

Distribution in homology classes and discrete fractal dimension*

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In this note, we examine the proportion of periodic orbits of Anosov flows that lie in an infinite zero density subset of the first homology group. We show that on a logarithmic scale we get convergence to a discrete fractal dimension.

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1. Introduction

There has been a considerable body of research on how closed geodesics on compact negatively curved manifolds and, more generally, periodic orbits of Anosov flows are distributed in homology classes, for example [1, 3, 11–14, 19, 21, 24]. To state these results more precisely, let $\phi^t : M \rightarrow M$ be a transitive Anosov flow such that the winding cycle associated with the measure of maximal entropy vanishes. This class of flows includes geodesic flows over compact negatively curved manifolds. The basic counting result is that the number of period orbits of length at most T and lying in a homology class $\alpha \in H_1(M, \mathbb{Z})$ is asymptotic to $(\text{constant}) \times e^{hT}/T^{1+k/2}$, where h is the topological entropy of the flow and $k \geq 0$ is the first Betti number of M . Furthermore, the distribution is Gaussian and the constant above is related to the variance [14, 17, 25].

It is also interesting to ask about the distribution of periodic orbits lying in a set $A \subset H_1(M, \mathbb{Z})$. If A is finite, the behaviour follows from that for single homology classes, so we suppose that A is infinite. This, of course, implies that $H_1(M, \mathbb{Z})$ is infinite, i.e. $k \geq 1$. Petridis and Risager [18] (for compact hyperbolic surfaces) and Collier and Sharp [6] (for Anosov flows for which the measure of maximal entropy has vanishing winding cycle) independently showed if A has positive density then the proportion of periodic orbits of length at most T lying

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in A converges to the density of A (with respect to an appropriate norm), as $T \rightarrow \infty$. To state this and our new results more precisely, let \mathcal{P} denote the set of prime periodic orbits for ϕ and, for $\gamma \in \mathcal{P}$, let $\ell(\gamma)$ denote the least period of γ and $[\gamma] \in H_1(M, \mathbb{Z})$ denote the homology class of γ . It is convenient to ignore any torsion in $H_1(M, \mathbb{Z})$, so we can think of $H_1(M, \mathbb{Z})$ as a lattice in $H_1(M, \mathbb{R}) \cong \mathbb{R}^k$. Write $\mathcal{P}_T = \{\gamma \in \mathcal{P} : \ell(\gamma) \leq T\}$, $\mathcal{P}_T(\alpha) = \{\gamma \in \mathcal{P}_T : [\gamma] = \alpha\}$, and $\mathcal{P}_T(A) = \bigcup_{\alpha \in A} \mathcal{P}_T(\alpha)$. Fixing a norm $\|\cdot\|$ on $H_1(M, \mathbb{R})$, write $\mathfrak{N}_A(r) = \#\{\alpha \in A : \|\alpha\| \leq r\}$ and $\mathfrak{N}(r) = \mathfrak{N}_{H_1(M, \mathbb{Z})}(r)$. We say that $A \subset H_1(M, \mathbb{Z})$ has density $d_{\|\cdot\|}(A)$ (with respect to $\|\cdot\|$) if

$$\lim_{r \rightarrow \infty} \frac{\mathfrak{N}_A(r)}{\mathfrak{N}(r)} = d_{\|\cdot\|}(A).$$

PROPOSITION 1.1 [6, 18]). *Let $\phi^t : M \rightarrow M$ be a transitive Anosov flow for which the winding cycle associated with the measure of maximal entropy vanishes. Then there exists a norm $\|\cdot\|$ on $H_1(M, \mathbb{R})$ such that if $A \subset H_1(M, \mathbb{Z})$ has density $d_{\|\cdot\|}(A)$ then*

$$\lim_{T \rightarrow \infty} \frac{\#\mathcal{P}_T(A)}{\#\mathcal{P}_T} = d_{\|\cdot\|}(A).$$

The norm (defined in § 2) is a Euclidean norm determined by the second derivative of a pressure function.

