b^j_n\rangle}\right\}\right\}, & \text{if } \langle V_j t_n - V_j b^j_n, h^j_n - b^j_n\rangle > 0, \\ \lambda_n, & \text{otherwise.} \end{cases}$$

6. Computational Experiments

In this section, we demonstrate the efficiency and applicability of our proposed method with two numerical examples. In all the experiments, we consider the case when $l = m = N = M = 5$.

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Example 6.1. Let $E_r = \mathbb{R}^m$, $r = 1, 2, 3$, equipped with induced norm $\|x\| = \sqrt{\sum_{i=1}^m |x_i|}$ and the inner product $\langle x, y\rangle = \sum_{i=1}^m x_i y_i$, for all $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ and $y = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m$. Let $C_i = Q_j = C$, where the feasible set C has the form

$$C = \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m_+ : |x_k| \leq 1, \quad k = 1, 2, \ldots, m\}.$$

Consider the following problem:

$$\text{Find } (\bar{x}, \bar{y}) \in \Upsilon := \Bigg\{ \bar{x} \in F(D_s) \cap \bigcap_{i=1}^{N} \mathrm{EP}(C_i, f_i), \\ \bar{y} \in F(G_t) \cap \bigcap_{j=1}^{M} \mathrm{EP}(Q_j, g_j) : A\bar{x} = B\bar{y} \Bigg\},$$

where $f_i : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is given by

$$f_i(x, y) = \sum_{k=1}^{n} (q_{ik} y_k^2 - q_{ik} x_k^2), \quad i = 1, 2, \ldots, N,$$

where $q_{ik} \in (0, 1)$ is randomly selected $\forall i = 1, 2, \ldots, N,\ k = 1, 2, \ldots, m$ and $D_s : \mathbb{R}^m \to \mathbb{R}^m$ is defined by

$$D_s(x) = \frac{x}{s+1}, \quad \forall s = 1, 2, \ldots, l.$$

In the same vein, let $g_j : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is given by

$$g_j(x, y) = \sum_{k=1}^{m} (q_{jk} y_k^2 - q_{jk} x_k^2), \quad j = 1, 2, \ldots, M,$$

where $q_{jk} \in (0, 1)$ is randomly selected $\forall j = 1, 2, \ldots, M,\ k = 1, 2, \ldots, 5$ and $G_t : \mathbb{R}^m \to \mathbb{R}^m$ is defined by

$$G_t(x) = \frac{x}{t+1}, \quad \forall t = 1, 2, \ldots, 5.$$

It is easy to see that conditions (C1)–(C4) of **Assumption A** are satisfied and D_s and G_t are Bregman quasi-nonexpansive mappings for $s = 1, 2, \ldots, l$ and $t = 1, 2, \ldots, m$, respectively, $I - D_s$ and $I - G_t$ are demiclosed at zero. Moreover, we define $A(x) = \frac{x}{2}$ and $B(x) = \frac{x}{3}$, then A and B are bounded linear operators. Furthermore, $\Upsilon = \{0\}$. In this example, we choose $\beta_n = \frac{3}{2n+3}$, $\kappa = 0.36$, $\tau_0 = 0.24$, $\epsilon = 0.5$, $\lambda_0 = 0.4$, $\alpha_{n,0} = \frac{3n}{8n+11}$, $\alpha_{n,s} = \frac{1}{5}(1 - \frac{3n}{8n+11})$, $s = 1, 2, \ldots, 5$, $\eta_{n,0} = \frac{2n}{4n+7}$, $\eta_{n,t} = \frac{1}{5}(1 - \frac{2n}{4n+7})$, $t = 1, 2, \ldots, 5$. Using $\frac{\|x_{n+1} - x_n\|}{\|x_2 - x_1\|} < 10^{-4}$ as our stopping criterion, we generate randomly different starting points $(\mu, \vartheta), (x_0, y_0) \in E_1 \times E_2$ for different cases of $m = 20,\ 50,\ 100$ and 500.