Now, suppose A has density zero. It is interesting to ask whether we can obtain more precise information about the behaviour of

$$\mathcal{D}(T, A) := \frac{\#\mathcal{P}_T(A)}{\#\mathcal{P}_T}$$

as $T \rightarrow \infty$. If we write $\rho_A(r) = \mathfrak{N}_A(r)/\mathfrak{N}(r)$, then the naive conjecture is that $\mathcal{D}(T, A)$ is of order $\rho_A(\sqrt{t})$, as $T \rightarrow \infty$, and this is consistent with case $A = \{\alpha\}$. It is too optimistic to hope that a precise asymptotic relation holds for general A . Nevertheless, one might hope for information on the logarithmic scale if we use some notion of discrete fractal dimension. We say that A has discrete mass dimension δ if

$$\lim_{r \rightarrow \infty} \frac{\log \mathfrak{N}_A(r)}{\log r} = \delta$$

or, equivalently, that if

$$\mathfrak{N}_A(r) = r^\delta \kappa_A(r) \tag{1.1}$$

then $\lim_{r \rightarrow \infty} \log \kappa_A(r)/\log r = 0$. (Note that this is independent of the choice of norm $\|\cdot\|$.) For a discussion of discrete fractal dimensions, see [4].

EXAMPLE 1.2. Suppose $A \subset \mathbb{Z}$ is given by $A = \{\pm m^2 : m \in \mathbb{N}\}$ then the discrete mass dimension of A is $1/2$. More interesting examples appear in percolation theory (see, e.g. [10]).

Our main result is the following.

THEOREM 1.3. *Let $\phi^t : M \rightarrow M$ be a transitive Anosov flow for which the winding cycle associated with the measure of maximal entropy vanishes. If $A \subset H_1(M, \mathbb{Z})$ has discrete mass dimension δ then*

$$\lim_{T \rightarrow \infty} \frac{\log \mathcal{D}(T, A)}{\log T} = \frac{\delta - k}{2}.$$

Let Σ be a compact orientable surface of genus $g \geq 2$ with a Riemannian metric g of negative curvature and let $T^1\Sigma$ denote the unit tangent bundle. Then the natural projection $p : T^1\Sigma \rightarrow \Sigma$ induces a homomorphism $p_* : H_1(T^1\Sigma, \mathbb{Z}) \rightarrow H_1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ whose kernel is the torsion subgroup, and induces a bijection between the prime periodic orbits of the geodesic flow and primitive closed geodesics on Σ such that $\ell(\gamma) = \text{length}_g(p(\gamma))$ and $p_*([\gamma]) = [p(\gamma)]$. If, for $A \subset H_1(\Sigma, \mathbb{Z})$, we define $\mathcal{D}_\Sigma(T, A)$ to be the proportion of closed primitive geodesics on Σ with g -length at most T and with homology class in A , then we have the following corollary.

COROLLARY 1.4. *Let Σ be a compact orientable surface of genus $g \geq 2$ with a Riemannian metric of negative curvature. If $A \subset H_1(\Sigma, \mathbb{Z})$ has discrete mass dimension δ then*

$$\lim_{T \rightarrow \infty} \frac{\log \mathcal{D}_\Sigma(T, A)}{\log T} = \frac{\delta - 2g}{2}.$$

2. Anosov flows

Let M be a compact Riemannian manifold and $\phi^t : M \rightarrow M$ be a transitive Anosov flow [2, 8]. We suppose that M has first Betti number $k \geq 1$ and ignore any torsion in $H_1(M, \mathbb{Z})$. Using the notation of the introduction, we say that ϕ is *homologically full* if the map $\mathcal{P} \rightarrow H_1(M, \mathbb{Z}) : \gamma \mapsto [\gamma]$ is a surjection. This automatically implies that the flow is weak-mixing (since an Anosov flow fails to be weak-mixing only when it is a constant suspension of an Anosov diffeomorphism [20], in which case it can have no null homologous periodic orbits) and hence that

$$\#\mathcal{P}_T \sim \frac{e^{hT}}{hT},$$

as $T \rightarrow \infty$, where $h > 0$ is the topological entropy of ϕ [15, 16]. There is a unique measure of maximal entropy μ for which the measure-theoretic entropy $h_\mu(\phi) = h$ [5]. (See [8] for the notions of topological and measure-theoretic entropy for ϕ .)