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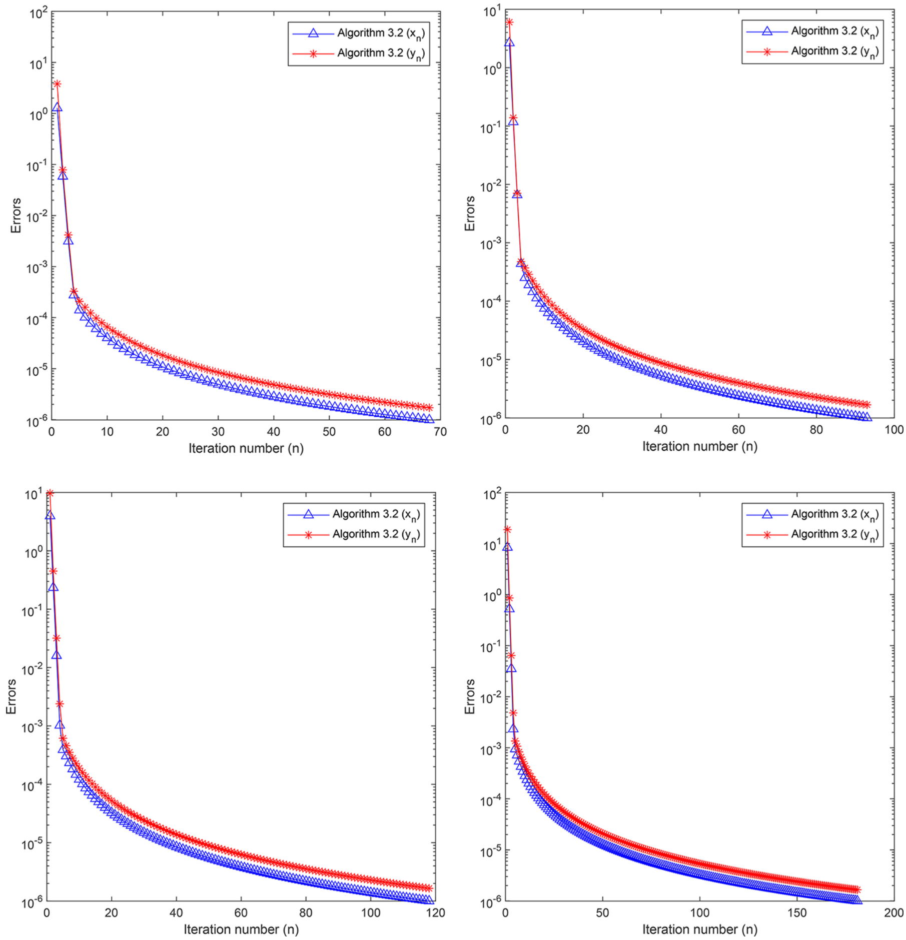

Figure 1. Top left: m = 20; top right: m = 50; bottom left: m = 100; bottom right: m = 500.

We plot the graphs $\|x_{n+1} - x_n\|$ against the number of iterations. The numerical results can be seen in Figure 1.

The next example is presented in an infinite dimensional space setting.

Example 6.2. Let $E_r = L^2([0,1])$, $r = 1,2,3$, with the induced norm given by $\|x\|_L = \int_0^1 |x(s)|^2 \, ds$ and the corresponding inner product $\langle x, y\rangle = \int_0^1 x(s)y(s) \, ds$. Let the feasible sets C_i and Q_j be defined as follows:

$$C_i := \{x \in H : \|x\|_L \leq 1\} \; i = 1,2,\ldots,5 \quad \text{and} \quad Q_j := \{x \in H : \|x\|_L \leq 1\} \; j = 1,2,\ldots,5.$$

Let $f_i(x,y) = \langle S_i x, y - x\rangle$ and $g_j(x,y) = \langle T_j x, y - x\rangle$ with the operators $(S_i x)(t) = \max\left\{0, \frac{x(t)}{i}\right\}$ for $i = 1,2,\ldots,5$ and $(T_j x)(t) = \max\left\{0, \frac{x(t)}{j}\right\}$ for $j = 1,2,\ldots,5$. Then, it is easy to see that each f_i is monotone (and by implication, pseudomonotone) on C_i. Similarly, g_j is pseudomonotone on Q_j. Furthermore, let $D_s : H \to H$ and $G_t : H \to H$

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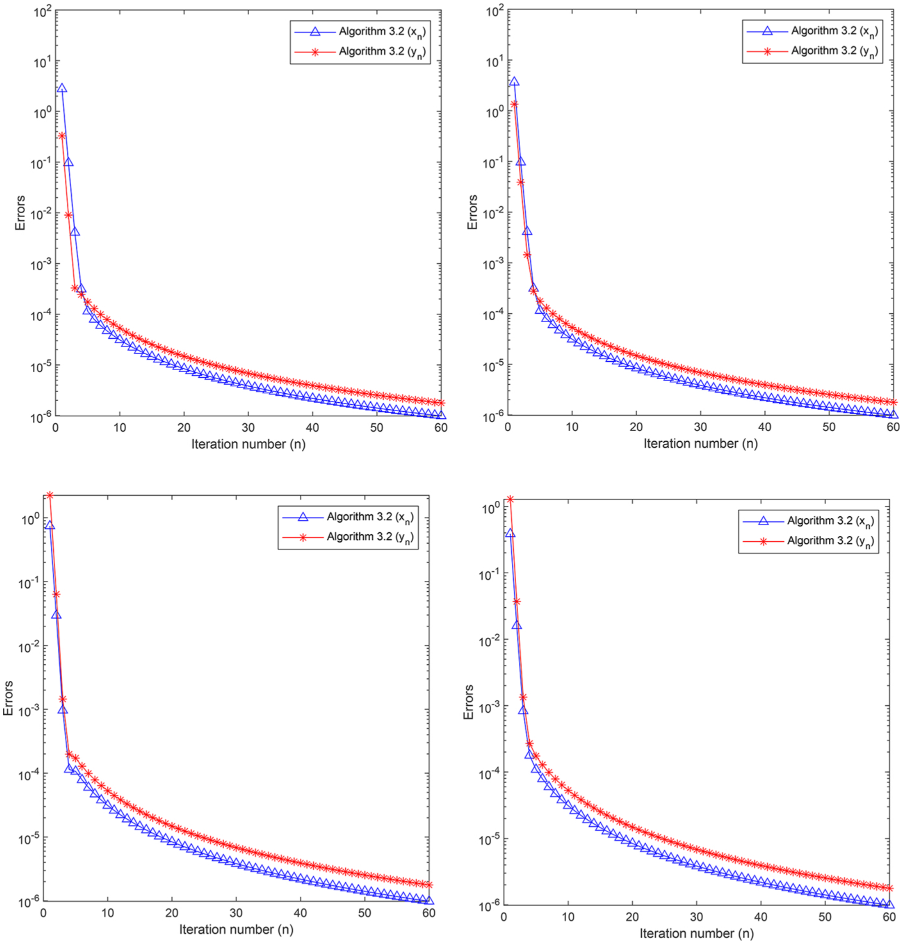

Figure 2. Top left: Case I; top right: Case II; bottom left: Case III; bottom right: Case IV.

be defined by $D_s(x)(t) = \frac{x(t)}{2s}$ and $G_t(x)(t) = \frac{x(t)}{2t}$, then the mappings D_s and G_t are quasi-nonexpansive $\forall s = 1, 2, \ldots, 5$ and $t = 1, 2, \ldots, 5$. Moreover, we define $A(x)(t) = \frac{x(t)}{3}$ and $B(x)(t) = \frac{x(t)}{5}$, then A and B are bounded linear operators. The solution set $\Upsilon = \{0\}$. We choose $\beta_n = \frac{1}{n+2}$, $\alpha_{n,0} = \frac{n+1}{2n+3}$, $\alpha_{n,s} = \frac{1}{5}(1 - \frac{n+1}{2n+3})$, $s = 1, 2, \ldots, 5$, $\eta_{n,0} = \frac{n+2}{2n+5}$, $\eta_{n,t} = \frac{1}{5}(1 - \frac{n+2}{2n+5})$, $t = 1, 2, \ldots, 5$, $\kappa = 0.54$, $\tau_0 = 0.63$, $\epsilon = 0.75$, $\lambda_0 = 0.83$, and using $\frac{\|x_{n+1} - x_n\|}{\|x_2 - x_1\|} < 10^{-4}$ as stopping criterion.

We choose fixed points $\mu = t^2 + 2$, $\vartheta = 4t^3 + 3$ and different starting points as follows:

Case I: $x_0 = t^2 + 4$, $\quad y_0 = t^3 + 2t + 1$
Case II: $x_0 = t^4 + 5$, $\quad y_0 = t^2 + t + 3$
Case III: $x_0 = \sin(2t)$, $\quad y_0 = \cos(5t)$
Case IV: $x_0 = \exp(t)$, $\quad y_0 = \exp(2t)$.

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We plot the graph of errors against the number of iterations in each case. The numerical results can be found in Figure 2.

7. Conclusion

In this article, using Bregman distance, we proposed and studied a new algorithm for approximating the common solution of multiple set split equality pseudomonotone EP and fixed points of Bregman quasi-nonexpansive mappings in real p-uniformly convex Banach spaces, which are also uniformly smooth. The algorithm under discourse is designed in such a way that its convergence does not rely on prior estimates of the Lipschitz constants of the pseudomonotone bifunctions as well as the prior knowledge of the norm of the bounded linear operators. We proved a strongly convergent theorem under some mild conditions on our parameters. We gave a theoretical application of our result and finally present two numerical examples to show the efficiency and applicability of our method. The result presented in this article extends numerous results in the literature in this direction of research.

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Competing Interests. The authors declare that they have no conflict of interest.

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