Let \mathcal{M}_ϕ denote the set of ϕ -invariant Borel probability measures on M . For a continuous function $f : M \rightarrow \mathbb{R}$, we define its *pressure* $P(f)$ by

$$P(f) = \sup \left\{ h_\nu(\phi) + \int f d\nu : \nu \in \mathcal{M}_\phi \right\}.$$

Given $\nu \in \mathcal{M}_\phi$, we can define the associated winding cycle $\Phi_\nu \in H_1(M, \mathbb{R})$ by

$$\langle \Phi_\nu, [\omega] \rangle = \int \omega(Z) d\nu,$$

where $[\omega]$ is the cohomology class of the closed 1-form ω , Z is the vector generating ϕ , and $\langle \cdot, \cdot \rangle$ is the duality pairing [23, 26]. Write $\mathcal{B}_\phi = \{\Phi_\nu : \nu \in \mathcal{M}_\phi\}$; this is a

compact and convex subset of $H_1(M, \mathbb{R})$. The assumption that ϕ is homologically full is equivalent to $0 \in \text{int}(\mathcal{B}_\phi)$ and implies that there are fully supported measures ν for which $\Phi_\nu = 0$. We will impose the more stringent condition that $\Phi_\mu = 0$, where μ is the measure of maximal entropy for ϕ . This class includes geodesic flows over compact negatively manifolds with negative sectional curvature. (In the case considered in corollary 1.4, \mathcal{B}_ϕ may be identified with the unit-ball for the Federer–Gromov stable norm on $H_1(\Sigma, \mathbb{R})$ [7, 9].)

Still assuming that $\Phi_\mu = 0$, there is an analytic pressure function $\mathfrak{p} : H^1(M, \mathbb{R}) \rightarrow \mathbb{R}$, defined by $\mathfrak{p}([\omega]) = P(\omega(Z))$ [12, 24]. This is a strictly convex function with positive definite Hessian; it has a unique minimum at 0. For $\xi \in H^1(M, \mathbb{R})$, we define $\sigma_\xi > 0$ by

$$\sigma_\xi^{2k} = \det \nabla^2 \mathfrak{p}(\xi)$$

and set $\sigma = \sigma_0$. There is also an analytic entropy function $\mathfrak{h} : \text{int}(\mathcal{B}_\phi) \rightarrow \mathbb{R}$ defined by $\mathfrak{h}(\rho) = \sup\{h_\nu(\phi) : \Phi_\nu = \rho\}$ such that \mathfrak{p} and $-\mathfrak{h}$ are Legendre conjugates (via the pairing $\langle \cdot, \cdot \rangle$) [22]. More precisely, $-\nabla \mathfrak{h} : \text{int}(\mathcal{B}_\phi) \rightarrow H^1(M, \mathbb{R})$ and $\nabla \mathfrak{p} : H^1(M, \mathbb{R}) \rightarrow \text{int}(\mathcal{B}_\phi)$ are inverses and

$$\mathfrak{h}(\rho) = \mathfrak{p}((\nabla \mathfrak{p})^{-1}(\rho)) - \langle (\nabla \mathfrak{p})^{-1}(\rho), \rho \rangle.$$

We write $\xi(\rho) = (\nabla \mathfrak{p})^{-1}(\rho)$. Then $-\nabla^2 \mathfrak{h}(\rho) = (\nabla^2 \mathfrak{p}(\xi(\rho)))^{-1}$. In particular, $\xi(0) = 0$, $\mathcal{H} := -\nabla^2 \mathfrak{h}(0) = (\nabla^2 \mathfrak{p}(0))^{-1}$ is positive definite and $\det \mathcal{H} = (\det \nabla^2 \mathfrak{p}(0))^{-1} = \sigma^{-2k}$. We use \mathcal{H} to define a norm $\|\cdot\|$ on $H_1(M, \mathbb{R})$ by

$$\|\rho\| = \langle \rho, \mathcal{H}\rho \rangle.$$

We note that

$$\mathfrak{N}(r) := \{\alpha \in H_1(M, \mathbb{Z}) : \|\alpha\| \leq r\} \sim \mathfrak{v}_k \sigma^k r^k,$$

where $\mathfrak{v}_k = \pi^{k/2} / \Gamma(k/2 + 1)$, the volume of the standard unit-ball in \mathbb{R}^k . For small ρ , Taylor’s theorem gives us the expansion

$$\mathfrak{h}(\rho) = h - \|\rho\|^2 / 2 + O(\|\rho\|^3). \tag{2.1}$$

We now consider the periodic orbits of ϕ^t . As above, we ignore the torsion in $H_1(M, \mathbb{Z})$ and treat it as a lattice in $H_1(M, \mathbb{R})$. We fix a fundamental domain \mathcal{F} and, for $\rho \in H_1(M, \mathbb{R})$, we define $[\rho] \in H_1(M, \mathbb{Z})$ by $\rho - [\rho] \in \mathcal{F}$.

PROPOSITION 2.1 (Theorem 3.5 of [3]). *Let $\phi^t : M \rightarrow M$ be a homologically full transitive Anosov flow. If $\rho \in \text{int}(\mathcal{B}_\phi)$ and $\alpha \in H_1(M, \mathbb{Z})$ then*

$$\#\{\gamma \in \mathcal{P}_T : [\gamma] = \alpha + [\rho T]\} \sim c(\rho) e^{\langle \xi(\rho), T\rho - [T\rho] - \alpha \rangle} \frac{e^{\mathfrak{h}(\rho)T}}{T^{1+k/2}},$$

as $T \rightarrow \infty$, uniformly for ρ in any compact subset of \mathcal{B}_ϕ , where $c(\rho) = 1 / ((2\pi)^{k/2} \sigma_{\xi(\rho)}^k \mathfrak{h}(\rho))$.

REMARK 2.2. The statement of Theorem 3.5 in [3] is for the more general class of hyperbolic flows and the hypotheses there are a condition called Assumption A

(see p. 18 of [3]) and that \mathcal{B}_ϕ contains zero in its interior. For transitive Anosov flows, it was proved in [24] that both of these are implied by the flow being homologically full. Counting results for a linearly varying homology class were originally proved by Lalley in the more restricted setting of geodesic flows over compact negatively curved surfaces [14].

If $\Phi_\mu = 0$, we can set $\rho = 0$ and recover the asymptotic

$$\#\mathcal{P}_T(\alpha) \sim \frac{1}{(2\pi)^{k/2}\sigma^k h} \frac{e^{hT}}{T^{1+k/2}},$$

originally proved by Katsuda and Sunada [12]. Furthermore, for all sufficiently small $\Delta > 0$, we have

$$\lim_{T \rightarrow \infty} \sup_{\|\alpha\| \leq \Delta T} \left| \frac{T^{1+k/2} \#\mathcal{P}_T(\alpha)}{c(\alpha/T) e^{h(\alpha/T)T}} - 1 \right| = 0. \tag{2.2}$$

3. Proof of theorem 1.3

3.1. Upper bound

In this section, we show that $(\delta - k)/2$ gives an upper bound for the limit in theorem 1.3. The main idea is to use proposition 2.1 and the Taylor expansion of $\mathfrak{h}(\rho)$ to replace $\mathcal{D}(T, A)$ with a sum of Gaussian terms over elements of A with norm bounded by $\eta\sqrt{T \log T}$, for $\eta > 0$ chosen sufficiently large that the resulting error decays faster than $T^{(\delta-k)/2}$.

We begin with the trivial observation that

$$\mathcal{D}(T, A) - e^{-hT} hT \#\mathcal{P}_T(A) = o(\mathcal{D}(T, A)),$$

so that it is sufficient to consider $e^{-hT} hT \#\mathcal{P}_T(A)$. We can make the following approximation.

LEMMA 3.1. *For any $\eta > 0$,*

$$\begin{aligned} \sum_{\substack{\alpha \in A \\ \|\alpha\| \leq \eta\sqrt{T \log T}}} & \left(\frac{hT \#\mathcal{P}_T(\alpha)}{e^{hT}} - \frac{e^{-\|\alpha\|^2/2T}}{(2\pi)^{k/2}\sigma^k T^{k/2}} \right) \\ & = o\left(T^{(\delta-k)/2} (\log T)^{\delta/2} \kappa_A(\eta\sqrt{T \log T})\right), \end{aligned}$$

where κ_A is defined by equation (1.1).

Proof. Let $\eta > 0$. Clearly, (2.2) still holds if we take the supremum over $\|\alpha\| \leq \eta\sqrt{T \log T}$. Over this set, we have $c(\alpha/T) = c(0) + O(\|\alpha\|/T) = c(0) +$

$O(\sqrt{\log T}/\sqrt{T})$ and

$$\mathfrak{h}\left(\frac{\alpha}{T}\right)T = ht - \frac{\|\alpha\|^2}{2T} + O\left(\frac{\|\alpha\|^3}{T^2}\right) = hT - \frac{\|\alpha\|^2}{2T} + O\left(\frac{(\log T)^{3/2}}{\sqrt{T}}\right).$$

Substituting these in, we obtain an estimate

$$\sup_{\|\alpha\| \leq \eta\sqrt{T \log T}} \left| \frac{hT \# \mathcal{P}_T(\alpha)}{e^{hT}} - \frac{e^{-\|\alpha\|^2/2T} e^{q(\alpha, T)}}{(2\pi)^{k/2} \sigma^k T^{k/2}} \right| = o(T^{-k/2}),$$

where $|q(\alpha, T)| \leq c'(\log T)^{3/2}T^{-1/2}$, for some $c' > 0$. A simple calculation then shows that we may remove the $q(\alpha, T)$ terms, while keeping the $o(T^{-k/2})$ error term. To complete the proof, we note that summing over $\|\alpha\| \leq \eta\sqrt{T \log T}$ involves $\mathfrak{N}_A(\eta\sqrt{T \log T}) = O(T^{\delta/2}(\log T)^{\delta/2} \kappa_A(\eta\sqrt{T \log T}))$ summands. \square

Next, we estimate the Gaussian part from the previous lemma.

LEMMA 3.2.

$$\sum_{\substack{\alpha \in A \\ \|\alpha\| \leq \eta\sqrt{T \log T}}} \frac{e^{-\|\alpha\|^2/2T}}{(2\pi)^{k/2} \sigma^k T^{k/2}} = O(T^{(\delta-k)/2}(\log T)^{\delta/2} \kappa_A(\eta\sqrt{T \log T})).$$

Proof. The result follows from the elementary estimate

$$\sum_{\substack{\alpha \in A \\ \|\alpha\| \leq \eta\sqrt{T \log T}}} \frac{e^{-\|\alpha\|^2/2T}}{(2\pi)^{k/2} \sigma^k T^{k/2}} = O\left(\frac{\mathfrak{N}_A(\eta\sqrt{T \log T})}{T^{k/2}}\right).$$

\square

The contribution from $\|\alpha\| > \eta\sqrt{T \log T}$ is estimated as follows.

LEMMA 3.3.

$$\sum_{\substack{\alpha \in A \\ \|\alpha\| > \eta\sqrt{T \log T}}} \frac{hT \# \mathcal{P}_T(\alpha)}{e^{hT}} = O(T^{-\eta^2/2}(\log T)^{3k/2-2}).$$

Proof. Applying proposition 2.1, we see that, for $x \in \mathbb{R}^k$ and $\mathcal{C}(\Delta)$ a cube of (small) side length Δ based at 0,

$$\begin{aligned} & hT e^{-hT} \# \left\{ \gamma \in \mathcal{P}_T : \frac{[\gamma]}{\sqrt{T \log T}} \in x + \mathcal{C}(\Delta) \right\} \\ & \sim h e^{-hT} c(x\sqrt{(\log T)/T}) \frac{e^{\mathfrak{h}(x\sqrt{(\log T)/T})T}}{T^{k/2}} (\Delta\sqrt{T \log T})^k \\ & \sim \frac{\Delta^k (\log T)^{k/2} e^{-(\|x\|^2 \log T)/2}}{(2\pi)^{k/2} \sigma^k}. \end{aligned}$$

Thus, we can estimate the sum in the statement by $(\log T)^{k/2} I_\eta(T)$, where $I_\eta(T)$ is the integral:

$$I_\eta(T) := \frac{1}{(2\pi)^{k/2} \sigma^k} \int_{B(\eta)} e^{-\|x\|^2 \log T/2} dx,$$

where $B(\eta) = \{x \in \mathbb{R}^k : \|x\| > \eta\}$. Substituting $u = x\sqrt{\log T}$ and passing to coordinates (r, θ) with $r > 0$ and $\|\theta\| = 1$, we obtain

$$I_\eta(T) = \frac{\text{Area}(\{\theta : \|\theta\| = 1\})}{(2\pi)^{k/2} \sigma^k} \int_{\eta\sqrt{T}}^\infty e^{-r^2/2} r^{k-1} dr = O(T^{-\eta^2/2} (\log T)^{k-2}),$$

where we have used standard asymptotics for the complementary error function $\text{erfc}(z)$. □

To complete the proof of the upper bound, choose $\eta > \sqrt{k - \delta}$. Then combining lemmas 3.1–3.3 and noting that

$$\lim_{T \rightarrow \infty} \frac{\log \kappa_A(\eta\sqrt{T \log T})}{\log T} = \lim_{T \rightarrow \infty} \frac{\log \kappa_A(\eta\sqrt{T \log T})}{\log(\eta\sqrt{T \log T})} \frac{\log(\eta\sqrt{T \log T})}{\log T} = 0$$

shows that

$$\limsup_{T \rightarrow \infty} \frac{\log \mathcal{D}(T, A)}{\log T} \leq \frac{\delta - k}{2}. \tag{3.1}$$

3.2. Lower bound

Since we seek a lower bound, we only need to consider

$$\sum_{\substack{\alpha \in A \\ \|\alpha\| \leq \sqrt{T}}} \frac{hT \# \mathcal{P}_T(\alpha)}{e^{hT}}.$$

The following result is almost identical to lemma 3.1 and we do not repeat the proof.

LEMMA 3.4.

$$\sum_{\substack{\alpha \in A \\ \|\alpha\| \leq \sqrt{T}}} \left(\frac{hT \# \mathcal{P}_T(\alpha)}{e^{hT}} - \frac{e^{-\|\alpha\|^2/2T}}{(2\pi)^{k/2} \sigma^k T^{k/2}} \right) = o\left(T^{(\delta-k)/2} \kappa_A(\sqrt{T})\right).$$

Since we have the bound

$$\sum_{\substack{\alpha \in A \\ \|\alpha\| \leq \sqrt{T}}} \frac{e^{-\|\alpha\|^2/2T}}{(2\pi)^{k/2} \sigma^k T^{k/2}} \geq \frac{e^{-2}}{(2\pi)^{k/2} \sigma^k T^{k/2}} \mathfrak{N}_A(\sqrt{T}) = \frac{e^{-2}}{(2\pi)^{k/2} \sigma^k} T^{(\delta-k)/2} \kappa_A(\sqrt{T}),$$

we conclude that

$$\liminf_{T \rightarrow \infty} \frac{\log \mathcal{D}(T, A)}{\log T} \geq \frac{\delta - k}{2}.$$

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